Theoretical Particle Physics I

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Exercise 1: Sine-Gordon field (7 points)

Consider a bar with pendula of mass m fixed to it at intervals d, as shown in fig. 1. Each pendulum can only swing in and out of the page. The bar is supported at the ends, but can rotate freely. If, e.g., pendula i and $i+1$ are not aligned, there is a force due to the torsion of the bar. The energy due to the torsion of two neighboring masses is $(\kappa/2)(\theta_i - \theta_{i+1})^2$, where θ_i is the angular displacement of mass i.

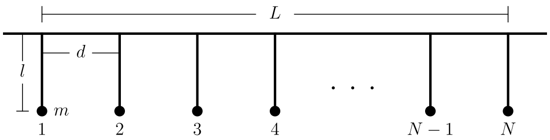

Figure 1: String of pendula

a) Write the Lagrangian of the system, and find the equations of motion for θ_i, where $i \in \{2, 3, \ldots, N-1\}$.

b) Determine the canonical momentum p_i conjugate to the variable θ_i and calculate the Hamiltonian $H[p_i, \theta_i]$ of the system.

c) Investigate the equation of motion in the continuum limit by substituting $\phi(x_i, t) = \theta_i$, and show that it assumes the form

$$\frac{\partial^2 \phi}{\partial t^2} - v^2 \frac{\partial^2 \phi}{\partial x^2} + w^2 \sin\phi = 0\,. \tag{1}$$

This equation is known as the Sine-Gordon equation (a pun on the Klein-Gordon equation, which has ϕ instead of $\sin\phi$). By introducing the mass density $\rho = m/d$ and the torsion modulus $\eta = \kappa d$, determine v and w.

d) Using the original Lagrangian, determine the Lagrangian density for the *continuum limit* and, using the Euler-Lagrange equation for a *field*, verify the equation of motion for ϕ from point (b).

e) Determine the field conjugate momentum $\pi(\phi, t)$ and calculate the Hamiltonian density $\mathcal{H}[\pi, \phi]$ of the continuous system.

Exercise 2: $SU(N)$ symmetry transformations (3 points)

A system of N complex scalar fields ϕ_i, $i = 1, \ldots, N$, is described by the following Lagrange density:

$$\mathcal{L} = \sum_{i=1}^{N} (\partial_\mu \phi_i^*)(\partial^\mu \phi_i) - m^2 \sum_{i=1}^{N} \phi_i^* \phi_i - \frac{\lambda^2}{2} \left(\sum_{i=1}^{N} \phi_i^* \phi_i \right)^2$$

a) Show that $\mathcal{L}$ is invariant under $SU(N)$ transformations

$$\phi_i \to \phi_i' = \sum_{j=1}^{N} U_{ij}\phi_j\,, \quad \phi_i^* \to \phi_i^{*\prime} = \sum_{j=1}^{N} U_{ij}^*\phi_j^*,$$

with $N \times N$ unitary constant matrices U, $U^\dagger \equiv (U^*)^T = U^{-1}$ and $\det U = 1$.
Hint: Write the Lagrange density and the $SU(N)$ transformation in terms of vectors of the scalar fields

$$\Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix}$$

and use the unitarity of the matrices U.

b) All $SU(N)$ matrices can be written in the following form

$$U = \exp\left(i \sum_a \epsilon_a T^a \right),$$

with the $SU(N)$ generators T^a and the real parameters ϵ_a. Consider infinitesimal transformations by expanding the exponential function in the limit $\epsilon_a \to 0$ up to the linear order. Show that the generators T^a are hermitian, $T^a = (T^a)^\dagger$, by using the unitarity of U. Derive from the condition $\det U = 1$ that the matrices T^a are traceless, $\mathrm{Tr}\, T^a = 0$.

Exercise 3: Vectorfield, covariant Maxwell equations

The Lagrange density of a massless, real vector field A^μ, interacting with the current j^μ, is given by:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\lambda}{2}(\partial_\mu A^\mu)^2 - j^\mu A_\mu\,,$$

where λ is a real parameter.

a) Use the Euler-Lagrange equations to derive the equations of motion for the vector field A^μ.

b) Now we set $\lambda = 0$. Show that the equations of motion are the covariant notation of the *inhomogeneous* Maxwell equations by replacing $F^{\mu\nu}$ with the fields $\vec{E}$ und $\vec{B}$. The four-current is given by the charge and current density as $j^\mu = (\rho, \vec{j})$.

c) Show that the *homogeneous* Maxwell equations can be written in covariant form as

$$\partial_\nu \tilde{F}^{\mu\nu} = 0 \quad \text{i.e.} \quad \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0\,,$$

with $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ and $\epsilon^{\mu\nu\rho\sigma}$ is the total antisymmetric tensor with $\epsilon^{0123} = +1$. Prove that the Ansatz $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ solves this equation.
