Theoretical Particle Physics I

Institut für Theoretische Teilchenphysik

Prof. Dr. Matthias Steinhauser, Dr. Daniel Stremmer, Dr. Marco Vitti, Pascal Reeck

Winter term 24/25 – Sheet 1 Due: 28.10.2024, Discussion: 30.10.2024

Exercise 1: Lorentz transformation (7 points)
Consider the Lorentz transformation

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu \, x^\nu \,, \qquad g_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\alpha\beta} \,, \tag{1}$$

where Λ is the Lorentz transformation matrix with Minkowski metric $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)_{\mu\nu}$.

a) The principles of special relativity imply that the spacetime interval is invariant for all observers, i.e.

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} dx'^\mu dx'^\nu = ds'^2 \,, \tag{2}$$

holds under a transformation of coordinates $x^\mu \to x'^\mu(x)$. Show that the spacetime interval invariance not only implies the Lorentz transformation, but also the Poincaré transformation $x'^\mu = \Lambda^\mu{}_\nu \, x^\nu + a^\mu$, with a^μ being the translation.

b) A generic element Λ of the proper, orthochronous Lorentz group $\mathrm{SO}(3,1)^+$ reads

$$\Lambda = \exp\left[-\frac{i}{2}\omega_{\mu\nu} L^{\mu\nu}\right] \,, \tag{3}$$

$$(L^{\mu\nu})^\rho{}_\sigma = i(g^{\mu\rho} g^\nu{}_\sigma - g^{\nu\rho} g^\mu{}_\sigma) \,, \tag{4}$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$. $L^{\mu\nu}$ are the generators of $\mathrm{SO}(3,1)^+$ having the form as in (4) when acting on four-vectors. Show that the first order expansion in ω of eq. (3) is equivalent to

$$\Lambda^\rho{}_\sigma = g^\rho{}_\sigma + \omega^\rho{}_\sigma + \mathcal{O}(\omega^2) \,. \tag{5}$$

c) How many degrees of freedom (i.e. real parameters) does the Lorentz group have? Try to show this using a symmetry argument.

d) The explicit form of ω has the following expression

$$(\omega)^\rho{}_\sigma = \begin{pmatrix} 0 & -\eta_1 & -\eta_2 & -\eta_3 \\ -\eta_1 & 0 & -\theta_3 & \theta_2 \\ -\eta_2 & \theta_3 & 0 & -\theta_1 \\ -\eta_3 & -\theta_2 & \theta_1 & 0 \end{pmatrix} = \sum_i (\theta_i L_i - \eta_i K_i)^\rho{}_\sigma \,. \tag{6}$$

Rewrite the angular momentum operators L_i and the boost operators K_i in terms of the generators $L^{\mu\nu}$ using equation (4).

e) Using your results obtained in (d), derive the expression for

$$\Delta x^\rho = \Lambda^\rho{}_\sigma x^\sigma - x^\rho \tag{7}$$

under the following transformations:

- a rotation by an angle θ_3 around the z-axis,
- a boost η_1 along the x-axis.

Exercise 2: Relativistic addition of velocities (3 points)

a) Write down the transformation matrix of a Lorentz boost as a function of the rapidity η (the velocity is given by $v = \tanh \eta$).

b) Consider the multiplication of two Lorentz transformation matrices with the rapidities η_1 and η_2 in the same direction. What is the rapidity of the combined Lorentz boost? Determine from this an equation for the relativistic addition of collinear velocities.

Exercise 3: Kinematics of a decay process

Consider the decay of a Z-boson with mass $M_Z = 91.19\,\mathrm{GeV}$ in its rest frame into a $\tau^+\tau^-$ pair ($m_\tau = 1.78\,\mathrm{GeV}$).

a) Compute the energy and the momentum (in GeV) of the decay products.

b) The mean lifetime of tau leptons is $2.9 \cdot 10^{-13}\,\mathrm{s}$. How far do the tau leptons travel on average?

Exercise 4: Proof that Poincaré group is a group

The Poincaré transformation is a generalisation of the Lorentz transformation and is given by

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu \, x^\nu + a^\mu, \tag{8}$$

where a^μ is a constant four-vector. Show that the set $\mathcal{P}$ of all Poincaré transformations is a group. If the elements of the group are the pairs (Λ, a) with composition rule given by

$$(\Lambda_1, a_1) \circ (\Lambda_2, a_2) = (\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1) \tag{9}$$

show that the following properties are satisfied

1. (Closure) The composition of any two elements of $\mathcal{P}$ is still an element of $\mathcal{P}$
2. (Associative Law) For any three elements T_1, T_2, T_3 within $\mathcal{P}$, the following holds

$$(T_1 \circ T_2) \circ T_3 = T_1 \circ (T_2 \circ T_3)$$

3. For every T, there exists an identity element I such that

$$(T \circ I) = (I \circ T) = T$$

4. For each T, there exists an inverse element T^{-1} such that

$$(T \circ T^{-1}) = (T^{-1} \circ T) = I$$
