Theoretical Particle Physics I

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Exercise 1: Interaction picture (6 points)

In quantum mechanics the time evolution of a state is described in the Schrödinger picture by the Schrödinger equation

$$i\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle^S = H|\psi(t)\rangle^S$$

with the Hamiltonian operator H, while an operator in the Schrödinger picture is in general time independent. In the interaction picture the Hamiltonian operator is split in two parts,

$$H = H_0 + H_W\,,$$

where H_0 describes free particles and H_W the interactions between them. The unitary time evolution operator $U_0(t, t_0)$ with

$$U_0(t, t_0) = e^{-iH_0(t-t_0)}$$

describes the time evolution of the states in the Schrödinger picture in the case of H_W=0. This operator is used for the transformation into the interaction picture:

$$|\psi(t)\rangle^W = U_0^\dagger(t, t_0)|\psi(t)\rangle^S$$

a) What must the operator $O^W(t)$ look like in the interaction picture to preserve matrix elements

$$^W\langle\psi_1(t)|\,O^W(t)\,|\psi_2(t)\rangle^W = {}^S\langle\psi_1(t)|\,O^S(t)\,|\psi_2(t)\rangle^S$$

under this transformation? Construct the operator H_0 in the interaction picture. Calculate $i\frac{\mathrm{d}}{\mathrm{d}t}O^W(t)$ and show that the following equation holds

$$i\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle^W = H_W^W(t)\,|\psi(t)\rangle^W,$$

where $H_W^W(t)$ is the operator H_W in the interaction picture.

b) Consider the Lagrange density $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_W$ with

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m^2}{2}\phi^2\,, \quad \mathcal{L}_W = -\frac{\lambda}{4!}\phi^4\,.$$

Calculate the parts H_0 and H_W of the Hamiltonian operator. Which differential equation describes the time evolution of the field operator $\phi^W(t)$ in the interaction picture? Do you know the solution of this differential equation?

Exercise 2: Gauge transformation

Show that the Lagrange density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \Big[(\partial_\mu + i\,e\,A_\mu)\,\phi\Big]^\star(\partial^\mu + i\,e\,A^\mu)\,\phi - m^2\phi^\star\,\phi$$

is invariant under $U(1)$ gauge transformation

$$\phi(x) \to \phi'(x) = e^{i\alpha(x)}\phi(x)\,, \quad A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \frac{1}{e}\,\partial_\mu \alpha(x).$$

Note that $\alpha(x)$ is not a constant and depends on the space-time coordinates.

Exercise 3: Mandelstam variables (4 points)

Consider the $2 \to 2$ scattering process

$$A(p_a) + B(p_b) \to C(p_c) + D(p_d), \tag{1}$$

where p_i are the four-momenta of the particles and their masses are denoted m_i. p_a and p_b are incoming momenta, and p_c and p_d are outgoing momenta.

a) How many Lorentz-invariant quantities can be constructed from the incoming and outgoing particle momenta? How many of them are independent, given the four-momentum conservation of the scattering process?

b) Show that, in addition to the particle masses, the following so-called *Mandelstam variables* are sufficient to describe the kinematics of the scattering process,

$$s = (p_a + p_b)^2, \quad t = (p_a - p_c)^2, \quad u = (p_a - p_d)^2. \tag{2}$$

c) Show that

$$s + t + u = \sum_i m_i^2. \tag{3}$$

d) Consider the $2 \to 2$ scattering process in the center-of-mass frame, where the momenta can be parametrized as

$$p_a^\mu = \frac{\sqrt{s}}{2}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \quad p_b^\mu = \frac{\sqrt{s}}{2}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}, \quad p_c^\mu = \begin{pmatrix}E_c\\p\sin\theta\\0\\p\cos\theta\end{pmatrix}, \quad p_d^\mu = \begin{pmatrix}E_d\\-p\sin\theta\\0\\-p\cos\theta\end{pmatrix},$$

with $m_a = m_b = 0$. Express $\cos\theta$ as a function of only the Mandelstam variables and the masses, and show that the squared transverse momentum $p_T^2 = p^2\sin^2\theta$ is given by

$$p_T^2 = \frac{ut - m_c^2 m_d^2}{s}.$$