TTP1
Lecture 2

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2 The scalar field

We consider a real-valued field $\phi(t, \vec{x}) = \phi(x)$, where $x^\mu = (ct, \vec{x})$. In what follows, I will choose units where $\hbar = c = 1$, so from now on $\hbar$ and c will appear rarely if at all.

To describe relativistic particles, this field should satisfy the Klein-Gordon equation

$$\left[\partial^\mu \partial_\mu + m^2\right] \phi(x) = 0. \tag{2.1}$$

We interpret this equation as a classical equation of motion for the field ϕ.

Any classical equation of motion can be derived from a variation of the action $S = \int \mathrm{d}t\ L$. The Lagrange function L is an integral of the Lagrange function density $L = \int \mathrm{d}^3\vec{x}\ \mathcal{L}$ so that

$$S = \int \mathrm{d}^4x\ \mathcal{L}. \tag{2.2}$$

Since we would like S be Lorentz scalar and since the space-time volume element d^4x is invariant under Lorentz transformations, the Lagrange density $\mathcal{L}$ should be invariant under Lorentz transformations as well. We will require that $\mathcal{L}$ depends on ϕ and its first derivative $\partial_\mu \phi$ and not on x itself. Hence,

$$S = \int \mathrm{d}^4x\ \mathcal{L}(\phi, \partial_\mu \phi). \tag{2.3}$$

To find equations of motion, we consider a variation of S and find

$$\Delta S = \int \mathrm{d}^4x\ \left[\frac{\delta \mathcal{L}}{\delta \phi(x)} \Delta\phi(x) + \frac{\delta \mathcal{L}}{\delta[\partial^\mu \phi(x)]} \partial^\mu \Delta\phi\right]. \tag{2.4}$$

We integrate the last term by parts, neglect the surface terms and find

$$\Delta S = \int \mathrm{d}^4x\ \left[\frac{\delta \mathcal{L}}{\delta \phi(x)} - \partial^\mu \frac{\delta \mathcal{L}}{\delta[\partial^\mu \phi(x)]}\right] \Delta\phi(x). \tag{2.5}$$

Equations of motion follow from the extremum of the action S. For this, we need $\Delta S = 0$ for *any* $\Delta\phi(x)$. Hence, we find

$$\partial^\mu \left[\frac{\delta \mathcal{L}}{\delta[\partial^\mu \phi(x)]}\right] = \frac{\delta \mathcal{L}}{\delta \phi(x)}. \tag{2.6}$$

For a free scalar field, we would like this equation to be the Klein-Gordon equation. This can be achieved if we choose

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - \frac{m^2}{2}\phi^2. \tag{2.7}$$

Given the Lagrangian, we can construct a Hamiltonian. In classical mechanics, a system described with the Lagrangian $L = L(\dot{q}, q)$, has the Hamiltonian

$$H = \frac{\partial L}{\partial \dot{q}}p - L, \tag{2.8}$$

where the canonical momentum p reads

$$p = \frac{\partial L}{\partial \dot{q}}\ . \tag{2.9}$$

In the current case, the Lagrangian is

$$L = \int \mathrm{d}^3\vec{x}\ \mathcal{L}. \tag{2.10}$$

The canonical momentum

$$\pi(x) = \frac{\delta L}{\delta[\partial^0\phi(x)]} = \partial^0\phi(x). \tag{2.11}$$

Hence,

$$H = \int \mathrm{d}^3\vec{x}\ \left[\frac{\pi^2(x)}{2} + \frac{(\vec{\nabla}\phi(x))^2}{2} + \frac{m^2\phi^2(x)}{2}\right]. \tag{2.12}$$

We will now quantize this theory. To do this, we note that in quantum mechanics the quantization rule amounts to declaring that the generalized coordinate q_i and the corresponding canonical momentum p_j are described by operators that satisfy equal-time commutation relations

$$[q_i(t), p_j(t)] = i\delta_{ij}, \quad [p_i(t), p_j(t)] = 0, \quad [q_i(t), q_j(t)] = 0. \tag{2.13}$$

For our system, the coordinate $q_i(t)$ is $\phi(t, \vec{x})$ and the canonical momentum $p_i(t)$ is $\pi(t, \vec{x}) = \partial_0\phi(t, \vec{x})$. Hence, we require

$$\begin{aligned} &[\phi(t, \vec{x}), \pi(t, \vec{y})] = i\delta(\vec{x} - \vec{y}), \\ &[\phi(t, \vec{x}), \phi(t, \vec{y})] = 0, \quad [\pi(t, \vec{x}), \pi(t, \vec{y})] = 0. \end{aligned} \tag{2.14}$$

To describe the quantum system with the Hamiltonian in Eq. (2.12), we need to solve the Schrödinger equation

$$H|\psi\rangle = E|\psi\rangle. \tag{2.15}$$

Since it is not quite obvious how to do this, we can take some guidance from the fact that $\phi(t,\vec{x})$ satisfies the Klein-Gordon equation. To construct its solution, we write

$$\phi(t,\vec{x}) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\phi(t,\vec{p})e^{i\vec{p}\cdot\vec{x}}, \tag{2.16}$$

use this ansatz in the Klein-Gordon equation and find

$$\left(\frac{\partial^2}{\partial t^2} + \omega_p^2\right)\phi(t,\vec{p}) = 0, \tag{2.17}$$

where $\omega_p = \sqrt{\vec{p}^2 + m^2}$. This is an equation of a harmonic oscillator with the frequency ω_p. The solutions to this equation are

$$\phi(t,\vec{p}) = \alpha_{\vec{p}}e^{-i\omega_p t} + \beta_{\vec{p}}e^{i\omega_p t}. \tag{2.18}$$

Hence,

$$\phi(t,\vec{x}) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\left[\alpha_{\vec{p}}e^{-i\omega_p t + i\vec{p}\vec{x}} + \beta_{\vec{p}}e^{i\omega_p t + i\vec{p}\vec{x}}\right]. \tag{2.19}$$

However, since $\phi(t,x)$ is a real-valued field, the coefficients α_p and β_p are not independent. In fact, from the reality condition $\phi(x)^* = \phi(x)$, it follows that

$$\beta_{\vec{p}} = \alpha^*_{-\vec{p}}. \tag{2.20}$$

Using this equation, we write

$$\phi(t,\vec{x}) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\left[\alpha_{\vec{p}}e^{-i\omega_p t + i\vec{p}\vec{x}} + \alpha^*_{\vec{p}}e^{i\omega_p t - i\vec{p}\vec{x}}\right]. \tag{2.21}$$

The canonical momentum $\partial_t\phi(t,\vec{x})$ is then

$$\pi(t,\vec{x}) = -i\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\omega_{\vec{p}}\left[\alpha_{\vec{p}}e^{-i\omega_p t + i\vec{p}\vec{x}} - \alpha^*_{\vec{p}}e^{i\omega_p t - i\vec{p}\vec{x}}\right]. \tag{2.22}$$

To satisfy the commutation relations in Eq. (2.14), we need to promote $\alpha_{\vec{p}}$ and $\alpha^*_{\vec{p}}$ to operators. These operators are identical to the creation and annihilation operators of the harmonic oscillator but they carry an additional index $\vec{p}$ to indicate the dependence of the oscillator's frequency on the momentum $\vec{p}$. Specifically, we choose $\alpha_{\vec{p}} \to a_{\vec{p}}/\sqrt{2\omega_{\vec{p}}}$ and $\alpha^*_{\vec{p}} \to a^+_{\vec{p}}/\sqrt{2\omega_{\vec{p}}}$, so that

$$\begin{aligned}\phi(t,\vec{x}) &= \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3\sqrt{2\omega_p}} \left[a_{\vec{p}} e^{-i\omega_p t + i\vec{p}\vec{x}} + a^+_{\vec{p}} e^{i\omega_p t - i\vec{p}\vec{x}}\right], \\ \pi(t,\vec{x}) &= -i\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\sqrt{\frac{\omega_{\vec{p}}}{2}} \left[a_{\vec{p}} e^{-i\omega_p t + i\vec{p}\vec{x}} - a^+_{\vec{p}} e^{i\omega_p t - i\vec{p}\vec{x}}\right].\end{aligned} \tag{2.23}$$

The commutation relations in Eq. (2.14) are satisfied provided that we choose

$$[a_{\vec{p}_1}, a^+_{\vec{p}_2}] = (2\pi)^3\delta^{(3)}(\vec{p}_1 - \vec{p}_2), \quad [a_{\vec{p}_1}, a_{\vec{p}_2}] = [a^+_{\vec{p}_1}, a^+_{\vec{p}_2}] = 0. \tag{2.24}$$

We will next express the Hamiltonian operator in Eq. (2.12) through the creation and annihilation operators. This is straightforward but requires some patience. Consider, as an example, a term proportional to m^2. We find

$$\begin{aligned}\int \mathrm{d}^3\vec{x}\;\phi(t,\vec{x})\phi(t,\vec{x}) &= \int \frac{\mathrm{d}^3\vec{p}_1\mathrm{d}^3\vec{p}_2}{(2\pi)^3\sqrt{2\omega_1}\sqrt{2\omega_2}}\Bigg[\left(a_{\vec{p}_1}a_{\vec{p}_2}e^{-2i\omega_1 t} + \right. \\ &\left. a^+_{\vec{p}_1}a^+_{\vec{p}_2}e^{2i\omega_1 t}\right)\delta^{(3)}(\vec{p}_1+\vec{p}_2) + \left(a_{\vec{p}_1}a^+_{\vec{p}_2} + a^+_{\vec{p}_1}a_{\vec{p}_2}\right)\delta^{(3)}(\vec{p}_1-\vec{p}_2)\Bigg] \\ &= \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2\omega_p}\left[a_{\vec{p}}a_{-\vec{p}}e^{-2i\omega_p t} + a^+_{\vec{p}}a^+_{-\vec{p}}e^{2i\omega_p t} + a_{\vec{p}}a^+_{\vec{p}} + a^+_{\vec{p}}a_{\vec{p}}\right].\end{aligned} \tag{2.25}$$

In writing the above formula, we have been using the notation $\omega_{1,2} = \omega_{p_1,p_2}$.

Similarly, we find

$$\int \mathrm{d}^3\vec{x}\;\pi^2(t,\vec{x}) = \int \frac{\mathrm{d}^3\vec{p}\;\omega_p^2}{(2\pi)^3 2\omega_p}\left[-a_{\vec{p}}a_{-\vec{p}}e^{-2i\omega_p t} - a^+_{\vec{p}}a^+_{-\vec{p}}e^{2i\omega_p t} + a_{\vec{p}}a^+_{\vec{p}} + a^+_{\vec{p}}a_{\vec{p}}\right], \tag{2.26}$$

and

$$\int \mathrm{d}^3\vec{x}\;(\vec{\nabla}\phi(t,\vec{x}))^2 = \int \frac{\mathrm{d}^3\vec{p}\;\;\vec{p}^2}{(2\pi)^3 2\omega_p}\left[a_{\vec{p}}a_{-\vec{p}}e^{-2i\omega_p t} + a^+_{\vec{p}}a^+_{-\vec{p}}e^{2i\omega_p t} + a_{\vec{p}}a^+_{\vec{p}} + a^+_{\vec{p}}a_{\vec{p}}\right], \tag{2.27}$$

Combining the above equations, we obtain the following result for the Hamiltonian

$$H = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \frac{\omega_p}{2} \left[a_p^+ a_p + a_p a_p^+ \right] = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \left[\omega_p \; a_p^+ a_p + \frac{\omega_p}{2} (2\pi)^3 \delta^{(3)}(\vec{0}) \right]. \tag{2.28}$$

Since $\delta^{(3)}(\vec{0}) = \infty$, the above result may look somewhat strange but it is easy to understand why that term is there. Indeed, to properly interpret this term, we write

$$(2\pi)^3 \delta^{(3)}(\vec{0}) = \int \mathrm{d}^3x \; e^{i\vec{p}\vec{x}}|_{\vec{p}=0} = \int \mathrm{d}^3x = V, \tag{2.29}$$

where V is the space volume. Hence,

$$H = E_{\mathrm{vac}} + \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \; \omega_p \; a_p^+ a_p, \tag{2.30}$$

where the infinite but constant contribution to the energy operator is

$$E_{\mathrm{vac}} = V \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \frac{\omega_p}{2}. \tag{2.31}$$

Note that this energy is infinite for two reasons. First, the volume V is infinitely large; this infinity is understandable – if there is an underlying energy density of the vacuum state, the total energy is proportional to the total volume which, if the volume is large, becomes large.

However, also the *energy density* E_{vac}/V is *infinite*

$$\rho_{\mathrm{vac}} = \frac{E_{\mathrm{vac}}}{V} = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \frac{\omega_p}{2} \sim \int \mathrm{d}^3\vec{p} \sqrt{\vec{p}^2 + m^2} \sim \int^{\Lambda} \mathrm{d}p \; p^3 \sim \Lambda^4. \tag{2.32}$$

In the above equation we introduce the upper cut-off for the integration over energies Λ to exhibit the degree of divergence. There is an important story behind this result and its connection to the so-called *cosmological constant problem* which, unfortunately, I cannot discuss here.

To determine the energy levels of this quantum system, we define the state with the lowest energy, the *vacuum* $|0\rangle$. This state is annihilated by all annihilation operators

$$a_{\vec{p}}|0\rangle = 0. \tag{2.33}$$

From Eq. (2.30) it follows that this state is the eigenstate of the Hamiltonian H and has *infinite* energy $E_{\rm vac}$. Since the absolute value of energy plays no role as long as we do not consider the force of gravity, we can safely set $E_{\rm vac}$ to zero.[1]

To construct the excited states, we compute the commutator of an operator $\vec{a}_p$ with the Hamiltonian H. We find

$$\begin{aligned}[H, a_{\vec{k}}] &= \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\, \omega_p\, [a_p^+ a_p, a_{\vec{k}}] \\ &= -\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\, \omega_p\, (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{k}) a_p = -\omega_k a_{\vec{k}}.\end{aligned} \tag{2.34}$$

Hence,

$$[H, a_{\vec{k}}] = -\omega_k a_{\vec{k}}, \quad [H, a_{\vec{k}}^+] = \omega_k a_{\vec{k}}^+. \tag{2.35}$$

Consider a state $|\vec{k}\rangle = a_{\vec{k}}^+|0\rangle$. This state is also an eigenstate of the Hamiltonian with the energy ω_k since

$$H|\vec{k}\rangle = H a_{\vec{k}}^+|0\rangle = [H, a_{\vec{k}}^+]|0\rangle = \omega_k a_{\vec{k}}^+|0\rangle = \omega_k |k\rangle. \tag{2.36}$$

Similarly,

$$\begin{aligned}H|\vec{k}_1, \vec{k}_2\rangle &= H a_{\vec{k}_1}^+ a_{\vec{k}_2}^+|0\rangle = [H, a_{\vec{k}_1}^+] a_{\vec{k}_2}^+|0\rangle + a_{\vec{k}_1}^+ H a_{\vec{k}_2}^+|0\rangle \\ &= (\omega_{k_1} + \omega_{k_2})|\vec{k}_1, \vec{k}_2\rangle.\end{aligned} \tag{2.37}$$

We will discuss later the construction of the energy-momentum tensor of the scalar field. It follows from that construction that the three-momentum operator of a scalar field reads

$$\vec{P} = -\int \mathrm{d}^3\vec{x}\; \pi(t,\vec{x}) \vec{\nabla}\phi(t,\vec{x}) \;. \tag{2.38}$$

A simple calculation, similar to what we have done for the Hamiltonian, gives[2]

$$\vec{P} = \frac{1}{2}\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\, \vec{p}\, \left(a_p^+ a_p + a_p \vec{a}_p^+\right) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} a_p^+ a_p + \int \mathrm{d}^3\vec{p}\vec{p}\delta^{(3)}(\vec{0}). \tag{2.39}$$

[1]Alternatively, we redefine the Hamiltonian $H \to H - E_{\rm vac}$.

[2]In contrast to the calculation for H, in this case terms with two creation and annihilation operators do not vanish right away. However, the integrand is an odd function of $\vec{p}$ and since we integrate over all values of $\vec{p}$, the result vanishes.

The last term integrates to zero since $\vec{p}$ is odd under parity and we conclude

$$\vec{P} = \int \frac{d^3\vec{p}}{(2\pi)^3}\, \vec{p}\, a_p^+ a_p. \tag{2.40}$$

Note that states $|\vec{k}\rangle$, $|\vec{k}_1, k_2\rangle$ etc. are eigenstates of the momentum operator $\vec{P}$. For example,

$$\vec{P}|\vec{k}\rangle = \vec{k}\, |\vec{k}\rangle. \tag{2.41}$$

Hence, the state $|\vec{k}\rangle$ is the state with the three-momentum $\vec{k}$ and the energy $\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$ which is a relativistic relation between particle's energy, three momentum and mass.

We conclude that this state describes a relativistic particle of mass m. Such a particle is *an excitation of the quantum field* $\phi(x)$. A state $|\vec{k}_1, \vec{k}_2\rangle$ describes two particles, with momenta $\vec{k}_1$ and $\vec{k}_2$ and energies ω_1 and ω_2 etc. Since the creation operators commute, this state is symmetric under the permutation of two particles; for this reason these particles are bosons. The Hilbert space of the theory contains states with arbitrary number of particles.

The quantum mechanical states are supposed to be normalized. We compute

$$\langle\vec{k}_2|\vec{k}_1\rangle = \langle 0|a_{\vec{k}_2} a^+_{\vec{k}_1}|0\rangle = \langle 0|[a_{\vec{k}_2}, a^+_{\vec{k}_1}]|0\rangle = (2\pi)^3\delta^{(3)}(\vec{k}_2 - \vec{k}_1). \tag{2.42}$$

This normalization is possible but not optimal because it is not invariant under Lorentz transformations. A Lorentz-invariant combination is the product of energy and the three-momentum δ-function, i.e. $2E_{\vec{k}_1}\delta^{(3)}(\vec{k}_1 - \vec{k}_2)$. Hence, we can choose to write states with an additional factor that involves the square root of the energy

$$|\vec{k}\rangle = (2E_{\vec{k}})^{1/2} a^+_{\vec{k}}|0\rangle. \tag{2.43}$$

Such states are then normalized as

$$\langle\vec{k}_2|\vec{k}_1\rangle = (2E_{\vec{k}_1})(2\pi)^3\delta^{(3)}(\vec{k}_1 - \vec{k}_2), \tag{2.44}$$

and this normalization does not change when a Lorentz boost is performed.