Theoretical Particle Physics I

Institut für Theoretische Teilchenphysik

Prof. Dr. Matthias Steinhauser, Dr. Daniel Stremmer, Dr. Marco Vitti, Pascal Reeck

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Exercise 1: Massless vector field (6 points)

Consider the Fourier expansion of a massless vector field

$$A_\mu(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3\, 2\omega_k} \sum_{\lambda=0}^{3} \left[e^{-ik\cdot x} \varepsilon_\mu^{(\lambda)}\, a^{(\lambda)}(\vec{k}) + e^{ik\cdot x} \varepsilon_\mu^{(\lambda)}\, a^{(\lambda)\,\dagger}(\vec{k}) \right] , \quad \omega_k = \sqrt{\vec{k}^2 + m^2}\,,$$

with real-valued polarization vectors which fulfill

$$\begin{aligned} \varepsilon_\mu^{(\lambda)} \varepsilon^{(\lambda),\mu} &= -\xi^{(\lambda)} \delta^{\lambda\lambda'} \\ \sum_{\lambda=0}^{3} \xi^{(\lambda)} \varepsilon_\mu^{(\lambda)} \varepsilon_\nu^{(\lambda)} &= -g_{\mu\nu} \end{aligned}$$

with $\xi^{(0)} = -1$ and $\xi^{(1)} = \xi^{(2)} = \xi^{(3)} = +1$.
The canonical commutation relations are

$$\begin{aligned} \left[A_\mu(x), \Pi_\nu(y)\right]_{x_0=y_0} &= -i g_{\mu\nu}\ \delta^{(3)}(\vec{x} - \vec{y}) \\ \left[A_\mu(x), A_\nu(y)\right]_{x_0=y_0} &= \left[\Pi_\mu(x), \Pi_\nu(y)\right]_{x_0=y_0} = 0 \end{aligned}$$

where $\Pi_\mu(x) = -\partial_0 A_\mu(x)$.
Show that starting from the canonical commutation relations one obtains

$$\begin{aligned} \left[a^{(\lambda)}(\vec{k}), a^{(\lambda')\dagger}(\vec{k}')\right] &= \delta_{\lambda\lambda'} \xi^{(\lambda)}\, 2\,\omega_k\, (2\pi)^3\, \delta^{(3)}(\vec{k} - \vec{k}')\,, \\ \left[a^{(\lambda)}(\vec{k}), a^{(\lambda')}(\vec{k}')\right] &= \left[a^{(\lambda)\dagger}(\vec{k}), a^{(\lambda')\dagger}(\vec{k}')\right] = 0\,. \end{aligned}$$

Exercise 2: Photon fields: gauge transformation

Consider the state $|\,\Psi_T\rangle$, which contains only transversal photons. We define a new state $|\Psi'_T\rangle$ as

$$|\,\Psi'_T\rangle = \left[1 + c \left(a^{(3)^\dagger}(\vec{k}) - a^{(0)^\dagger}(\vec{k}) \right) \right] |\,\Psi_T\rangle \quad \text{with}\, c = \text{constant.}$$

Show that the replacement from $|\,\Psi_T\rangle$ to $|\,\Psi'_T\rangle$ is equivalent to a gauge transformation, i.e.

$$\langle \Psi'_T\, | A^\mu(x) |\, \Psi'_T\rangle = \langle \Psi_T\, | A^\mu(x) + \partial^\mu \Lambda(x) |\, \Psi_T\rangle\,,$$

where

$$\Lambda(x) = \frac{2}{k_0} \mathrm{Re}(ic e^{-ikx})\,.$$

Exercise 3: Equations of motion and energy-momentum tensor (4 points)

Consider the following Lagrangian density, the "Proca" Lagrangian,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) + \frac{m^2}{2} A_\mu(x) A^\mu(x)\,, \tag{1}$$

where $A^\mu(x)$ is a real vector field and $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$. In exercise 3 on sheet 3 we have calculated the equations of motions for a massless vector field. For a massive vector field they are given by

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) + m^2 A^\nu = 0.$$

a) Derive the energy-momentum tensor $T^{\mu\nu}$ for this Lagrangian such that $T^{\mu\nu}$ is symmetric in the indices.

b) Consider the transformation

$$A^\mu(x) \to A^\mu(x) - \partial^\mu f(x)\,,$$

where $f(x)$ is an arbitrary smooth function. Under what condition is this transformation a symmetry of Eq. (1)?

Exercise 4: Polarizations of massive vector fields

Show that the polarization sum for a massive vector boson ($m > 0$) can be written as

$$P_{\mu\nu}(\vec{k}) \equiv \sum_{\lambda=1}^{3} \varepsilon_\mu^{(\lambda)}(\vec{k})\, \varepsilon_\nu^{*(\lambda)}(\vec{k}) = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2}\,.$$

Proceed as follows:

- Which (normalized) polarization vectors are possible in the rest frame of the massive vector boson?

  Hint: $k^\mu \varepsilon_\mu^{(\lambda)}(\vec{k}) = 0$.

- Perform a Lorentz boost for all vectors into a frame, where the massive vector boson is moving (for simplicity consider a Lorentz boost in z-direction). Using the new vectors, calculate the upper relation for $P_{\mu\nu}$.
