Theoretical Particle Physics I

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Exercise 1: Energy-momentum tensor

The action $S = \int \mathrm{d}^4x\, \mathcal{L}$ of the real scalar field with $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2$ is invariant under translations. From this follows that $\partial_\mu T^{\mu\nu} = 0$ where the energy-momentum tensor is given by

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,(\partial^\nu\phi) - g^{\mu\nu}\mathcal{L} = (\partial^\mu\phi)(\partial^\nu\phi) - g^{\mu\nu}\mathcal{L}\,.$$

It follows that the components of the four momentum $P^\mu = \int \mathrm{d}^3x\, T^{0\mu}$, the energy $H = P^0$ and the momentum $\vec{P}$, are conserved.

Use for the scalar field $\phi(x)$ the Fourier representation

$$\phi(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3\, 2k_0} \left[e^{ik\cdot x}a^\dagger(\vec{k}) + e^{-ik\cdot x}a(\vec{k})\right]\,, \quad k_0 = \sqrt{\vec{k}^2 + m^2}\,,$$

and show that the four momentum operator can be written as

$$P^\mu = \int \frac{\mathrm{d}^3k}{(2\pi)^3\, 2k_0}\, \frac{1}{2}\, k^\mu \left[a^\dagger(\vec{k})a(\vec{k}) + a(\vec{k})a^\dagger(\vec{k})\right]\,.$$

Note: Usually the irrelevant but infinite vacuum energy is neglected and the *normal order* of the operators is used:

$$:P^\mu: = \int \frac{\mathrm{d}^3k}{(2\pi)^3\, 2k_0}\, k^\mu\, a^\dagger(\vec{k})a(\vec{k})\,.$$

Exercise 2: Quantization of the free complex scalar field (10 points)

Consider the following Lagrangian density

$$\mathcal{L} = \partial_\mu\phi^\dagger(x)\partial^\mu\phi(x) - m^2\phi^\dagger(x)\phi(x)\,, \tag{1}$$

where $\phi(x)$ is a complex scalar field. The field operators $\phi(x), \phi^\dagger(x)$ are expressed as

$$\phi(x) = \int D\vec{k}\left[b(\vec{k})e^{-ikx} + a^\dagger(\vec{k})e^{ikx}\right]$$
$$\phi^\dagger(x) = \int D\vec{k}\left[b^\dagger(\vec{k})e^{ikx} + a(\vec{k})e^{-ikx}\right]\,,$$

where $\int D\vec{k} = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2\omega_k}$ and $\omega_k = \sqrt{\vec{k}^2 + m^2}$. $a^\dagger(\vec{k})$ is the creation operator of the particle with momentum $\vec{k}$, $b^\dagger(\vec{k})$ is the creation operator of the anit-particle with momentum $\vec{k}$, $a(\vec{k})$ is the annihilation operator of the particle with momentum $\vec{k}$, $b(\vec{k})$ is the annihilation operator of the anit-particle with momentum $\vec{k}$. The creation and annihilation operators satisfy the following commuting relations

$$[a(\vec{k}), a(\vec{k}')] = 0, \quad [a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = 0, \quad [a(\vec{k}), a^\dagger(\vec{k}')] = 2\omega_k(2\pi)^3\delta(\vec{k} - \vec{k})$$
$$[b(\vec{k}), b(\vec{k}')] = 0, \quad [b^\dagger(\vec{k}), b^\dagger(\vec{k}')] = 0, \quad [b(\vec{k}), b^\dagger(\vec{k}')] = 2\omega_k(2\pi)^3\delta(\vec{k} - \vec{k})$$
$$[a(\vec{k}), b(\vec{k}')] = 0, \quad [a^\dagger(\vec{k}), b^\dagger(\vec{k}')] = 0, \quad [a(\vec{k}), b^\dagger(\vec{k}')] = 0\,.$$

The n-particle $\bar{n}$-anti-particle state is expressed as

$$|\Psi_{n,\bar{n}}\rangle = |\vec{k}_1, ..., \vec{k}_n; \vec{k}'_1, ..., \vec{k}'_{\bar{n}}\rangle = a^\dagger(\vec{k}_1)\cdots a^\dagger(\vec{k}_n) b^\dagger(\vec{k}'_1)\cdots b^\dagger(\vec{k}'_{\bar{n}})\,|0\rangle \ , \tag{2}$$

where $|0\rangle$ is the vacuum state.

a) Derive the following five formulae.

$$\left(\int D\vec{k}\left\{f(\vec{k})a^\dagger(\vec{k})a(\vec{k}) + g(\vec{k})b^\dagger(\vec{k})b(\vec{k})\right\}\right)|\Psi_{n,\bar{n}}\rangle = \left(\sum_{j=1}^{n} f(\vec{k}_j) + \sum_{j=1}^{\bar{n}} g(\vec{k}_j)\right)|\Psi_{n,\bar{n}}\rangle \tag{3}$$

$$\begin{aligned}
&\int \mathrm{d}^3x D\vec{k}D\vec{k}'\left\{f(\vec{k}')b^\dagger(\vec{k}')e^{ik'x} + g(\vec{k}')a(\vec{k}')e^{-ik'x}\right\}\left\{c(\vec{k})b(\vec{k})e^{-ikx} + d(\vec{k})a^\dagger(\vec{k})e^{ikx}\right\} \\
&= \int D\vec{k}\,\frac{1}{2\omega_k}\left\{c(-\vec{k})g(\vec{k})a(\vec{k})b(-\vec{k})e^{-2i\omega_k t} + d(\vec{k})f(-\vec{k})b^\dagger(-\vec{k})a^\dagger(\vec{k})e^{2i\omega_k t}\right\} \\
&\quad + \int D\vec{k}\,\frac{1}{2\omega_k}\left\{d(\vec{k})g(\vec{k})a(\vec{k})a^\dagger(\vec{k}) + c(\vec{k})f(\vec{k})b^\dagger(\vec{k})b(\vec{k})\right\}
\end{aligned} \tag{4}$$

$$\begin{aligned}
&\left[\left(\int D\vec{k}\left\{f(\vec{k})a^\dagger(\vec{k})a(\vec{k}) + g(\vec{k})b^\dagger(\vec{k})b(\vec{k})\right\}\right), \phi(x)\right] \\
&= \int D\vec{k}\left\{-g(\vec{k})b(\vec{k})e^{-ikx} + f(\vec{k})a^\dagger(\vec{k})e^{ikx}\right\}
\end{aligned} \tag{5}$$

$$\left[a^\dagger(\vec{k})a(\vec{k})a^\dagger(\vec{k}')a(\vec{k}'), a^\dagger(\vec{k}')a(\vec{k}')a^\dagger(\vec{k})a(\vec{k})\right] = 0 \tag{6}$$

$$\left[b^\dagger(\vec{k})b(\vec{k})b^\dagger(\vec{k}')b(\vec{k}'), b^\dagger(\vec{k}')b(\vec{k}')b^\dagger(\vec{k})b(\vec{k})\right] = 0 \tag{7}$$

In the following questions, you can use the above formulae without explanation.

b) The number operator N is defined as

$$N = \int D\vec{k}\left\{a^\dagger(\vec{k})a(\vec{k}) + b^\dagger(\vec{k})b(\vec{k})\right\} .$$

Apply N to $|\Psi_{n,\bar{n}}\rangle$ and interpret the result briefly.

c) Express the Hamiltonian H in terms of the creation and annihilation operators. Apply it to $|\Psi_{n,\bar{n}}\rangle$ and interpret the result briefly.

d) The energy-momentum operator P^μ is defined as

$$P^\mu = \int \mathrm{d}^3x\ T^{0\mu} ,$$

where $T^{\nu\mu}$ is the energy-momentum tensor for a complex scalar field.
Express P^μ in terms of the creation and annihilation operators. Apply it to $|\Psi_{n,\bar{n}}\rangle$ and interpret the result briefly.

e) Calculate

$$[iP^\mu, \phi(x)]$$

and express the result in terms of $\phi(x), \phi^\dagger(x)$ and derivatives of them.

f) Calculate the Noether charge Q associated with the symmetry

$$\phi(x) \to e^{i\theta}\phi(x)$$

and express it in terms of the creation and annihilation operators.
Apply it to $|\Psi_{n,\bar{n}}\rangle$ and interpret the result briefly.

g) Calculate

$$[Q, \phi(x)], \quad [Q, \phi^\dagger(x)]$$

and express the results in terms of $\phi(x), \phi^\dagger(x)$ and derivatives of them. Interpret the results.

h) Calculate and interpret

$$[Q, H], \quad [N, H], \quad [P^\mu, H]\,.$$
