TTP1
Lecture 6

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6 Quantization of a Dirac field

As we explained earlier, the Lagrange density in case of the fermion field ψ reads

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi. \tag{6.1}$$

We would like to quantize this theory by repeating what we did for the scalar field, namely write down canonical commutation relations, solve them by expressing ψ in terms of creation and annihilation operators and then proceed with computing the Hamiltonian and determining the quantum states.

An important difference between fermions and bosons is that multi-fermionic states should be antisymmetric under permutations of different particles. As such states will be constructed by acting with creation operators on the vacuum, fermion creation and annihilation operators have to *anticommute*. By analogy with the quantization of the scalar field, we expect that the Dirac field ψ is a linear combination of various creation and annihilation operators and, since such operators anticommute, we expect that Dirac fields anti-commute as well.

To quantize the theory, we treat $\bar{\psi}$ and ψ as independent degrees of freedom and compute a canonical momentum of the field ψ. We find

$$\pi_\alpha = \frac{\delta L}{\delta \partial_0 \psi_\alpha} = (\bar{\psi} i \gamma^0)_\alpha = i\psi_\alpha^+. \tag{6.2}$$

We then require that the following *anti-commutation* relations hold

$$\begin{aligned} \{\psi_\beta(t,\vec{y}), \pi_\alpha(t,\vec{x})\} &= i\delta^{(3)}(\vec{x}-\vec{y}) \Rightarrow \\ \{\psi_\alpha(t,\vec{x}), \psi_\beta^+(t,\vec{y})\} &= \delta_{\alpha\beta}\, \delta^{(3)}(\vec{x}-\vec{y}), \end{aligned} \tag{6.3}$$

and

$$\{\psi_\alpha(t,\vec{x}), \psi_\beta(t,\vec{y})\} = \{\psi_\alpha^+(t,\vec{x}), \psi_\beta^+(t,\vec{y})\} = 0. \tag{6.4}$$

To proceed further, we compute the Hamiltonian density. We write

$$\mathcal{H} = \sum_{\alpha=1}^{4} \pi_\alpha \partial_0 \psi_\alpha - L = i\psi^+ \partial_0 \psi - \bar{\psi}\left(i\gamma^0 \partial_0 + i\vec{\gamma}\cdot\vec{\nabla} - m\right)\psi, \tag{6.5}$$

and obtain

$$\mathcal{H} = \bar{\psi}\left(-i\vec{\gamma}\cdot\vec{\nabla} + m\right)\psi. \tag{6.6}$$

The Hamiltonian is obtained by integrating $\mathcal{H}$ over $\mathrm{d}^3\vec{x}$

$$H = \int \mathrm{d}^3\vec{x}\,\mathcal{H} = \int \mathrm{d}^3\vec{x}\,\bar{\psi}\,\left(-i\vec{\gamma}\cdot\vec{\nabla} + m\right)\psi. \tag{6.7}$$

To (eventually) diagonalize the Hamiltonian operator, we write Ψ and $\bar{\Psi}$ as a linear combination of creation and annihilation operators. Similar to the case of the *complex* scalar field, the field Ψ is a linear combination of particular annihilation operators multiplied with positive energy solutions and different creation operators multiplied with negative energy solutions

$$\psi(t,\vec{x}) = \sum_{s=1}^{2}\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec{p}}}}\left[a_{\vec{p},s}u(p,s)e^{-ip_\mu x^\mu} + b^+_{\vec{p},s}v(p,s)e^{ip_\mu x^\mu}\right]. \tag{6.8}$$

The conjugate field reads

$$\psi^+(t,\vec{x}) = \sum_{s=1}^{2}\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec{p}}}}\left[a^+_{\vec{p},s}u^+(p,s)e^{ip_\mu x^\mu} + b_{\vec{p},s}v^+(p,s)e^{-ip_\mu x^\mu}\right]. \tag{6.9}$$

To satisfy simple anti-commutation relations, we assume that

$$\{a_{\vec{p}_1,s_1}, a_{\vec{p}_2,s_2}\} = \{b_{\vec{p}_1,s_1}, b_{\vec{p}_2,s_2}\} = \{a_{\vec{p}_1,s_1}, b_{\vec{p}_2,s_2}\} = \{a_{\vec{p}_1,s_1}, b^+_{\vec{p}_2,s_2}\} = 0. \tag{6.10}$$

Further relations of a similar type are obtained by writing down conjugate versions of these anticommutation relations. It is easy to see that the above anticommutation relations ensure that, at equal times, any ψ anticommutes with any ψ and any ψ^+ anticommutes with any ψ^+. We then need to compute an anticommutator of ψ and ψ^+. We write

$$\begin{aligned}\left\{\psi_\alpha(t,\vec{x}_1), \psi^+_\beta(t,\vec{x}_2)\right\} = \sum_{s_1,s_2}\int \frac{\mathrm{d}^3\vec{p}_1\mathrm{d}^3\vec{p}_2}{(2\pi)^6\sqrt{2E_{p_1}2E_{p_2}}}\Bigg[\\ u_\alpha(p_1,s_1)u^+_\beta(p_2,s_2)e^{-ip_1x_1+ip_2x_2}\left\{a_{\vec{p}_1,s_1}, a^+_{\vec{p}_2,s_2}\right\} \\ +v_\alpha(p_1,s_1)v^+_\beta(p_2,s_2)e^{ip_1x_1-ip_2x_2}\left\{b^+_{\vec{p}_1,s_1}, b_{\vec{p}_2,s_2}\right\}\Bigg].\end{aligned} \tag{6.11}$$

Given what we know about creation and annihilation operators for the scalar field, a natural choice for the anticommutators is

$$\left\{a_{\vec{p}_1,s_1}, a^+_{\vec{p}_2,s_2}\right\} = \left\{b_{\vec{p}_1,s_1}, b^+_{\vec{p}_2,s_2}\right\} = (2\pi)^3\delta^{(3)}(\vec{p}_1 - \vec{p}_2)\delta_{s_1s_2}. \tag{6.12}$$

We will now show that these anticommutation relations give the desired anticommutation relation of ψ and ψ^+.

We use Eq. (6.12) in Eq. (6.11) and find

$$\begin{aligned}\left\{\psi_\alpha(t,\vec{x}_1),\psi_\beta^+(t,\vec{x}_2)\right\} = \sum_s \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E_p}\Bigg[& u_\alpha(p,s)u_\beta^+(p,s)e^{i\vec{p}(\vec{x}_1-\vec{x}_2)} \\ & + v_\alpha(p,s)v_\beta^+(p,s)e^{-i\vec{p}(\vec{x}_1-\vec{x}_2)}\Bigg].\end{aligned} \tag{6.13}$$

To simplify Eq. (6.12) we need to sum over polarizations. To do this, we can use the density matrices for positive and negative energy solutions that we computed earlier. We find

$$\begin{aligned}\sum_s u_\alpha(p,s)u_\beta^+(p,s) &= \left((p_\mu\gamma^\mu + m)\gamma^0\right)_{\alpha\beta} \\ \sum_s v_\alpha(p,s)v_\beta^+(p,s) &= \left((p_\mu\gamma^\mu - m)\gamma^0\right)_{\alpha\beta}.\end{aligned} \tag{6.14}$$

Substituting these results into Eq. (6.13) and changing $\vec{p} \to -\vec{p}$ in the second term, we obtain

$$\left\{\psi_\alpha(t,\vec{x}_1),\psi_\beta^+(t,\vec{x}_2)\right\} = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E_p}\, 2E_p\delta_{\alpha\beta}e^{i\vec{p}(\vec{x}_1-\vec{x}_2)} = \delta_{\alpha\beta}\delta^{(3)}(\vec{x}_1-\vec{x}_2), \tag{6.15}$$

as desired.

As the next step, we express the Hamiltonian operator in terms of creation and annihilation operators. This is done in a relatively straightforward way; the following equations are helpful

$$(\vec{\gamma}\cdot\vec{p}+m)u(p) = p_0\gamma_0 u(p), \quad (-\vec{\gamma}\cdot\vec{p}+m)v(p) = -p_0\gamma_0 v(p), \tag{6.16}$$

as well as

$$\begin{aligned}&\bar{u}(\vec{p},s_1)\gamma_0 v(-\vec{p},s_2) = 0, \quad \bar{v}(\vec{p},s_1)\gamma_0 u(-\vec{p},s_2) = 0, \\ &\bar{u}(\vec{p},s_1)\gamma_0 u(\vec{p},s_2) = 2E_p\delta_{s_1,s_2}, \quad \bar{v}(\vec{p},s_1)\gamma_0 v(\vec{p},s_2) = 2E_p\delta_{s_1,s_2}.\end{aligned} \tag{6.17}$$

Using these equations, one can show that

$$
\begin{aligned}
H &= \sum_s \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\; E_{\vec{p}} \left[a^+_{\vec{p},s} a_{\vec{p},s} - b_{\vec{p},s} b^+_{\vec{p},s} \right] \\
&= \sum_s \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\; E_{\vec{p}} \left[a^+_{\vec{p},s} a_{\vec{p},s} + b^+_{\vec{p},s} b_{\vec{p},s} \right] ,
\end{aligned}
\tag{6.18}
$$

where in the last step we neglected the infinite vacuum energy.

The eigenstates of the Hamiltonian are constructed by acting with creation operators $a^+_{\vec{p},s}$ and $b^+_{\vec{p},s}$ on the vacuum state of the theory. Both "a"- and "b" particles with equal momentum have the same energy and the same mass. We will see later that particles created by a^+ operators describe electrons and particle created with b-operators describe positrons, i.e. "electrons with the positive electric charge". The label s in both cases describes the polarization state of the created fermion; this label assumes two values as electrons and positrons are spin 1/2 particles.

Similar to what we did for the scalar field, we will study the various Green's functions of the Dirac equation. Consider $\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle$. Using the fact that annihilation operators annihilate the vacuum state, we find

$$
\begin{aligned}
\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle = &\sum_{s_1,s_2} \int \frac{\mathrm{d}^3\vec{p}_1\ \mathrm{d}^3\vec{p}_2}{(2\pi)^6\sqrt{2E_{p_1}2E_{p_2}}} \langle 0|a_{\vec{p}_1,s_1} a^+_{\vec{p}_2,s_2}|0\rangle \times \\
& u_\alpha(\vec{p}_1,s_1)\bar{u}_\beta(\vec{p}_2,s_2) e^{-ip_1x+ip_2y}.
\end{aligned}
\tag{6.19}
$$

We use

$$
\langle 0|a_{\vec{p}_1,s_1} a^+_{\vec{p}_2,s_2}|0\rangle = (2\pi)^3 \delta_{s_1s_2}\delta^{(3)}(\vec{p}_1-\vec{p}_2)
\tag{6.20}
$$

and results for the density matrix to find

$$
\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E_p} (p_\mu\gamma^\mu + m)_{\alpha\beta} e^{-ip(x-y)}.
\tag{6.21}
$$

We can write

$$
\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle = (i\partial_{x\,\mu}\gamma^\mu + m)_{\alpha\beta} \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E_p} e^{-ip(x-y)},
\tag{6.22}
$$

where we indicated that the derivative w.r.t. x^μ is taken.

A similar computation gives

$$\langle 0|\bar{\psi}_\beta(y)\psi_\alpha(x)|0\rangle = -(i\partial_\mu\gamma^\mu + m)_{\alpha\beta}\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E_p} e^{+ip(x-y)}. \qquad (6.23)$$

We can use these results to construct various Green's functions for the Dirac equation; in doing so, we benefit from our knowledge of Green's functions calculated for the Klein-Gordon equation. Consider a Feynman propagator for fermion fields. It is defined as follows

$$\langle 0|T\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle = \theta(x_0-y_0)\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle - \theta(y_0-x_0)\langle 0|\bar{\psi}_\beta(y)\psi_\alpha(x)|0\rangle. \qquad (6.24)$$

Note the relative minus sign between the two terms. This sign is part of a definition of the T-product for fermion fields; it is related to the fact that fermion fields anticommute and, to determine whether a particular term enters with a plus or a minus sign, we need to count the number of permutations that are required to bring the ordering of fermion fields back to the original ordering.

We use our previous results to write

$$\begin{aligned}
\langle 0|T\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle &= \theta(x_0-y_0)(i\partial_\mu\gamma^\mu + m)_{\alpha\beta}\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E_p} e^{-ip(x-y)} \\
&+ \theta(y_0-x_0)(i\partial_\mu\gamma^\mu + m)_{\alpha\beta}\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E_p} e^{+ip(x-y)} \\
&= (i\partial_\mu\gamma^\mu + m)_{\alpha\beta}\Bigg[\theta(x_0-y_0)\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E_p} e^{-ip(x-y)} \\
&+ \theta(y_0-x_0)\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E_p} e^{+ip(x-y)}\Bigg] \\
&- i\delta(x_0-y_0)\gamma^0_{\alpha\beta}\left[\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E_p} e^{-ip(x-y)} - \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E_p} e^{+ip(x-y)}\right].
\end{aligned} \qquad (6.25)$$

By replacing $\vec{p} \to -\vec{p}$, we can easily see that the term proportional to $\delta(x_0 - -y_0)$ vanishes and the two terms that multiply $(i\partial_\mu\gamma^\mu + m)_{\alpha\beta}$ combine into the Feynman Greens's function of a scalar field Eq. (**??**). We obtain

$$\langle 0|T\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle = (i\partial_\mu\gamma^\mu + m)_{\alpha\beta}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2 - m^2 + i\epsilon}e^{-ip(x-y)}$$
$$= \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{(p_\mu\gamma^\mu + m)_{\alpha\beta}}{p^2 - m^2 + i\epsilon}e^{-ip(x-y)}. \tag{6.26}$$

The above equation gives us the Feynman Green's function for the Dirac fermion.