Theoretical Particle Physics I

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Exercise 1: Interpretation of the propagators (5 points)
Consider the following Lagrange density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi(x)\partial^\mu \phi(x) - \frac{1}{2}m^2\phi^2(x)\,,$$

where $\phi(x)$ is a real scalar field. The field operator $\phi(x)$ is expressed as

$$\phi(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\frac{1}{2\omega_k}\left[e^{-ikx}a(\vec{k}) + e^{ikx}a^\dagger(\vec{k})\right]\,,$$

where $\omega_k = \sqrt{\vec{k}^2 + m^2}$. The creation and annihilation operators satisfy the following commuting relations (cf. lecture)

$$[a(\vec{k}), a(\vec{k}')] = 0, \quad [a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = 0, \quad [a(\vec{k}), a^\dagger(\vec{k}')] = 2\omega_k(2\pi)^3\delta(\vec{k}-\vec{k}')\,.$$

You may use the following relations without any derivation

$$\int_0^\infty \frac{\kappa\sin(\kappa r)\mathrm{d}\kappa}{\sqrt{\kappa^2+m^2}} = \frac{2^{\frac{1}{2}}m}{\pi^{\frac{1}{2}}}\texttt{BesselK}(1, mr), \qquad \int_0^\infty \frac{\kappa\sin(\kappa r)\mathrm{d}\kappa}{\sqrt{2\sqrt{\kappa^2+m^2}}} = \frac{2^{\frac{1}{4}}}{\Gamma(\frac{1}{4})}\frac{\texttt{BesselK}(\frac{5}{4}, mr)}{m^{-\frac{5}{4}}r^{\frac{1}{4}}}\,, \quad (1)$$

where the Bessel function `BesselK` can be evaluated by using, e.g., `Mathematica`.

Property of the one-particle state (3 points) In this subsection, we consider a time-independent state. The one-particle state with momentum $\vec{k}$ is defined as $|\vec{k}\rangle = a^\dagger(\vec{k})|0\rangle$ and this is an eigenstate of the 4-momentum operator. In a similar way, let us define the one-particle state localized at $\vec{x}$ as the eigenstate of the position operator. The properties of these states are

$$\vec{P}\,|\vec{k}\rangle = \vec{k}\,|\vec{k}\rangle\,, \qquad \langle\vec{k}'|\vec{k}\rangle = 2\omega_k(2\pi)^3\delta(\vec{k}'-\vec{k}), \qquad \vec{X}\,|\vec{x}\rangle = \vec{x}\,|\vec{x}\rangle\,, \qquad \langle\vec{x}'|\vec{x}\rangle = \delta(\vec{x}'-\vec{x})\,. \quad (2)$$

$|\vec{k}\rangle$ and $|\vec{x}\rangle$ span the Hilbert space of the one-particle states:

$$\mathbb{1}_{\text{one-particle}} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\frac{1}{2\omega_k}|\vec{k}\rangle\langle\vec{k}|\,, \qquad \mathbb{1}_{\text{one-particle}} = \int \mathrm{d}^3 x\,|\vec{x}\rangle\langle\vec{x}|\,. \quad (3)$$

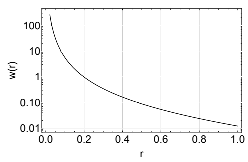

Figure 1: The behavior of $w(r)$ in Eq. (5). We set $m = 1$.

The canonical commutation relation, $[X_a, P_b] = i\delta_{ab}$, leads to the following differential equation for $\langle\vec{x}|\vec{k}\rangle$

$$\vec{k}\,\langle\vec{x}|\vec{k}\rangle = \langle\vec{x}|\,\vec{P}\,|\vec{k}\rangle = \langle\vec{x}|\left(-i\frac{\partial}{\partial\vec{x}}\right)|\vec{k}\rangle = -i\frac{\partial\langle\vec{x}|\vec{k}\rangle}{\partial\vec{x}}\,. \quad (4)$$

- **a)** Obtain the solution of the differential equation (4) including the proper normalization factor. The proper normalization factor should be consistent with all of the above equations.

b) Let us express the state $\hat{\phi}(t = 0, \vec{x})\,|0\rangle$ in terms of $|\vec{x}\rangle$, which allows us to get a more intuitive physical interpretation. For this purpose, we use the completeness relation of $|\vec{x}\rangle$ in Eq. (3) and express the state as the superposition of the localized states $|\vec{x}'\rangle$ with a weight function:

$$\phi(t = 0, \vec{x})\,|0\rangle = \int \mathrm{d}^3x'\,|\vec{x}'\rangle\,\langle\vec{x}'|\,\hat{\phi}(t = 0, \vec{x})\,|0\rangle = \int \mathrm{d}^3x' w(r)\,|\vec{x}'\rangle\ , \tag{5}$$

where $r = |\vec{x} - \vec{x}'|$. Determine $w(r)$. The result should look like Fig. 1.

Correlation function (2 points) Consider the correlation of two states separated in the spatial direction:

$$D(\vec{x} - \vec{y}) = \langle 0|\,\phi(t = 0, \vec{y})\phi(t = 0, \vec{x})\,|0\rangle\ . \tag{6}$$

a) Show that

$$D(\vec{x} - \vec{y}) = D(\vec{y} - \vec{x}). \tag{7}$$

b) Calculate $D(\vec{x} - \vec{y})$ and express the result as a function of $r = |\vec{x} - \vec{y}|$.

[Optional] Causality In a quantum theory, information about a state cannot be obtained without some explicit measurement. We introduce an operator $\mathcal{P}(t, \vec{x}) = [\phi(t, \vec{x})]^2$ and call it a "probe", since it enables us to probe a state. Successive probes are expressed as

$$\mathcal{P}(t_2, \vec{x}_2)U(t_2 - t_1)\mathcal{P}(t_1, \vec{x}_1)U(t_1 - t_0)\,|\Psi(t_0)\rangle\ , \tag{8}$$

where $|\Psi(t_0)\rangle$ is the initial state at $t = t_0$ (it is a multi-particle state in general) and $U(t)$ is the time evolution operator.

Let us consider the case of two simultaneous probes at different points. This can be expressed as

$$\mathcal{P}(t, \vec{x})\mathcal{P}(t, \vec{y})U(t - t_0)\,|\Psi(t_0)\rangle\ , \tag{9}$$

where $\vec{x} \neq \vec{y}$. Since the probes act on the different points, we could also consider

$$\mathcal{P}(t, \vec{y})\mathcal{P}(t, \vec{x})U(t - t_0)\,|\Psi(t_0)\rangle\ . \tag{10}$$

If Eq. (9) and Eq. (10) yield different results, it means that the measurement at $\vec{y}$ affects the measurement at $\vec{x}$ no matter how far they are separated. This is a breakdown of causality.

Derive the sufficient condition for

$$\mathcal{P}(t, \vec{x})\mathcal{P}(t, \vec{y})U(t - t_0)\,|\Psi(t_0)\rangle = \mathcal{P}(t, \vec{y})\mathcal{P}(t, \vec{x})U(t - t_0)\,|\Psi(t_0)\rangle \tag{11}$$

and show that it is indeed satisfied.

[Optional] Feynman propagator In this subsection, we define the Feynman propagator as (see lecture)

$$D_F(x - y) = \langle 0|\,\mathrm{T}\,[\phi(x)\phi(y)]\,|0\rangle \tag{12}$$

and show the equivalence with the definition via the method of Green's functions. The time ordering operator T acts as

$$\mathrm{T}\,[\phi(x)\phi(y)] = \theta(x^0 - y^0)\phi(x)\phi(y) + \theta(y^0 - x^0)\phi(y)\phi(x) \tag{13}$$

where $\theta(t)$ is the step function.

a) The step function can be expressed as

$$\theta(t) = -\int_{-\infty}^{\infty} \frac{\mathrm{d}\xi}{2\pi i} \frac{e^{-i\xi t}}{\xi + i0^+} \tag{14}$$

where 0^+ is an infinitesimal positive number. Confirm that this expression indeed works.

b) Use the representation (14) and determine the integrand $\mathcal{I}$ of the following equation.

$$\theta(x^0 - y^0) \langle 0| \phi(x)\phi(y) |0\rangle = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{-ik(x-y)} \mathcal{I} \,. \tag{15}$$

$\mathcal{I}$ should not contain an integral.

c) Calculate $\theta(y^0 - x^0) \langle 0| \phi(y)\phi(x) |0\rangle$ in analogy to (b), and obtain the Feynman propagator by substituting the results into Eq. (12). The result should agree with the one obtained by the method of the Green's function (see lecture):

$$D_F(x-y) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i0^+} e^{-ik(x-y)} \,. \tag{16}$$

Make sure that the sign of $i0^+$ is correct.

Exercise 2: Quantised Dirac field (5 points)

We have a quantised Dirac field given by

$$\psi(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3\, 2k_0} \sum_{s=\pm} \left[e^{ik\cdot x} v_s(\vec{k}) b_s^\dagger(\vec{k}) + e^{-ik\cdot x} u_s(\vec{k}) a_s(\vec{k}) \right] \,, \quad k_0 = \sqrt{\vec{k}^2 + m^2} \,.$$

a) Derive from the anti-commutation relations of the fields

$$\begin{aligned} \left\{ \psi_\alpha(x), \psi_\beta^\dagger(y) \right\}_{x_0 = y_0} &= \delta_{\alpha\beta}\, \delta^{(3)}(\vec{x} - \vec{y}) \,, \\ \left\{ \psi_\alpha(x), \psi_\beta(y) \right\}_{x_0 = y_0} &= \left\{ \psi_\alpha^\dagger(x), \psi_\beta^\dagger(y) \right\}_{x_0 = y_0} = 0 \,, \end{aligned}$$

the anti-commutation relation for the creation and annihilation operators

$$\begin{aligned} \left\{ a_s(\vec{k}), a_{s'}^\dagger(\vec{k}') \right\} &= \left\{ b_s(\vec{k}), b_{s'}^\dagger(\vec{k}') \right\} = 2k_0\, \delta_{ss'}\, (2\pi)^3\, \delta^{(3)}(\vec{k} - \vec{k}') \,, \\ \left\{ a_s(\vec{k}), a_{s'}(\vec{k}') \right\} &= \left\{ b_s(\vec{k}), b_{s'}(\vec{k}') \right\} = \left\{ a_s^\dagger(\vec{k}), a_{s'}^\dagger(\vec{k}') \right\} = \left\{ b_s^\dagger(\vec{k}), b_{s'}^\dagger(\vec{k}') \right\} = 0 \,, \\ \left\{ a_s(\vec{k}), b_{s'}(\vec{k}') \right\} &= \left\{ a_s(\vec{k}), b_{s'}^\dagger(\vec{k}') \right\} = \left\{ a_s^\dagger(\vec{k}), b_{s'}(\vec{k}') \right\} = \left\{ a_s^\dagger(\vec{k}), b_{s'}^\dagger(\vec{k}') \right\} = 0 \,. \end{aligned}$$

Hint: $u_s^\dagger(\vec{k}) u_{s'}(\vec{k}) = v_s^\dagger(\vec{k}) v_{s'}(\vec{k}) = 2k_0\, \delta_{ss'}$, $u_s^\dagger(\vec{k}) v_{s'}(-\vec{k}) = v_s^\dagger(\vec{k}) u_{s'}(-\vec{k}) = 0$.

b) Show that the momentum operator of the free Dirac fields, $P^\mu = \int \mathrm{d}^3 x\, \psi^\dagger\, i\partial^\mu \psi$, can be written in normal ordering as

$$:\!P^\mu\!: = \int \frac{\mathrm{d}^3 k}{(2\pi)^3\, 2k_0}\, k^\mu \sum_{s=\pm} \left[a_s^\dagger(\vec{k}) a_s(\vec{k}) + b_s^\dagger(\vec{k}) b_s(\vec{k}) \right]$$

Hint: $:\!a_s a_s^\dagger\!: = -a_s^\dagger a_s$ etc. for the Dirac fields.
Calculate the commutators $[P^\mu, \psi(x)]$ and $[P^\mu, \bar{\psi}(x)]$.

c) Determine the charge Q (in normal ordering) of the conserved current $j^\mu = \bar{\psi}\gamma^\mu\psi$ (see exercise 1 on sheet 4).
Calculate the commutators $[Q, \psi(x)]$, $[Q, \bar{\psi}(x)]$ and $[Q, P^\mu]$.

d) What happens if the fermion field is quantised using a commutator relation?
