Theoretical Particle Physics I

Institut für Theoretische Teilchenphysik

Prof. Dr. Matthias Steinhauser, Dr. Daniel Stremmer, Dr. Marco Vitti, Pascal Reeck

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Exercise 1: Noether theorem with Dirac fields (5 points)
Consider the following Lagrange density of N Dirac fields with the mass m:

$$\mathcal{L} = \sum_{i=1}^{N} \bar{\psi}_i \left(i\partial\!\!\!/ - m\right)\psi_i = \sum_{i=1}^{N} \bar{\psi}_i \left(i\gamma^\mu \partial_\mu - m\right)\psi_i\,, \quad \text{with} \quad \bar{\psi}_i = \psi_i^\dagger \gamma^0.$$

a) Use the Euler-Lagrange equations to derive the equations of motion.

b) Under which of the following global transformations is $\mathcal{L}$ invariant?

(i) $\psi_i \to \psi_i' = e^{i\alpha}\psi_i\,,$ [$U(1)$-Transformation]

(ii) $\psi_i \to \psi_i' = \exp(i\epsilon_a T^a)_{ij}\psi_j\,,$ [$SU(N)$-Transformation]

(iii) $\psi_i \to \psi_i' = e^{i\alpha\gamma_5}\psi_i\,,$

(iv) $\psi_i \to \psi_i' = \exp(i\epsilon_a T^a \gamma_5)_{ij}\psi_j\,,$

with $\alpha \in \mathbb{R}$, $\epsilon_a \in \mathbb{R}$ and the hermitian traceless $SU(N)$ generators T^a. It is enough to consider infinitesimal transformations. (*Hint:* $\gamma_5^\dagger = \gamma_5$.)
How does the answer change for massless Dirac fields?

c) Find the corresponding Noether currents

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i)}\,\delta\psi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi}_i)}\,\delta\bar{\psi}_i$$

to the transformations above.

Exercise 2: Symmetry and Noether Theorem
Consider the following Lagrange density

$$\begin{aligned}\mathcal{L} = {}& \partial_\mu \phi_+^*(x)\partial^\mu \phi_+(x) - m_+^2 \phi_+^*(x)\phi_+(x) + \partial_\mu \phi_-^*(x)\partial^\mu \phi_-(x) - m_-^2 \phi_-^*(x)\phi_-(x) \\ & - \frac{1}{8}\phi_+(x)^2\phi_-(x)^2 - \frac{1}{8}\phi_+^*(x)^2\phi_-^*(x)^2\,, \end{aligned} \tag{1}$$

where $\phi_+(x)$ and $\phi_-(x)$ are complex scalar fields.

a) Assume that $m_+ \neq m_-$. Consider the transformation

$$\phi_+(x) \to e^{i\tau}\phi_+(x).$$

How must the field $\phi_-(x)$ transform, for the combined transformation of $\phi_+(x)$ and $\phi_-(x)$ to be a symmetry of the Lagrangian?

b) Calculate the Noether current of the above symmetry.

c) Now assume that $m_+ = m_-$ and that ϕ_+, ϕ_- are real scalar fields. Define the Lagrangian density $\mathcal{L}_2 = \mathcal{L} - \frac{1}{8}(\phi_+^4 + \phi_-^4)$. Consider the transformation

$$\phi_+(x) \to \cos(\theta)\,\phi_+(x) + \sin(\theta)\,\phi_-(x)\,.$$

How must the field $\phi_-(x)$ transform, for the combined transformation of $\phi_+(x)$ and $\phi_-(x)$ to be a symmetry of $\mathcal{L}_2$? Compute the Noether current of this symmetry.

Exercise 3: Noether Theorem with Grassmann Variables (5 points)

The commutator $[A, B]$ and the anti-commutator $\{A, B\}$ are defined as follows,

$$[A, B] = AB - BA, \qquad \{A, B\} = AB + BA\,.$$

The Grassmann variables Θ_1, Θ_2 are variables that anti-commute:

$$\{\Theta_1, \Theta_2\} = 0\,.$$

Here we consider the following Lagrangian

$$L = \frac{1}{2}\dot{x}(t)^2 - \frac{1}{2}x(t)^2 + \frac{i}{2}\left(\psi(t)^*\dot{\psi}(t) - \dot{\psi}^*(t)\psi(t)\right) - \psi(t)^*\psi(t)\,,$$

and the corresponding action

$$S = \int \mathrm{d}t L$$

where x is a real variable, ψ is a complex Grassmann variable and ψ^* is its complex conjugate. Note that since ψ and ψ^* and their derivatives anti-commute, differentiation with respect to a Grassmann variable should be performed carefully. For example,

$$\frac{\partial}{\partial\psi}(\psi^*\psi) = -\psi^*\frac{\partial}{\partial\psi}\psi = -\psi^*\,.$$

a) Derive the equations of motion for x, ψ, ψ^*.

b) The action S is symmetric under the infinitesimal transformation

$$x \to x + \delta x, \quad \psi \to \psi + \delta\psi, \quad \psi^* \to \psi^* + \delta\psi^* \tag{2}$$

where

$$\delta x = \xi\psi^* - \xi^*\psi, \quad \delta\psi = \xi(i\dot{x} - x). \tag{3}$$

ξ is a complex Grassmann-variable parameter and ξ^* is its complex conjugate, so

$$\{\psi, \xi\} = 0, \quad \{\xi, \xi^*\} = 0, \quad \text{etc.}$$

Determine $\delta\psi^*$ in Eq. (2).

c) Since there are two transformation parameters ξ, ξ^* in Eq. (3), there are two Noether charges associated with them; Q with ξ, and $\bar{Q}$ with ξ^*. Derive Q and $\bar{Q}$.

d) Derive the Hamiltonian associated with this Lagrangian.