Theoretical Particle Physics I

Institut für Theoretische Teilchenphysik

Prof. Dr. Matthias Steinhauser, Dr. Daniel Stremmer, Dr. Marco Vitti, Pascal Reeck

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Exercise 1: Two-Particle Phase Space (3 points)

Consider the $2 \to 2$ scattering process $A(q_1) + B(q_2) \to C(p_1) + D(p_2)$. The associated phase-space integral

$$\int \mathrm{dLIPS} = \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3 2p_1^0} \int \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2p_2^0}\ (2\pi)^4\, \delta^{(4)}(p_1 + p_2 - q_1 - q_2)$$

was computed in the lectures, for the case where the final state particles have equal masses, $m_C = m_D = m$. Repeat the calculation, for the case where $m_C \neq m_D$. You may find the following relation helpful,

$$\int \mathrm{d}\rho\, g(\rho)\, \delta(f(\rho)) = \sum_i \frac{g(\rho_i^*)}{|f'(\rho_i^*)|}, \quad \text{where each } f(\rho_i^*) = 0.$$

Check that your result reduces correctly to the equal mass case.

Exercise 2: Two-Particle Phase Space in d dimensions

Consider the $2 \to 2$ scattering process $A(q_1) + B(q_2) \to C(p_1) + D(p_2)$ with $m_C = m_D = 0$. Calculate the two-particle phase space volume in d dimensions. The associated phase-space integral can be written as

$$\int \mathrm{dLIPS} = \int \frac{\mathrm{d}^{d-1} p_1}{(2\pi)^{d-1} 2p_1^0} \int \frac{\mathrm{d}^{d-1} p_2}{(2\pi)^{d-1} 2p_2^0}\ (2\pi)^d\, \delta^{(d)}(p_1 + p_2 - q_1 - q_2).$$

In d dimensions the momenta are written as $p^\mu = (p^0, \vec{p})$, where $\vec{p}$ is of dimension $d-1$ with the scalar product $p_\mu q^\mu = p_0 q_0 - \vec{p} \cdot \vec{q}$. Use for the volume integration spherical coordinates in $d-1$ dimensions where the integral measure can written as

$$\mathrm{d}^{d-1} p = \mathrm{d}|\vec{p}|\,|\vec{p}|^{d-2} \mathrm{d}\Omega_{d-1},$$

where $\mathrm{d}\Omega_d$ is the differential solid angle of the d dimensional unit sphere. In order to perform the angular integration, show that

$$\int \mathrm{d}\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)},$$

by starting from the following equality

$$(\sqrt{\pi})^d = \left(\int_{-\infty}^{\infty} dx\, e^{-x^2} \right)^d.$$

Exercise 3: Muon Pair Production from Electron-Positron Annihilation (7 points)

Consider the $2 \to 2$ scattering process

$$e^+ + e^- \to \mu^+ + \mu^-,$$

where the electron and positron have a mass m_e and the muon and anti-muon a mass m_m.

a) Draw the Feynman diagram for this process, and write down the associated matrix element.

b) Compute the squared matrix element, averaging over incoming particle spins and summing over outgoing particle spins. Express your result in terms of the particle masses (keeping the dependence on the electron and positron mass), and the Mandelstam variables defined above. You may use that

$$\sum_{\text{spin s}} u_s(p)\bar{u}_s(p) = \not{p} + m, \qquad \sum_{\text{spin s}} v_s(p)\bar{v}_s(p) = \not{p} - m.$$

c) Work in the centre-of-mass frame, and denote the scattering angle between the incoming and outgoing particles as θ. Express the answer to part (b) in terms of the centre-of-mass energy E and θ.

d) Using the formula from the lecture

$$\sigma = \frac{1}{64\pi^2 s}\frac{\sqrt{1-4m_m^2/s}}{\sqrt{1-4m_e^2/s}}\int \mathrm{d}\Omega \left[\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2\right],$$

compute the total cross section for this scattering process.