TTP1
Lecture 3

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25. April 2023

3 The Klein-Gordon field in space time

We have seen earlier that quantized scalar field can be written as

$$\phi(t,\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2\omega_{\vec{p}}}} \left[a_p e^{-i\omega_p t + i\vec{p}\vec{x}} + a_p^+ e^{i\omega_p t - i\vec{p}\vec{x}} \right]. \tag{3.1}$$

We will now discuss an alternative way to describe the time-dependence of the field ϕ.

To this end, we recall that the Hamiltonian H and the creation and annihilation operators satisfy the following commutation relations

$$[H, a_{\vec{p}}] = -\omega_{\vec{p}} a_{\vec{p}}, \quad [H, a_{\vec{p}}^+] = \omega_{\vec{p}} a_{\vec{p}}^+. \tag{3.2}$$

Consider the last equation and write it as

$$H a_p^+ = a_p^+ (H + \omega_p). \tag{3.3}$$

It follows that

$$H^2 a_p^+ = H a_p^+ (H + \omega_p) = a_p^+ (H + \omega_p)^2, \tag{3.4}$$

and, consequently,

$$H^n a_p^+ = a_p^+ (H + \omega_p)^n, \tag{3.5}$$

for all values of n. A similar equation for a_p reads

$$H^n a_p = a_p (H - \omega_p)^n. \tag{3.6}$$

Then,

$$\begin{aligned} e^{iHt}\phi(0,\vec{x})e^{-Ht} &= \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2\omega_p}} \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \left[H^n a_p e^{i\vec{p}\vec{x}} + H^n a_p^+ e^{-i\vec{p}\vec{x}} \right] e^{-Ht} \\ &= \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2\omega_p}} \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \left[a_p (H-\omega_p)^n e^{i\vec{p}\vec{x}} + a_p^+ (H+\omega_p) e^{-i\vec{p}\vec{x}} \right] e^{-Ht} \\ &= \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2\omega_p}} \left[a_p e^{-i\omega t + i\vec{p}\vec{x}} e^{iHt} + a_p^+ e^{i\omega_p t - i\vec{p}\vec{x}} e^{iHt} \right] e^{-Ht} = \phi(t,\vec{x}). \end{aligned} \tag{3.7}$$

Hence,

$$\phi(t,\vec{x}) = e^{iHt}\phi(0,\vec{x})e^{-iHt}. \tag{3.8}$$

Similarly, it is straightforward to check that

$$e^{-i\vec{P}\cdot\vec{x}}\phi(t,0)e^{i\vec{P}\cdot\vec{x}} = \phi(t,\vec{x}), \tag{3.9}$$

where $\vec{P}$ is an operator of total momentum defined in Eq. (**??**). Combining the above equations, we can write

$$\phi(x^\mu) = e^{iP_\mu x^\mu}\phi(0)e^{-iP_\mu x^\mu}, \tag{3.10}$$

where $x^\mu = (t,\vec{x})$ and

$$P^\mu = (H,\vec{P}) \tag{3.11}$$

is a four-vector composed of operators of the total energy (H) and the three momentum ($\vec{P}$).

Using Eq. (3.8), we compute the time derivative of the field ϕ. We find

$$i\frac{\partial\phi(t,\vec{x})}{\partial t} = [\phi(t,\vec{x}), H] \tag{3.12}$$

Similarly,

$$i\frac{\partial\pi(t,\vec{x})}{\partial t} = [\pi(t,\vec{x}), H]. \tag{3.13}$$

To compute the commutator in Eq. (3.13), we use the fact that the Hamiltonian H is time-independent. For this reason we can write it using operators π and ϕ at the time t. The Hamiltonian reads

$$H = \frac{1}{2}\int \mathrm{d}^3\vec{y}\ \left(\pi^2(t,\vec{y}) + \left(\vec{\nabla}_y\phi(t,\vec{y})\right)^2 + m^2\phi(t,\vec{y})^2\right). \tag{3.14}$$

Then, using equal-time commutation relations

$$\begin{aligned} &[\pi(t,\vec{x}),\phi(t,\vec{y})] = -i\delta^{(3)}(\vec{x}-\vec{y}),\\ &[\pi(t,\vec{x}),\pi(t,\vec{y})] = 0,\\ &[\phi(t,\vec{x}),\phi(t,\vec{y})] = 0, \end{aligned} \tag{3.15}$$

we obtain

$$[\pi(t,\vec{x}), H] = -i\left(-\vec{\nabla}^2\phi(t,\vec{x}) + m^2\phi(t,\vec{x})\right). \tag{3.16}$$

Next, using $\pi = \partial\phi/\partial t$, we derive

$$\left[\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + m^2\right]\phi(t,\vec{x}) = 0. \tag{3.17}$$

This is the Klein-Gordon equation but this time for an *operator* ϕ.

In variance with Quantum Mechanics where Schrödinger equation is the main object to study, in Quantum Field Theory we almost never solve the Schrödinger equation and typically work with either Green's functions or somewhat related objects called scattering amplitudes. To introduce the idea of Green's functions, consider the following object

$$D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle. \tag{3.18}$$

When the field $\phi(y)$ acts on the vacuum state $|0\rangle$, it creates a state build of particles. When we compute the scalar product of the state $\phi(y)|0\rangle$ with $\langle 0|\phi(x)$, we calculate the probability amplitude that some of the created particles arrive at the point x and are absorbed back into the vacuum.

To calculate $D(x-y)$ we use the representation of the quantum field

$$\phi(x) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec{p}}}}\left(a_{\vec{p}}e^{-ip_\mu x^\mu} + a^+_{\vec{p}}e^{ip_\mu x^\mu}\right), \tag{3.19}$$

and find

$$D(x-y) = \int \frac{\mathrm{d}^3\vec{q}}{(2\pi)^3\sqrt{2E_{\vec{q}}}}\,\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec{p}}}}e^{-ip_\mu x^\mu + iq_\mu y^\mu}\langle 0|a_{\vec{p}}a^+_{\vec{q}}|0\rangle. \tag{3.20}$$

Since

$$\langle 0|a_{\vec{p}}a^+_{\vec{q}}|0\rangle = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}), \tag{3.21}$$

we integrate over $\vec{q}$ and find

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3\,2E_{\vec{p}}}\;e^{-ip_\mu(x^\mu - y^\mu)}. \tag{3.22}$$

To compute this integral, we consider two cases $(x-y)^2 > 0$ and $(x-y)^2 < 0$. In the first case, we can use the fact that the integral in Eq. (3.34) is Lorentz-invariant and choose a frame where $x - y = (\tau, \vec{0})$. Then we find

$$\langle 0|\phi(x)\phi(y)|0\rangle = \frac{1}{4\pi^2}\int\limits_m^\infty \mathrm{d}E_p\,\sqrt{E_p^2 - m^2}\;e^{-iE_p\tau}. \tag{3.23}$$

The function depends on the ratio of the particle's Compton wavelength $\lambda = 1/m$ and the interval τ. Although the integral in the above equation is too difficult to compute, we can obtain its value at large times, i.e. $\tau \gg \lambda$ or $m\tau \gg 1$. We find

$$\lim_{\tau m \to \infty} \langle 0|\phi(x)\phi(y)|0\rangle \approx \frac{\sqrt{2m} e^{-3i\pi/4}\Gamma(3/2)}{4\pi^2 \tau^{3/2}} e^{-im\tau}, \quad \tau = \sqrt{(x-y)^2}. \tag{3.24}$$

It follows that the probability amplitude for a particle to reach a faraway point decreases like a power of τ.

The second case is when $(x-y)^2 < 0$. Then, we choose a reference frame where $(x-y) = (0, \vec{r})$. In the limit $mr \gg 1$, correlator evaluates to

$$\lim_{rm \to \infty} \langle 0|\phi(x)\phi(y)|0\rangle \approx \frac{\sqrt{2m}\Gamma(3/2)}{4\pi^2 r^{3/2}} e^{-mr}. \tag{3.25}$$

Hence, in this case, the correlator is *exponentially* suppressed for interval values that exceed the Compton wave length of the particle. Nevertheless, it should be stressed that in this case points x and y are spatially separated which means that in order to be created at the point x and be absorbed at the point y, a particle has to travel *faster than light* which cannot happen in classical physics. However, this is not a problem in Quantum Mechanics and in Quantum Field Theory where interpretation in terms of traveling times and trajectories does not exist. What we need to check though is that *measurements* of the field at points y and x do not influence each other provided that the two points are separated by *a space-like interval*.

As we know from Quantum Mechanics, whether observables described by two operators can simultaneously be measured depends on whether they commute. Hence, we need to compute the commutator of $\phi(x)$ and $\phi(y)$. We find

$$\begin{aligned}
&[\phi(x), \phi(y)] = \\
&\int \frac{\mathrm{d}^3\vec{p}_1 \mathrm{d}^3\vec{p}_2}{(2\pi)^3\sqrt{2E_{p_1}}(2\pi)^3\sqrt{2E_{p_2}}} \left([a_{\vec{p}_1}, a^+_{\vec{p}_2}] e^{-ip_1x} e^{ip_2y} + [a^+_{\vec{p}_1}, a_{\vec{p}_2}] e^{ip_1x} e^{-ip_2y}\right) \\
&= \int \frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3 2E_{\vec{p}_1}} (2\pi)^3 \delta^{(3)}(\vec{p}_1 - \vec{p}_2) \left(e^{-ip_1(x-y)} - e^{ip_1(x-y)}\right) \\
&= D(x-y) - D(y-x).
\end{aligned} \tag{3.26}$$

It follows from our previous discussion that for space-like separation $D(x-y)$ only depends on $r=\sqrt{|(x-y)^2|}$. Then $D(x-y)-D(y-x)=0$. On the contrary, for time-like separation, $D(x-y)$ depends on the sign of τ. Then $D(x-y)\sim e^{-im\tau}$ whereas $D(y-x)\sim e^{im\tau}$ so that $D(x-y)-D(y-x)\neq 0$. Hence,

$$[\phi(x),\phi(y)]_{(x-y)^2<0}=0, \tag{3.27}$$

which is the necessary requirement for the causality of the theory.

We can now introduce the concept of a "propagator". We consider

$$\langle 0|[\phi(x),\phi(y)]|0\rangle = D(x-y)-D(y-x). \tag{3.28}$$

Using the representation for $D(x-y)$ derived above, we obtain

$$\begin{aligned}\langle 0|[\phi(x),\phi(y)]|0\rangle &= \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}}\left(e^{-ip(x-y)}-e^{ip(x-y)}\right)\\ &= \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\left(\frac{1}{2E_{\vec{p}}}e^{-ip(x-y)}+\frac{1}{(-2E_{\vec{p}})}e^{-i(-E_p(x-y)_0+\vec{p}(\vec{x}-\vec{y}))}\right).\end{aligned} \tag{3.29}$$

We can now change the integration variable $\vec{p}\to-\vec{p}$ in the second term in the sum. Then we find

$$\langle 0|[\phi(x),\phi(y)]|0\rangle = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\left[\frac{1}{2p_0}e^{-ip(x-y)}\bigg|_{p_0=E_{\vec{p}}}+\frac{1}{2p_0}e^{-ip(x-y)}\bigg|_{p_0=-E_{\vec{p}}}\right]. \tag{3.30}$$

We can represent this quantity as an integral over $\mathrm{d}^4p=\mathrm{d}p_0\,\mathrm{d}^3\vec{p}$ along a particular integration contour. Consider

$$G(x-y)=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2-m^2}e^{-ip(x-y)}. \tag{3.31}$$

This integral is poorly defined because there are two poles on the real axis, at $p_0=\pm\sqrt{\vec{p}^2+m^2}$. To define G we have to provide a prescription for going around these poles in p_0 complex plane. Suppose we specify the contour by adding a small imaginary part to the denominator of $1/(p^2-m^2)$. There are different ways to do that; for example

$$\frac{1}{p^2-m^2}\to\frac{1}{p^2-m^2+i\epsilon\,\mathrm{sgn}(p_0)}. \tag{3.32}$$

In this formula ϵ is an infinitesimal positive quantity that we are supposed to take to zero at the end of the calculation.

We would like to compute G by integrating over p_0 first with the help of Cauchy's theorem. To use the theorem, we need to consider p_0 integration as the integration in the complex plane and we need to determine poles of the integrand. The poles are found by solving the equation

$$p^2 - m^2 + i\epsilon\,\mathrm{sgn}(p_0) = p_0^2 - (\vec{p}^2 + m^2) + i\epsilon\,\mathrm{sgn}(p_0) = 0. \tag{3.33}$$

It is easy to see that this equation has two solutions,

$$p_0 = \pm\sqrt{\vec{p}^2 + m^2} - i\epsilon, \tag{3.34}$$

which means that both poles lie below the real axis.

To compute G using Cauchy's theorem, we would like to close the integration contour by integrating over infinitely remote half-circle. Whether this half-circle should lie in the upper or in the lower complex half-plane, depends on the sign of $(x_0 - y_0)$. If $(x_0 - y_0) > 0$, we have to close the integration contour in the lower half-plane and in this case the residues at the two poles in Eq. (3.34) contribute. However, if $(x_0 - y_0) < 0$, we have to close the integration contour in the upper half plane in which case the two poles do not contribute as they are outside of the integration contour. Computing the residues we find

$$\begin{aligned} G_R(x-y) &= \theta(x_0 - y_0) \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \left\{ \left.\frac{e^{-ip(x-y)}}{2p_0}\right|_{p_0\to E_{\vec{p}}} + \left.\frac{e^{-ip(x-y)}}{2p_0}\right|_{p_0\to -E_{\vec{p}}} \right\} \\ &= \theta(x_0 - y_0)\langle 0|[\phi(x), \phi(y)]|0\rangle. \end{aligned} \tag{3.35}$$

From the integral representation in Eq. (3.31) it follows that $G_R(x-y)$ is the Green's function of the Klein-Gordon equation. Indeed,

$$(\partial_\mu\partial^\mu + m^2)G_R(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i(-p^2 + m^2)}{p^2 - m^2 + i\epsilon\,\mathrm{sgn}(p_0)} e^{-ip(x-y)} = -i\delta^{(4)}(x-y). \tag{3.36}$$

Such a Green's function is called *retarded*, because it vanishes for $(x-y)_0 < 0$. If we solve the inhomogeneous Klein-Gordon equation

$$(\partial_\mu\partial^\mu + m^2)\phi(x) = j(x), \tag{3.37}$$

using the retarded Green's function, we obtain the result

$$\phi(x) = i \int \mathrm{d}^4 y \, G_R(x-y)\, j(y). \tag{3.38}$$

An important feature of this solution is that $\phi(x_0, \vec{x})$ depends on the values of the source $j(y_0, \vec{y})$ at *earlier times* $y_0 < x_0$. Although this feature of the solution is appealing, we will see that in Quantum Field Theory a different type of Green's function is required.

To introduce it, we will again consider the Green's function defined by Eq. (3.31), but we will shift p_0 poles away from the real axis in the following way

$$G_F(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}. \tag{3.39}$$

This is also a Green's function of the Klein-Gordon equation; it is called the *Feynman propagator*. To understand what it does, we again compute the integral over p_0 using Cauchy's theorem. The poles are

$$p_0 = \pm\sqrt{\vec{p}^2 + m^2} \mp i\epsilon. \tag{3.40}$$

For $(x-y)_0 > 0$, we need to close the contour in the complex lower half-plane; for $(x-y)_0 < 0$ we should close the contour in the upper half-plane. In the first case the pole at $p_0 = \sqrt{\vec{p}^2 + m^2}$ contributes and in the second case the pole at $p_0 = -\sqrt{\vec{p}^2 + m^2}$. We find

$$\begin{aligned} G_F(x-y) &= \theta(x_0 - y_0) \int \frac{\mathrm{d}^3 \vec{p}}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)} \\ &\quad + \theta(y_0 - x_0) \int \frac{\mathrm{d}^3 \vec{p}}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)} \Big|_{p_0 \to -E_{\vec{p}}}. \end{aligned} \tag{3.41}$$

We transform the last term by writing

$$e^{-ip(x-y)} \Big|_{p_0 \to -E_{\vec{p}}} = e^{-ip(y-x)}_{p_0 \to E_{\vec{p}}, \vec{p} \to -\vec{p}} \,. \tag{3.42}$$

Hence,

$$\begin{aligned} G_F(x-y) &= \theta(x_0 - y_0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y_0 - x_0)\langle 0|\phi(y)\phi(x)|0\rangle \\ &= \langle 0|T\phi(x)\phi(y)|0\rangle, \end{aligned} \tag{3.43}$$

where we introduced the time-order product of two operators

$$T\phi(x)\phi(y) = \theta(x_0 - y_0)\phi(x)\phi(y) + \theta(y_0 - x_0)\phi(y)\phi(x). \tag{3.44}$$