Theoretical Particle Physics I

Institut für Theoretische Teilchenphysik

Prof. Dr. Matthias Steinhauser, Dr. Daniel Stremmer, Dr. Marco Vitti, Pascal Reeck

Winter term 24/25 – Sheet 2 Due: 4.11.2024, Discussion: 6.11.2024

Exercise 1: Dirac Equation

The γ matrices in the Dirac representation are given by

$$\gamma^0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix},$$

with the Pauli matrices σ^k. 1_2 is the 2×2 identity matrix.

a) The Dirac equation is given in momentum space by

$$(\not{p} - m)u(p) = 0,$$

where $\not{p} = p^\mu \gamma_\mu$. Construct all four solutions, $u_{1,2,3,4}(p)$.

b) Construct all four solutions $v_{1,2,3,4}(p)$ of the anti-particle Dirac equation

$$(\not{p} + m)v(p) = 0.$$

c) Obtain the relation between $v_{1,2}(p)$ and $u_{3,4}(p)$.

d) Derive the spin of $u_{1,2,3,4}(p)$ and $v_{1,2,3,4}(p)$.

Exercise 2: Dirac spinors (5 points)

Consider the following representation of the Dirac spinors

$$u_s(k) = \sqrt{k_0 + m} \begin{pmatrix} \chi_s \\ \frac{\vec{k}\cdot\vec{\sigma}}{k_0+m}\chi_s \end{pmatrix}, \quad v_s(k) = \sqrt{k_0 + m} \begin{pmatrix} \frac{\vec{k}\cdot\vec{\sigma}}{k_0+m}\eta_s \\ \eta_s \end{pmatrix},$$

with $k_0 = \sqrt{\vec{k}^2 + m^2}$, $s = \pm$ is the spin orientation and

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \eta_+ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \eta_- = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

a) Show that these spinors satisfy the Dirac equation, i.e. that the following equations hold

$$(\not{k} - m)\, u_s(k) = 0\,, \quad (\not{k} + m)\, v_s(k) = 0\,.$$

b) Calculate the spin sum of

$$\Lambda_+(k) = \sum_{s=\pm} u_s(k)\bar{u}_s(k)\,, \quad \Lambda_-(k) = \sum_{s=\pm} v_s(k)\bar{v}_s(k)\,,$$

and construct Λ_+^2, Λ_-^2, $\Lambda_+\Lambda_-$, $\Lambda_-\Lambda_+$.
How can the matrices $\pm\frac{1}{2m}\Lambda_\pm$ be consequently interpreted?

c) Show that the spinors $u_s(k)$ and $v_s(k)$ are charge conjugated to each other, i.e. that the following relations hold

$$u_s^C(k) \equiv C\, \bar{u}_s^T(k) = v_s(k)\,, \quad v_s^C(k) \equiv C\, \bar{v}_s^T(k) = u_s(k)\,,$$

with the charge conjugation matrix $C = i\gamma^2\gamma^0$ (in the Dirac representation).

d) Derive the following relations:

$$\bar{u}_s(k)u_{s'}(k) = 2m\,\delta_{ss'}\,, \quad \bar{v}_s(k)v_{s'}(k) = -2m\,\delta_{ss'}\,, \quad \bar{u}_s(k)v_{s'}(k) = \bar{v}_s(k)u_{s'}(k) = 0\,,$$
$$\bar{u}_s(k)\gamma^\mu u_{s'}(k) = \bar{v}_s(k)\gamma^\mu v_{s'}(k) = 2k^\mu\,\delta_{ss'}\,.$$

Exercise 3: Calculations with Dirac matrices (5 points)

Proof the following equations by using the anti-commutation relation of the Dirac matrices,

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\,,$$

without using an explicit representation of the γ-matrices.
Definitions: $\not{a} = \gamma^\mu a_\mu\,,\ \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]\,,\ \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\,.$

a) $\not{a}\not{b} = a\cdot b - i\sigma^{\mu\nu}a_\mu b_\nu\,,$

b) $\gamma_\mu\not{a}\gamma^\mu = -2\not{a}\,, \quad \gamma_\mu\not{a}\not{b}\gamma^\mu = 4a\cdot b\,,$

c) $\mathrm{Tr}(\not{a}\not{b}) = 4a\cdot b\,,$

d) $\mathrm{Tr}(\not{a}_1\not{a}_2\not{a}_3\not{a}_4) = 4\left[(a_1\cdot a_2)(a_3\cdot a_4) + (a_1\cdot a_4)(a_2\cdot a_3) - (a_1\cdot a_3)(a_2\cdot a_4)\right],$

e) $\{\gamma_5, \gamma^\mu\} = 0\,, \quad (\gamma_5)^2 = 1\,, \quad \mathrm{Tr}\,\gamma_5 = 0\,, \quad \mathrm{Tr}(\gamma_5\not{a}\not{b}) = 0\,,$

f) $\mathrm{Tr}(\gamma_5\not{a}_1\not{a}_2\not{a}_3\not{a}_4) = -4i\,\epsilon^{\mu\nu\rho\sigma}a_{1\mu}a_{2\nu}a_{3\rho}a_{4\sigma}\,,$

g) $\mathrm{Tr}(\not{a}_1\not{a}_2\cdots\not{a}_r) = 0$ if r is odd.

Exercise 4: Dirac Algebra

a) Compute c_1 and c_2,

$$\gamma_{\nu_1}\gamma_{\nu_2}\gamma_{\mu_1}\gamma_{\nu_3}\gamma^{\nu_1}\gamma_{\mu_2}\gamma^{\nu_3}\gamma^{\nu_2} = c_1 g_{\mu_1\mu_2} + c_2\gamma_{\mu_1}\gamma_{\mu_2}\,.$$

b) Let

$$f(x) = \frac{1 + x + x^2 + 3x^3}{1+x}\,.$$

Compute d_1 and d_2, where $\not{p} = p^\mu\gamma_\mu$,

$$f(\not{p}) = d_1 + d_2\not{p}\,.$$

Compute the left derivative of $f(\not{p})$,

$$(\mathrm{d}/\mathrm{d}\not{p})f(\not{p}) \equiv \lim_{\delta\to 0}\not{\delta}^{-1}\left[f(\not{p}+\not{\delta}) - f(\not{p})\right]$$

and the right derivative of $f(\not{p})$,

$$\mathrm{d}f(\not{p})/\mathrm{d}\not{p} \equiv \lim_{\delta\to 0}\left[f(\not{p}+\not{\delta}) - f(\not{p})\right]\not{\delta}^{-1}$$

assuming $[\not{p}, \not{\delta}] = 0$.
