TTP1
Lecture 4

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4 Lorentz transformations

There are good experimental reasons to believe that laws of Nature are identical in all inertial frames; for our purposes this means that theories that we consider should be invariant under Lorentz transformations. A Lorentz transformation of coordinates and other four-vector is described by a four-by-four matrix $\Lambda^{\mu}{}_{\nu}$ such that four-vectors in two frames are related by

$$x_1^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}. \tag{4.1}$$

Scalar products of four-vectors are invariant under Lorentz transformations, e.g.

$$g_{\mu\nu} x_1^{\mu} y_1^{\nu} = g_{\mu\nu} x^{\mu} y^{\nu}, \tag{4.2}$$

provided that $y_1^{\mu} = \Lambda^{\mu}{}_{\nu} y^{\nu}$, and x_1 and x are related by Eq. (4.1).

The matrix $\Lambda^{\mu}{}_{\nu}$ obeys the following equation

$$g_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}. \tag{4.3}$$

From this equation, matrix of an inverse Lorentz transformation can be deduced. It reads

$$\bar{\Lambda}^{\mu}{}_{\nu} = \Lambda_{\nu}{}^{\mu}, \tag{4.4}$$

so that

$$x^{\mu} = \bar{\Lambda}^{\mu}{}_{\nu}\, x_1^{\nu} = \Lambda_{\nu}{}^{\mu} x_1^{\nu}. \tag{4.5}$$

We will use the above formulas in what follows.

We will now discuss how Lorentz transformations affect quantum fields. We have said in the previous lecture(s) that the field ϕ is the scalar field; this means that the field is the same in all reference frames. Mathematically, this means

$$\phi_1(x_1) = \phi(\Lambda^{-1} x_1), \tag{4.6}$$

provided that x_1 and x are related by a Lorentz transformation

$$x_1^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}. \tag{4.7}$$

Eq. (4.6) will be sufficient to specify transformation properties of the classical field. Since in the quantum theory the field is an operator, the Lorentz invariance of the theory implies that matrix elements of the field operator computed in different reference frames are the same. Quantum states in different frames are related by transformations, e.g.

$$|\alpha_1\rangle = U(\Lambda)|\alpha\rangle, \quad \langle\alpha_1| = \langle\alpha|U^{-1}(\Lambda), \tag{4.8}$$

where $U(\Lambda)$ "represents" the action of Lorentz transformation on the state. Then, we require

$$\langle\alpha_1|\phi(x_1)|\beta_1\rangle = \langle\alpha|\phi(x)|\beta\rangle. \tag{4.9}$$

We re-write the left-hand side of this equation by expressing $|\beta_1\rangle$ and $|\alpha_1\rangle$ through $|\alpha\rangle$ and $|\beta\rangle$ and find

$$\langle\alpha|U^{-1}\phi(x_1)U|\beta\rangle = \langle\alpha|\phi(x)|\beta\rangle. \tag{4.10}$$

Since this result should be valid for all external states, we obtain

$$U^{-1}\phi(\Lambda x)U = \phi(x), \tag{4.11}$$

or, equivalently,

$$\phi(\Lambda x) = U\phi(x)U^{-1}. \tag{4.12}$$

To understand consequences of the above equation, we write an expansion of the field ϕ in terms of creation and annihilation operators and obtain

$$\begin{aligned} U(\Lambda)\phi(x)U^{-1}(\Lambda) &= \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \left[U a_{\vec{p}} U^{-1} e^{-ip_\mu x^\mu} + U a^+_{\vec{p}} U^{-1} e^{ip_\mu x^\mu}\right], \\ \phi(\Lambda x) = \phi(x_1) &= \int \frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3\sqrt{2E_{\vec{p}_1}}} \left[a_{\vec{p}_1} e^{-ip_{1,\mu}x_1^\mu} + a^+_{\vec{p}_1} e^{ip_{1,\mu}x_1^\mu}\right], \end{aligned} \tag{4.13}$$

where $x_1 = \Lambda x$ and $\vec{p}_1$ is a dummy integration variable. To force the two equations look similar to each other, we inspect phases of exponential functions in the second equation. We find

$$p_{1,\mu}x_1^\mu = p_{1,\mu}\Lambda^\mu{}_\nu x^\nu = p_{1,\mu}\bar{\Lambda}_\nu{}^\mu x^\nu = x_\nu\, \bar{\Lambda}^\nu{}_\mu p_1^\mu = x_\nu p^\nu, \tag{4.14}$$

where we introduced a four-vector

$$p^\mu = \bar{\Lambda}^\mu{}_\nu p_1^\nu. \tag{4.15}$$

The matrix $\bar{\Lambda}$ is the matrix of the inverse Lorentz transformation.

We can interpret the above equation as the integration-variable transformation. Since this equation represents a Lorentz transformation of a four-vector p_1^μ and since $\mathrm{d}^3\vec{p}/(2E_{\vec{p}})$ is Lorentz-invariant, we easily find

$$\frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3\sqrt{2E_{\vec{p}_1}}} = \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec{p}}}}\sqrt{\frac{E_{\vec{p}_1}}{E_{\vec{p}}}}. \tag{4.16}$$

Hence, we can write

$$\phi(x_1) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3\sqrt{E_{\vec{p}}}}\sqrt{\frac{E_{\Lambda\vec{p}}}{E_{\vec{p}}}}\left[a_{\Lambda\vec{p}}e^{-ipx} + a^+_{\Lambda\vec{p}}e^{ipx}\right], \tag{4.17}$$

Requiring that Eq. (4.12) holds, we find the transformation rules for creation and annihilation operators

$$U(\Lambda)a_{\vec{p}}U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda\vec{p}}}{E_{\vec{p}}}}a_{\Lambda\vec{p}}, \quad U(\Lambda)a^+_{\vec{p}}U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda\vec{p}}}{E_{\vec{p}}}}a^+_{\Lambda\vec{p}}. \tag{4.18}$$

To understand what these equations imply, consider single particle state $|\vec{p}\rangle$. Such a state is constructed by acting on the vacuum state with the relevant creation operator

$$|\vec{p}\rangle = (2E_{\vec{p}})^{1/2}a^+_{\vec{p}}|0\rangle. \tag{4.19}$$

Note that we use relativistically-invariant normalization of the state $|\vec{p}\rangle$.

Then Lorentz transformation of this state leads to

$$U(\Lambda)|\vec{p}\rangle = (2E_{\vec{p}})^{1/2}\sqrt{\frac{E_{\Lambda\vec{p}}}{E_{\vec{p}}}}a^+_{\Lambda\vec{p}}|0\rangle = |\Lambda\vec{p}\rangle, \tag{4.20}$$

which is a quantum state with the boosted momentum. Note that we assumed that the vacuum state is invariant under Lorentz transformations

$$U(\Lambda)|0\rangle = |0\rangle. \tag{4.21}$$

To move further, we recall that Lorentz transformations form what mathematicians call a "group". This means that a product of two Lorentz transformations is a Lorentz transformation and that for every transformation there is an inverse one. As with most of the continuous groups, any Lorentz transformations can be constructed from infinitesimal transformations by applying them multiple times. Consider a matrix of Lorentz transformations $\Lambda^{\mu}{}_{\nu}$. A "small" Lorentz transformation is written as

$$\Lambda^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}. \tag{4.22}$$

Constraints on $\omega^{\mu}{}_{\nu}$ follows from Eq. (4.3). Working to first order in ω, we find that $\omega_{\mu\nu}$ is anti-symmetric,

$$\omega_{\mu\nu} = -\omega_{\nu\mu}. \tag{4.23}$$

In the four-dimensional space, an anti-symmetric rank-two tensor has six independent components. These components can be thought of as independent Lorentz transformation that leave scalar products of two four-vectors $x^0 y^0 - \vec{x} \cdot \vec{y}$ invariant; the independent transformations are – three Lorentz-boosts (one in each directions) and three rotations (one around each of the three coordinate axes).

To find explicit expressions for the tensor $\omega_{\mu\nu}$, we start with rotations. Rotations do not change time-like components, so we assume that, for rotations, ω_{0i} vanishes. For spatial components, we write

$$\omega_{ij} = -\epsilon_{ijk} n_k \theta, \tag{4.24}$$

where $\vec{n}$ is the unit vector that describes the direction of the rotation axis and θ is the infinitesimal rotation angle of the reference frame.

Lorentz-boosts are described by the components of the matrix $\omega^{\mu\nu}$ where either μ or ν equals to zero. If the Lorentz transformation describes a boost to a reference system that moves with a small velocity $\beta\vec{n}$ relative to an original system, we find[1]

$$\omega_{i0} = n_i \beta. \tag{4.25}$$

We have already seen that Lorentz symmetry has important consequences for the scalar field and its representation in terms of creation and annihilation operators. We have seen that in this case Lorentz transformations are represented by unitary operators acting in the Hilbert space of the theory. These operators should furnish representation of the Lorentz group, i.e.

$$U(\Lambda_1\Lambda_2) = U(\Lambda_1)U(\Lambda_2), \quad U(\hat{1}) = \hat{I}, \tag{4.26}$$

[1] Note that if a particle was at rest in the original system, it will be moving with the velocity $-\beta\vec{n}$ in the new one.

where I is the identity operator and

$$U(\Lambda^{-1}) = U^{-1}(\Lambda). \tag{4.27}$$

Consider an infinitesimal transformation parameterized by a matrix ω. Then

$$U(1+\omega) \approx \hat{I} + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}, \tag{4.28}$$

where $M^{\mu\nu}$ are generators of the Lorentz algebra. What exactly these objects are depends on the space where operators U act or, in other words, they depend on the representation of the group and/or algebra that we are interested in.

We will start by dealing with these operators as abstract objects and find their transformation properties which are representation-independent. To do so, consider

$$U^{-1}(\Lambda)U(\Lambda')U(\Lambda) = U(\Lambda^{-1}\Lambda'\Lambda). \tag{4.29}$$

We then consider the transformation Λ' to be infinitesimal and write

$$\Lambda' = 1 + \omega'. \tag{4.30}$$

Then

$$U^{-1}(\Lambda)\left(I + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right)U(\Lambda) = U(\Lambda^{-1}(1+\omega)\Lambda). \tag{4.31}$$

It is straightforward to obtain

$$\left(\Lambda^{-1}(1+\omega)\Lambda\right)^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} + \Lambda^{\alpha\mu}\Lambda^{\rho}{}_{\nu}\omega_{\alpha\rho}. \tag{4.32}$$

Writing $U(g^{\mu}{}_{\nu} + \Lambda^{\alpha\mu}\Lambda^{\rho}{}_{\nu}\omega_{\alpha\rho})$ in terms of algebra generators, we find that generators of Lorentz algebra should transform in the following way

$$U^{-1}(\Lambda)\; M^{\mu\nu}\; U(\Lambda) = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\alpha}M^{\rho\alpha}. \tag{4.33}$$

By extrapolation, we can say that the above transformation applies to tensor operators of any rank. Hence, for example, the operator of four-momentum $P^{\mu} = (H, \vec{P})$ gets transformed as follows

$$U(\Lambda^{-1})P^{\mu}U(\Lambda) = \Lambda^{\mu}{}_{\nu}P^{\nu}, \tag{4.34}$$

which is what one expects from a four-vector operator.

As a further consequence of the fact that Lorentz transformations form a group, one can show that generators of Lorentz algebra satisfy commutation relations. Consider Eq. (4.33) and take the Lorentz transformation in that equation to be infinitesimal. Then we write

$$
\begin{aligned}
&U(\Lambda) = I + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}, \quad U(\Lambda^{-1}) = I - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}, \\
&\Lambda^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu},
\end{aligned}
\tag{4.35}
$$

and obtain

$$
[M^{\mu\nu}, M^{\rho\sigma}] = i\left[g^{\mu\sigma}M^{\rho\nu} + g^{\nu\sigma}M^{\mu\rho} - g^{\mu\rho}M^{\sigma\nu} - g^{\nu\rho}M^{\mu\sigma}\right]. \tag{4.36}
$$

Although these commutation relations look complicated, we know that we should be able to identify three generators of rotations and three generators of Lorentz boosts. One can check that by using

$$
J^i = \frac{1}{2}\epsilon^{ijk}M_{jk}, \tag{4.37}
$$

and

$$
K^i = M^{0i}, \tag{4.38}
$$

instead of $M^{\mu\nu}$, one obtains the following commutation relations

$$
[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [K_i, K_j] = -i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijm}K_m. \tag{4.39}
$$

We note that the commutation relations for J_i are identical to that of the angular momentum operator in Quantum Mechanics.

A similar exercise applied to Eq. (4.34) allows us to find commutation relations of the generators of Lorentz group with the Hamiltonian and three momenta operators. The generic form reads

$$
[P^{\mu}, M^{\rho\sigma}] = i\left(g^{\mu\sigma}P^{\rho} - g^{\mu\rho}P^{\sigma}\right). \tag{4.40}
$$

Using J^i and K^i to parameterize generators $M^{\mu\nu}$, we find

$$
[J^i, H] = 0, \quad [J_i, P_j] = i\epsilon_{ijk}P_k, \quad [K_i, H] = iP_i, \quad [K_i, P_j] = i\delta_{ij}H. \tag{4.41}
$$

Finally, we also know that

$$[P_i, P_j] = 0, \quad [P_i, H] = 0. \tag{4.42}$$

Together, the above commutation relations describe the Lie algebra of *Poincare' group*.

We will now describe transformation properties of fields that we use to construct Lagrangians in Quantum Field Theories. We have already talked about the scalar field ϕ. According to the previous discussion the transformation rule reads

$$U^{-1}(\Lambda)\, \phi(x)\, U(\Lambda) = \phi(\Lambda^{-1}x). \tag{4.43}$$

A transformation rule for a *vector field* $V^\mu(x)$ is similar

$$U^{-1}(\Lambda)\, V^\mu(x)\, U(\Lambda) = \phi(\Lambda^{-1}x) = \Lambda^\mu_{\ \nu}\, V^\nu(\Lambda^{-1}x). \tag{4.44}$$

The generalization to the case of tensor fields of an arbitrary rank are clear.

We have seen that Lorentz group contains rotations as eligible transformations and we know from Quantum Mechanics that representations of the rotation group involve objects with either integer or half-integer total angular momentum. We have so far discussed objects that transform as scalars, vectors, etc. which, from rotations point of view, corresponds to integer spins. To see how objects with half-integer spin appear in a relativistic Quantum Field Theory, we go back to the commutation relations in Eq. (4.39) and try to analyze them. Our goal should be to determine the minimal set of generators that can be diagonalized simultaneously; we will then use this information to determine eligible states and their quantum numbers.

To simplify the commutation relations in Eq. (4.39) it is convenient to introduce new operators

$$I_m^\pm = \frac{1}{2}\left(J_m \pm i K_m\right). \tag{4.45}$$

It is easy to check that commutation relations of these operators decouple

$$[I_i^+, I_j^+] = i\epsilon_{ijk} I_k^+, \quad [I_i^-, I_j^-] = i\epsilon_{ijk} I_k^-, \quad [I_i^+, I_j^-] = 0. \tag{4.46}$$

Since the above commutation relations are just two identical copies of the commutation relations for the angular momentum operators in quantum mechanics, we can easily understand how to classify them.

Indeed, the Hilbert state is then constructed out of states that are direct products of eigenstates of I^+ and I^- operators

$$|\psi\rangle = |\psi^+\rangle \otimes |\psi^-\rangle. \tag{4.47}$$

The states $\psi^\pm$ are eigenstates of the following operators

$$(I^\pm)^2 = \sum_{i=1}^{3} (I_i^\pm)^2, \quad I_3^\pm, \tag{4.48}$$

so that

$$(I^\pm)^2|\psi^\pm\rangle = j_\pm(j_\pm+1)|\psi^\pm\rangle, \quad I_3^\pm|\psi^\pm\rangle = m_\pm|\psi_\pm\rangle, \tag{4.49}$$

with $-j_\pm \leq m_\pm \leq j_{pm}$ and $j_\pm = 0, 1/2, 1, 3/2, \ldots$.

Terms with $j_\pm = 0$ are the scalar fields that we already discussed. Next possibility is to take $j_+ = 1/2$ and $j_- = 0$ or vice versa. Consider, for definiteness, $j_+ = 1/2$ and $j_- = 0$. The Hilbert space is two dimensional; the generators can be chosen to be Pauli matrices

$$I_i^+ = \frac{1}{2}(J_i + iK_i) = \frac{1}{2}\,\sigma_i, \quad I_i^- = \frac{1}{2}(J_i - iK_i) = 0. \tag{4.50}$$

We solve this system to find

$$J_i = \frac{1}{2}\sigma_i, \quad K_i = -\frac{i}{2}\sigma_i. \tag{4.51}$$

For $j_- = 0$ and $j_+ = 1/2$ the situation is similar but we will need generators to be

$$J_i = \frac{1}{2}\sigma_i \quad K_i = \frac{i}{2}\sigma_i. \tag{4.52}$$

Each of the two representations $(1/2, 0)$ and $(0, 1/2)$ can be described by a two-component spinor, as we do in Quantum Mechanics. We can also combine the two two-component spinors into a four-component object

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \tag{4.53}$$

which will simultaneously furnish $(1/2, 0)$, $(0, 1/2)$ and $(1/2, 1/2)$ representations of the Lorentz group. Group generators are then represented by four-by-four matrices

$$J_i = \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}, \quad K_i = \frac{i}{2}\begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}. \tag{4.54}$$

As we will see later on, in Nature all representations $(1/2, 0)$, $(0, 1/2)$ and $(1/2, 1/2)$ are realized in the spectrum of elementary particles. However, the first representation we will need is $(1/2, 1/2)$ and we will use it to describe an electron. This is somewhat peculiar since, as we know from Quantum Mechanics, an electron has spin 1/2 and, therefore, is described by a two component spinor. Hilbert space of $(1/2, 1/2)$ representation are four-component spinors, as follows from the above discussion, so it appear that we have two degrees of freedom too much. The reason we still need $(1/2, 1/2)$ representation to describe an electron is parity. If we perform a parity transformation, $\vec{J} \to \vec{J}$ and $\vec{K} \to -\vec{K}$. This implies that $I^+ \to I^-$ and vice versa. A parity-invariant theory therefore cannot be based on either $(1/2, 0)$ or $(0, 1/2)$ representation and requires both.

Now, to describe an electron we need an equation. This equation was guessed by Dirac and we will try to guess it as well. Suppose we take a four-component spinor

$$\Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}, \tag{4.55}$$

to describe an electron in *its rest frame*. Since we need two and not four component, an equation that should define an admissible solution for an electron can be thought of as projection on two relevant degrees of freedom. According to our discussion about parity, such an equation should state that for a physical electron $\psi = \chi$. We then write an equation for the electron in its rest frame which makes this statement manifest. The equation reads

$$\frac{1}{2}\left(1 - \gamma^0\right)\Psi = 0, \tag{4.56}$$

where γ^0 is a four-by-four matrix defined as

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.57}$$

Since we have an equation for the electron at rest, we can obtain an equation for the electron in an arbitrary frame by performing a boost. A boost is described by an operator

$$U = e^{-i\vec{n}\cdot\vec{K}\eta}, \tag{4.58}$$

where η is the so-called rapidity and $\vec{n}$ is the direction of the boost. Rapidity is defined as

$$\cosh \eta = \frac{E}{m}, \tag{4.59}$$

where E is the energy of the electron in the new frame. The transformed equation reads

$$U(\gamma^0 - 1)U^{-1}\Psi(\vec{p}), \tag{4.60}$$

where

$$\Psi(p) = U\Psi. \tag{4.61}$$

One can show that

$$U\gamma^0 U^{-1} = \frac{\gamma^\mu p_\mu}{m}, \tag{4.62}$$

where γ^0 is given above and

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}. \tag{4.63}$$

The matrices $\gamma^0, \gamma^1, ..$ are known as Dirac matrices (in the Weyl representation). The spinor $\Psi(p)$ is the Dirac spinor and the resulting equation

$$(\gamma^\mu p_\mu - m)\,\Psi(p) = 0, \tag{4.64}$$

is the Dirac equation. In the position space, the Dirac equation reads

$$(i\partial_\mu \gamma^\mu - m)\,\Psi(x) = 0. \tag{4.65}$$

The transformation rules for the Dirac field $\Psi(x)$ reads

$$U^{-1}(\Lambda)\ \Psi(x)\ U(\Lambda) = \hat{\Lambda}_{1/2}\ \Psi(\Lambda^{-1}x), \tag{4.66}$$

where

$$\hat{\Lambda}_{1/2} = e^{-i\vec{K}\cdot\vec{\eta} - i\vec{J}\cdot\vec{\varphi}}. \tag{4.67}$$

This transformation rule has important consequences for using Dirac fields to construct quantities which transform in a particular way under Lorentz transformations. The simplest quantity that we may want to construct is the the scalar "field". A natural thing to try would be $\Psi^+(x)\Psi(x)$ where the sum over spinor indices is assumed. Under Lorentz transformation conjugate field Ψ^+ transforms as follows

$$U^{-1}(\Lambda)\Psi^+(x)U(\Lambda) = \Psi^+(\Lambda^{-1}x)\Lambda^+_{1/2}. \tag{4.68}$$

The important point is that

$$\Lambda^+_{1/2} = e^{-i\vec{K}\cdot\vec{\eta}+i\vec{J}\cdot\vec{\varphi}}, \tag{4.69}$$

so that

$$\Lambda^+_{1/2} \neq \Lambda^{-1}_{1/2}. \tag{4.70}$$

This means that $\Psi^+(x)\Psi(x)$ is not transformed in the right way.

To construct a proper conjugate field, we can use the matrix γ^0. Indeed, consider

$$U^{-1}(\Lambda)\Psi^+(x)\gamma_0 U(\Lambda) = \Psi^+(\Lambda^{-1}x)\Lambda^+_{1/2}\gamma_0. \tag{4.71}$$

Using explicit representation of the matrix γ_0 and boost operators K_i, it is easy to see that[2]

$$\Lambda^+_{1/2}\gamma_0 = e^{-i\vec{K}\cdot\vec{\eta}+i\vec{J}\cdot\vec{\varphi}}\gamma_0 = \gamma_0 e^{-i\vec{K}\cdot\vec{\eta}-i\vec{J}\cdot\vec{\varphi}} = \gamma_0\Lambda^{-1}_{1/2}. \tag{4.72}$$

Writing $\Psi^+\gamma_0 = \bar{\Psi}$, we obtain

$$U^{-1}(\Lambda)\bar{\Psi}(x)U(\Lambda) = \bar{\Psi}(x)\Lambda^{-1}_{1/2}. \tag{4.73}$$

Therefore, we find

$$U^{-1}\bar{\Psi}(x)\Psi(x)U = \bar{\Psi}(\Lambda^{-1}x)\Psi(\Lambda^{-1}x), \tag{4.74}$$

which is indeed a transformation rule of a scalar field.

Similarly, we can take $\bar{\Psi}(x)\gamma^\mu\Psi(x)$. In this case, the transformation rule is

$$U^{-1}\bar{\Psi}(x)\gamma^\mu\Psi(x)U = \bar{\Psi}(\Lambda^{-1}x)\Lambda^{-1}_{1/2}\,\gamma^\mu\Lambda_{1/2}\Psi(\Lambda^{-1}x). \tag{4.75}$$

One can show that

$$\Lambda^{-1}_{1/2}\,\gamma^\mu\Lambda_{1/2} = \Lambda^\mu{}_\nu\gamma^\nu \tag{4.76}$$

[2] The reason for this is that $[\vec{J},\gamma_0] = 0$ and $\{\vec{K},\gamma_0\} = 0$, as can be verified using the explicit form of these operators.

which implies that the transformation rules of $\bar{\Psi}\gamma^{\mu}\Psi$ are the same as of a *vector field*.

This understanding allows us to write Lorentz-invariant action for fermions. A Lorentz-invariant action with the smallest number of fermion fields and derivatives reads

$$S = \int \mathrm{d}^4 x \bar{\Psi} \left(\partial_\mu \gamma^\mu - m\right) \Psi. \tag{4.77}$$

Variation of this action w.r.t. to the field $\bar{\Psi}$ gives the Dirac equation.