Theoretical Particle Physics I

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Exercise 1: Wick's theorem

Wick's theorem is a useful tool for the perturbative calculation of vacuum expectation values and matrix elements. It transforms time-ordered into normal-ordered products. We consider here the simple case of a real scalar field. The *contraction* of two field operators $\phi(x_1)$ and $\phi(x_2)$ is defined as

$$\underbrace{\phi(x_1)\phi}(x_2) = \langle 0|\, T\phi(x_1)\phi(x_2)\,|0\rangle = D_F(x_1 - x_2)\,,$$

where additional field operators can appear on the left, right or between the two contracted field operators, which are not affected by this contraction. Wick's theorem states

$$\begin{aligned}
&T\phi(x_1)\cdots\phi(x_n)\\
&\quad = :\phi(x_1)\cdots\phi(x_n):\\
&\qquad + \text{all combinations of contractions in } :\phi(x_1)\cdots\phi(x_n):\\
&\quad = :\phi(x_1)\cdots\phi(x_n):\\
&\qquad + :\underbrace{\phi(x_1)\phi}(x_2)\phi(x_3)\cdots\phi(x_n): + \ldots + :\phi(x_1)\cdots\underbrace{\phi(x_i)\cdots\phi}(x_j)\cdots\phi(x_n): + \ldots\\
&\qquad + \ldots + :\phi(x_1)\cdots\underbrace{\phi(x_i)\cdots\phi}(x_j)\cdots\underbrace{\phi(x_k)\cdots\phi}(x_l)\cdots\phi(x_n):\\
&\qquad + \ldots + :\underbrace{\phi(x_1)\phi}(x_2)\underbrace{\phi(x_3)\phi}(x_4)\cdots:
\end{aligned}$$

Prove Wick's theorem by induction. Show first the case of $n = 2$ field operators and the induction step from n to $n+1$ operators. *Hint*: Split up the field ϕ into the parts $\phi^{(+)}$ and $\phi^{(-)}$ depending only on the annihilation and creation operator, respectively, and notice that

$$\underbrace{\phi(x_1)\phi}(x_2) = \begin{cases} \left[\phi^{(+)}(x_1), \phi^{(-)}(x_2)\right] & (x_1^0 > x_2^0) \\ \left[\phi^{(+)}(x_2), \phi^{(-)}(x_1)\right] & (x_2^0 > x_1^0) \end{cases} \,.$$

Exercise 2: Interaction in ϕ^4 theory (10 points)

Consider the Lagrange density $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_W$ in ϕ^4 theory with

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m^2}{2}\phi^2\,, \quad \mathcal{L}_W = -\frac{\lambda}{4!}\phi^4\,.$$

Expand the S matrix element

$$S_{fi} = \langle f|\, T\exp\left(i\int \mathrm{d}^4x\, \mathcal{L}_W(x)\right)|i\rangle\,, \quad |i\rangle = |\vec{p}_1, \vec{p}_2\rangle\,, \quad |f\rangle = |\vec{p}_3, \vec{p}_4\rangle\,,$$

in the coupling constant λ and find with the help of Wick's theorem all contributions to S_{fi} up to λ^2.

Which Feynman diagrams correspond to the individual contributions? Consider only contributions with fully connected Feynman diagrams. Take care of how often you get the single Feynman diagrams from the individual contractions from Wick's theorem. How does the Feynman rules look like for the ϕ^4 theory in position space (i.e. as a function of $x, \ldots$ and the vertices)? Insert the propagator

$$D_F(x-y) = \langle 0| T\phi(x)\phi(y) |0\rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, e^{-ik(x-y)}\, \frac{i}{k^2 - m^2 + i\epsilon}$$

and perform the integrals in position space $\int \mathrm{d}^4x$. Which conditions result from this for the momenta? Specify the Feynman rules in momentum space.

Exercise 3: Introduction to FORM

You can get FORM from `https://github.com/vermaseren/form`. A good way to get started with FORM is to have a look at the tutorial "FORM for Pedestrians", which can be found at

`https://www.nikhef.nl/~form/maindir/documentation/tutorial/tutorial.html`.

An online version of the manual, where you can find detailed information about the functions implemented in FORM, you can find at

`https://www.nikhef.nl/~form/maindir/documentation/reference/online/online.html`.

You can also find the manual on git as a pdf-file.
Hint: This exercise might be useful for the next exercise sheets.

a) Install FORM following the instructions in `INSTALL`.

b) Read pages 1-5 of the tutorial "FORM for Pedestrians". This should give you an idea how you can write short FORM programs.

c) Write your first FORM program `Example1.frm`. Define two symbols x and y and calculate $(x+y)^n$ for different values of n. Print your result in the end. You can execute your program by typing `form Example1.frm` in your shell.

d) We want to use FORM to do calculations with γ matrices (see for example "FORM for Pedestrians" page 55). Write a FORM program `Example2.frm` which calculates the following traces of γ matrices:

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right], \qquad \mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\right], \qquad \mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right].$$

Start by defining the indices μ, ν, ρ, σ and use the γ matrices `g_(...)` implemented in FORM. To calulate the trace, you can use the function `trace4`;

e) Per default, FORM treats the Gamma matrices as 4-dimensional objects. However, it is also possible to choose their dimension arbitrarily by using `dimension`. Write a FORM program `Example3.frm` to calculate the three traces of the previous exercise in d dimensions (here you have to use `tracen` instead of `trace4`). Does the result change?
Calculate $\mathrm{Tr}\left[\gamma^\mu\gamma_\mu\right]$ in d dimensions. What result do you expect?

f) The matrix element of the process $e^+(p_1)\, e^-(p_2) \to \mu^+(p_3)\, \mu^-(p_4)$ is given by

$$\mathcal{M} = e^2\, \bar{v}_{s_1}(p_1)\gamma^\mu u_{s_2}(p_2) \frac{g_{\mu\nu}}{(p_1+p_2)^2} \bar{u}_{s_3}(p_3)\gamma^\nu v_{s_4}(p_4),$$

with $m_1 = m_2 = m_e$ and $m_3 = m_4 = m_\mu$. Calculate the squared matrix element and express the result in terms of Mandelstam variables using FORM.
