TTP1
Lecture 5

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5 Solutions of the Dirac equation

In this lecture we will discuss solutions of the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0. \tag{5.1}$$

The four matrices $\gamma^0, \vec{\gamma}$ are the Dirac matrices. In the Weyl representation, they read

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}. \tag{5.2}$$

Using these matrices, we write the Dirac equation as

$$\begin{pmatrix} -m & i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla}) \\ i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla}) & -m \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix} = 0. \tag{5.3}$$

The spinors ψ and χ are two-component spinors; they are used to construct a four-component spinor Ψ and are called the left- and the right-handed spinors. They satisfy a system of differential equations

$$\begin{aligned} i\left(\partial_0 + \vec{\sigma}\cdot\vec{\nabla}\right)\chi &= m\psi. \\ i\left(\partial_0 - \vec{\sigma}\cdot\vec{\nabla}\right)\psi &= m\chi. \end{aligned} \tag{5.4}$$

These equations can be re-written using the following notation

$$\sigma^\mu = (1, \vec{\sigma}), \quad \bar{\sigma}^\mu = (1, -\vec{\sigma}). \tag{5.5}$$

Then, Eq. (5.4) becomes

$$i\sigma^\mu\partial_\mu\chi = m\psi, \quad i\bar{\sigma}^\mu\partial_\mu\psi = m\chi. \tag{5.6}$$

These equations have an interesting property – they decouple from each other if $m = 0$. This implies that ψ and χ become independent in the massless limit.

To construct solutions of the Dirac equation, recall that any solution of the Dirac equation must be also a solution of the Klein-Gordon equation. We therefore write

$$\Psi = u(p)e^{-ip_\mu x^\mu}. \tag{5.7}$$

The four-momentum p^μ is such that $p^2 = m^2$. Also, $p^\mu = (E_{\vec{p}}, \vec{p})$ and $E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$. The equation for $u(p)$ follows from the Dirac equation. We find

$$(\gamma^\mu p_\mu - m)\,u(p) = 0. \tag{5.8}$$

As we already discussed, in the rest frame of p, the Dirac equation ensures that two upper and two lower components of the spinor $u(p)$ are the same. We then write

$$u_{\rm rest} = \sqrt{m} \left(\begin{array}{c} \xi_s \\ \xi_s \end{array} \right), \tag{5.9}$$

where ξ denotes a two-dimensional spinor which describes a particular polarization of a fermion in the rest frame.

By choosing the spinor ξ_s to be normalized as $\xi_s^+ \xi_s = 1$, we can ensure that

$$\bar u_{\rm rest}(p) u_{\rm rest}(p) = m(\xi_s^+ \xi_s + \xi_s^+ \xi_s) = 2m. \tag{5.10}$$

To obtain the generic solution, we need to boost $\Psi(x)$ to a reference frame where the fermion's momentum is p_μ. For the spinor $u(p)$ this implies

$$u(p,s) = e^{-i\vec K \cdot \vec \eta} u_{\rm rest}(s). \tag{5.11}$$

Using the definition of $\vec K$

$$\vec K = \left(\begin{array}{cc} -\frac{i}{2}\vec \sigma & 0 \\ 0 & \frac{i}{2}\vec\sigma \end{array} \right), \tag{5.12}$$

we find

$$e^{-i\vec K \vec \eta} = \left(\begin{array}{cc} \cosh\frac{\eta}{2} - \vec\sigma \cdot \vec n \sinh \frac{\eta}{2} & 0 \\ 0 & \cosh\frac{\eta}{2} + \vec\sigma \cdot \vec n \sinh \frac{\eta}{2} \end{array} \right), \tag{5.13}$$

where $\vec n = \vec \eta / |\vec \eta| = \vec p / |\vec p|$, $\cosh \eta = E_p/m$ and $\sinh \eta = |\vec p|/m$.

To elucidate the structure of this matrix, we write it as

$$e^{-i\vec K \vec \eta} = \cosh \frac{\eta}{2} \, \hat 1 + \sinh \frac{\eta}{2} \left(\begin{array}{cc} -\vec\sigma \cdot \vec n & 0 \\ 0 & \vec \sigma \cdot \vec n \end{array} \right). \tag{5.14}$$

We then use the fact that the following two-by-two matrices

$$\mathcal{P}_\pm = \frac{1 \pm \vec n \cdot \vec \sigma}{2} \tag{5.15}$$

are operators that project an arbitrary two-component spinors on states with either $+1/2$ or $-1/2$ spin projection along the $\vec n$ axis. A simple algebra leads to

$$u(p,s) = \left(\begin{array}{cc} \sqrt{E+p}\, \mathcal{P}_- + \sqrt{E-p}\, \mathcal{P}_+ & 0 \\ 0 & \sqrt{E+p}\, \mathcal{P}_+ + \sqrt{E-p}\, \mathcal{P}_- \end{array} \right) \left(\begin{array}{c} \xi_s \\ \xi_s \end{array} \right). \tag{5.16}$$

One can write this expression in a more compact form using the fact that

$$\begin{aligned}\sqrt{E+p}\mathcal{P}_- + \sqrt{E-p}\mathcal{P}_+ &= \sqrt{p_\mu \sigma^\mu}\\ \sqrt{E+p}\mathcal{P}_+ + \sqrt{E-p}\mathcal{P}_- &= \sqrt{p_\mu \bar{\sigma}^\mu}.\end{aligned} \tag{5.17}$$

To prove this, we need to recognize that the expression on the right hand side must be linear[1] in $\vec{\sigma}$ and then compare the eigenvalues of the left hand side and the right hand side.

We now take

$$u(p,s) = \left(\begin{array}{c} \sqrt{p_\mu \sigma^\mu}\, \xi_s \\ \sqrt{p_\mu \bar{\sigma}^\mu}\, \xi_s \end{array} \right) \tag{5.18}$$

as expressions for spinors that provide solutions to the Dirac equation and investigate their properties.

For example, we can check that these solutions remain properly normalized after the boost. We start by writing

$$\bar{u}(p,s) = \left(\xi_s^+ \sqrt{p_\mu \sigma^\mu}, \xi_s^+ \sqrt{p_\mu \bar{\sigma}^\mu} \right) \gamma_0 = \left(\xi_s^+ \sqrt{p_\mu \bar{\sigma}^\mu},\ \xi_s^+ \sqrt{p_\mu \sigma^\mu} \right). \tag{5.19}$$

Then,

$$\bar{u}(p,s_1)\ u(p,s_2) = \xi_{s_1}^+ \left(\sqrt{p_\mu \bar{\sigma}^\mu}\sqrt{p_\nu \sigma^\nu} + \sqrt{p_\mu \sigma^\mu}\sqrt{p_\nu \bar{\sigma}^\nu} \right) \xi_{s_2} \tag{5.20}$$

To simplify products of square roots, we use Eq. (5.17). We then find

$$\sqrt{p_\mu \sigma^\mu}\sqrt{p_\nu \bar{\sigma}^\nu} + \sqrt{p_\mu \bar{\sigma}^\mu}\sqrt{p_\nu \sigma^\nu} = 2\sqrt{(E^2-p^2)} = 2m. \tag{5.21}$$

Hence,

$$\bar{u}(p,s_1)\ u(p,s_2) = 2m\, \delta_{s_1 s_2}. \tag{5.22}$$

Another important quantity that we require is the so-called density matrix. It is defined as follows

$$\rho_{\alpha\beta} = \sum_{s=1}^{2} u_\alpha(p,s)\ \bar{u}_\beta(p,s), \tag{5.23}$$

where α and β are indices that denote components of the spinors. We use explicit expressions for u and $\bar{u}$ and write

$$\begin{aligned}\rho_{\alpha\beta} &= \sum_{s=1}^{2} \left(\begin{array}{c} \sqrt{p_\mu \sigma^\mu}\, \xi_s \\ \sqrt{p_\mu \bar{\sigma}^\mu}\, \xi_s \end{array} \right)_\alpha \left(\xi_s^+ \sqrt{p_\nu \bar{\sigma}^\nu}, \xi_s^+ \sqrt{p_\nu \sigma^\nu} \right)_\beta \\ &= \sum_{s=1}^{2} \left(\begin{array}{cc} \sqrt{p_\mu \sigma^\mu}\, \xi_s \otimes \xi_s^+ \sqrt{p_\nu \bar{\sigma}^\nu} & \sqrt{p_\mu \sigma^\mu}\, \xi_s \otimes \xi_s^+ \sqrt{p_\nu \sigma^\nu} \\ \sqrt{p_\mu \bar{\sigma}^\mu}\, \xi_s \otimes \xi_s^+ \sqrt{p_\nu \bar{\sigma}^\nu} & \sqrt{p_\mu \bar{\sigma}^\mu}\, \xi_s \otimes \xi_s^+ \sqrt{p_\nu \sigma^\nu} \end{array} \right)_{\alpha\beta}.\end{aligned} \tag{5.24}$$

[1]Any function of $\vec{n}\cdot\vec{\sigma}$ is linear in this quantity as its square is one.

This expression can be further simplified if we recognize that

$$\sum_{s=1}^{2} \xi_s^+ \xi_s = \hat{1}, \tag{5.25}$$

where the matrix on the right-hand side is the two-by-two identity matrix. Then, using $\sqrt{p_\mu \sigma^\mu}\sqrt{p_\nu \bar{\sigma}^\nu} = m$, we find

$$\rho_{\alpha\beta} = \begin{pmatrix} m & p_\mu \sigma^\mu \\ p_\mu \bar{\sigma}^\mu & m \end{pmatrix}_{\alpha\beta} = (p_\mu \gamma^\mu + m)_{\alpha\beta}. \tag{5.26}$$

There exists another important class of solutions of the Dirac equation; these are solutions with "negative" energies. To construct them, we write $\Psi(x) = v(p) e^{i p_\mu x^\mu}$, with $p_0 > 0$. The equation for $v(p)$ becomes

$$(p_\mu \gamma^\mu + m)\, v(p) = 0. \tag{5.27}$$

We construct $v(p)$ following the same steps we used to construct $u(p)$. In fermion's rest frame, we find

$$v(p,s) = \sqrt{m} \begin{pmatrix} \eta_s \\ -\eta_s \end{pmatrix}, \tag{5.28}$$

where η_s is a two-component spinor. Applying the boost operator, we find

$$v(p,s) = \begin{pmatrix} \sqrt{p_\mu \sigma^\mu}\eta_s \\ -\sqrt{p_\mu \bar{\sigma}^\mu}\eta_s \end{pmatrix}. \tag{5.29}$$

It is straightforward to show that

$$\bar{v}(p,s_1) v(p,s_2) = -2m\delta_{s_1 s_2}, \tag{5.30}$$

and that the density matrix for v spinors reads

$$\rho_{\alpha\beta} = \sum_{s=1}^{2} v_\alpha(p,s)\bar{v}_\beta(p,s) = (p_\mu \gamma^\mu - m)_{\alpha\beta} \tag{5.31}$$

As the final comment, we note that spinors $v(p)$ und $u(p)$ are "orthogonal" to each other

$$\bar{v}(p,s_1) u(p,s_2) = 0. \tag{5.32}$$

We know that in the Weyl representation, two upper components of the four-component spinor furnish $(1/2, 0)$ (left, L) representation of the Lorentz group and two lower components the $(0, 1/2)$ (right, R) representation. We can construct projection operators on these two representations by using the matrix that is called γ_5. It reads

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.33}$$

The projection operators are written as

$$\mathcal{P}_{L,R} = \frac{1 \pm \gamma_5}{2}. \tag{5.34}$$

The matrix γ_5 is not fully independent of the other Dirac matrices. In fact, we can write

$$\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{5.35}$$

The matrix γ_5 *anti-commutes* with all Dirac matrices

$$\gamma^\mu\gamma_5 + \gamma_5\gamma^\mu = 0, \tag{5.36}$$

and its square is the identity matrix

$$\gamma_5^2 = 1. \tag{5.37}$$

As we said earlier, in the massless case, left- and right-handed two-component fermions are independent and satisfy the following (Weyl) equations

$$p_\mu\bar{\sigma}^\mu\psi_L = 0, \quad p_\mu\sigma^\mu\psi_R = 0. \tag{5.38}$$

Since for the massless fermion $p_0 = |\vec{p}|$, we write

$$p_\mu\sigma^\mu = p_0\left(1 - \frac{\sigma\cdot\vec{p}}{|\vec{p}|}\right), \quad p_\mu\bar{\sigma}^\mu = p_0\left(1 + \frac{\sigma\cdot\vec{p}}{|\vec{p}|}\right), \tag{5.39}$$

and find

$$\frac{\sigma\cdot\vec{p}}{|\vec{p}|}\psi_L = -\psi_L, \quad \frac{\sigma\cdot\vec{p}}{|\vec{p}|}\psi_R = +\psi_R, \tag{5.40}$$

Hence, Weyl equations simply state that right-handed fermions are always polarized along the direction of their momentum and left-handed fermions

– in the opposite direction. This quantity – the projection of particle's spin on the direction of its momentum – is called *helicity*. The four-component *massless* spinors are then written as

$$u(p, L) = \sqrt{2E_p} \begin{pmatrix} \xi_- \\ 0 \end{pmatrix}, \quad u(p, R) = \sqrt{2E_p} \begin{pmatrix} 0 \\ \xi_+ \end{pmatrix}, \tag{5.41}$$

where $\xi_\pm$ satisfy

$$\frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|} \xi_\pm = \pm \xi_\pm. \tag{5.42}$$