TTP1
Lecture 7

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7 Symmetries in Quantum Field Theory

7.1 Continuous symmetries in Quantum Field Theory

Symmetries in physics play an important role as they, among other things, allow us to find "integrals of motion", i.e. quantities that in a closed dynamical system cannot change with time. Plenty of such quantities are known from classical mechanics and electrodynamics, i.e. energy, momentum, angular momentum, electric charge and, perhaps, a few others. In this lecture we will discuss how to understand and use symmetries in a quantum field theory.

We can start this discussion by simply extrapolating what one usually does in mechanics to a field theory. Indeed, in mechanics, we derive integrals of motion by requiring that the action remains invariant under redefinition of canonical coordinates and the time variable. In case of field theory, fields play the role of a canonical coordinate in mechanics, and x^μ the role of the time variable.

We consider a field theory characterized by the action

$$S = \int \mathrm{d}^4 x \mathcal{L}(\phi, \partial_\mu \phi). \tag{7.1}$$

and imagine performing a field transformation

$$\phi(x) = f(\tilde{\phi}(x)), \tag{7.2}$$

where f is a function of a new field $\tilde{\phi}$. In general, after a transformation in Eq. (7.5) we will find

$$\mathcal{L}(\phi(x), \partial_\mu \phi(x)) = \mathcal{L}_1(\tilde{\phi}(x), \partial_\mu \tilde{\phi}(x)), \tag{7.3}$$

with no obvious relation between $\mathcal{L}$ and $\mathcal{L}_1$. If we then use $\mathcal{L}_1$ to derive equations of motion for the field $\tilde{\phi}$, they will be different from the equations of motion for the field ϕ and this will be the end of the story.

However, imagine a situation where

$$\mathcal{L}_1(\tilde{\phi}, \partial_\mu \tilde{\phi}) = \mathcal{L}(\tilde{\phi}, \partial_\mu \tilde{\phi}) + \partial_\mu K^\mu(x). \tag{7.4}$$

Since addition of the total derivative to a Lagrange density does not change equations of motion, equations of motion for the fields ϕ and $\tilde{\phi}$ *are the same*; transformations that satisfy these conditions are called *symmetries*.

Suppose we found a field transformation that satisfies Eq. (7.4). To work out its consequences, we consider an infinitesimal version of the transformation and write

$$\phi = \tilde{\phi} + \lambda \Delta \tilde{\phi} + \mathcal{O}(\lambda^2). \tag{7.5}$$

Substituting this expression into $\mathcal{L}$ and expanding in λ, we obtain

$$\begin{aligned} \mathcal{L}(\phi, \partial_\mu \phi) &= \mathcal{L}(\tilde{\phi} + \Delta\tilde{\phi}, \partial_\mu \tilde{\phi} + \partial_\mu \Delta \tilde{\phi}) \\ &= \mathcal{L}(\tilde{\phi}, \partial_\mu \tilde{\phi}) + \lambda \, \frac{\delta \mathcal{L}}{\delta \tilde{\phi}} \Delta \tilde{\phi} + \lambda \frac{\delta \mathcal{L}}{\delta [\partial^\mu \tilde{\phi}]} \partial^\mu \Delta \tilde{\phi} + \mathcal{O}(\lambda^2). \end{aligned} \tag{7.6}$$

Rewriting derivative in the third term on the r.h.s. we obtain

$$\begin{aligned} &\mathcal{L}(\phi, \partial_\mu \phi) = \mathcal{L}(\tilde{\phi}, \partial_\mu \tilde{\phi}) \\ &+ \lambda \Delta \tilde{\phi} \left[\frac{\delta \mathcal{L}}{\delta \tilde{\phi}} - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta [\partial^\mu \tilde{\phi}]} \right) \right] + \lambda \partial^\mu \left[\frac{\delta \mathcal{L}}{\delta [\partial^\mu \tilde{\phi}]} \Delta \tilde{\phi} \right] + \mathcal{O}(\lambda^2). \end{aligned} \tag{7.7}$$

The second term on the r.h.s. vanishes if the field $\tilde{\phi}$ satisfies equations of motion. Since, according to Eq. (7.4), the difference between $\mathcal{L}$ computed with ϕ and $\tilde{\phi}$ fields is $\partial_\mu K^\mu$, we obtain

$$\partial_\mu K^\mu = \lambda \partial^\mu \left[\frac{\delta \mathcal{L}}{\delta [\partial^\mu \tilde{\phi}]} \Delta \tilde{\phi} \right]. \tag{7.8}$$

From the above equation it follows that

$$\partial_\mu J^\mu = 0, \tag{7.9}$$

where

$$J^\mu = \frac{\delta \mathcal{L}}{\delta [\partial^\mu \phi]} \Delta \tilde{\phi} - K^\mu. \tag{7.10}$$

The relation between the symmetry of the Lagrangian and the existence of the conserved current Eqs. (7.9,7.10) is the essence of the so-called Noether's theorem. The conserved current J^μ is often referred to as the Noether's current.

To see how this works, consider the Lagrangian of a free massless scalar field

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi. \tag{7.11}$$

In this case, a transformation $\phi = \tilde{\phi} + a$, where a is a constant, leaves $\mathcal{L}$ unchanged, so $K^\mu = 0$. The field $\Delta\phi$ is just 1 and we find

$$J^\mu = \partial^\mu \phi. \tag{7.12}$$

The current conservation

$$\partial_\mu J^\mu = \partial_\mu \partial^\mu \phi = 0 \tag{7.13}$$

is the Klein-Gordon equation for the massless field.

As another example, consider the theory of *two* identical, interacting scalar fields

$$\mathcal{L}(\phi_1, \phi_2) = \frac{\partial_\mu \phi_1 \, \partial^\mu \phi_1}{2} + \frac{\partial_\mu \phi_2 \, \partial^\mu \phi_2}{2} - \frac{m^2 \, (\phi_1^2 + \phi_2^2)}{2} - V(\phi_1^2 + \phi_2^2), \tag{7.14}$$

where V is an arbitrary function that only depends on $\phi_1^2 + \phi_2^2$.

The Lagrangian density in Eq. (7.14) does not change if, instead of $\phi_{1,2}$ we use two other fields $\tilde{\phi}_{1,2}$ defined as follows

$$\begin{aligned} \phi_1 &= \quad \cos\theta \, \tilde{\phi}_1 + \sin\theta \, \tilde{\phi}_2, \\ \phi_2 &= -\sin\theta \, \tilde{\phi}_1 + \cos\theta \, \tilde{\phi}_2. \end{aligned} \tag{7.15}$$

The infinitesimal version of these transformations reads

$$\phi_1 = \tilde{\phi}_1 + \theta \tilde{\phi}_2, \quad \phi_2 = \tilde{\phi}_2 - \theta \tilde{\phi}_1, \tag{7.16}$$

so that

$$\Delta \tilde{\phi}_1 = \tilde{\phi}_2, \quad \Delta \tilde{\phi}_2 = -\tilde{\phi}_1. \tag{7.17}$$

Since the Lagrangian is invariant, $K^\mu = 0$ and the conserved current reads

$$J^\mu = \sum_{i=1}^{2} \frac{\delta \mathcal{L}}{\delta \partial^\mu \phi_i} \Delta \phi_i = (\partial^\mu \phi_1) \, \phi_2 - (\partial^\mu \phi_2) \, \phi_1. \tag{7.18}$$

It is instructive to re-derive this result using somewhat different notations. Suppose that instead of $\phi_{1,2}$ we introduce two complex fields

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}, \quad \phi^* = \frac{\phi_1 - i\phi_2}{\sqrt{2}}. \tag{7.19}$$

The Lagrangian density in Eq. (7.14) becomes

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^* - V(\phi \phi^*). \tag{7.20}$$

The Lagrangian depends on the product of ϕ and ϕ^*. For this reason it is invariant under the following transformation

$$\phi = e^{i\lambda} \tilde{\phi}, \quad \phi^* = e^{-i\lambda} \tilde{\phi}^*. \tag{7.21}$$

The infinitesimal transformations lead to

$$\Delta \tilde{\phi} = i \tilde{\phi}, \quad \Delta \tilde{\phi}^* = -i \tilde{\phi}^*. \tag{7.22}$$

We use these formulas to write the conserved current

$$J^\mu = \phi^* (\partial_\mu \phi) - (\partial_\mu \phi^*)\, \phi. \tag{7.23}$$

Finally, consider the transformation which, at first sight, has nothing to do with the transformation of fields discussed above. Namely, we will perform a *coordinate transformation*

$$x^\mu = x_1^\mu + \lambda a^\mu, \tag{7.24}$$

where a^μ is a constant four-vector. Although this transformation does not appear to be a field transformation, fields do change under this transformation as well. We write

$$\phi(x) = \phi(x_1 + \lambda a) = \tilde{\phi}(x_1). \tag{7.25}$$

We then find

$$\phi(x) = \tilde{\phi}(x) - \lambda a^\mu \partial_\mu \tilde{\phi}(x), \tag{7.26}$$

so that

$$\Delta \tilde{\phi}(x) = -a^\mu \partial_\mu \tilde{\phi}. \tag{7.27}$$

Also, according to Eq. (7.25) the following equation is valid

$$\mathcal{L}(\phi(x), \partial_\mu \phi(x)) = \mathcal{L}(\tilde{\phi}(x_1), \partial_\mu \tilde{\phi}(x_1)) \tag{7.28}$$

Although this equation appears to tell us that the Lagrangian written with old and new fields are equal and, therefore $K^\mu = 0$, it is to be noted that the two Lagrangians are computed *at different space time points*. However, to make use of Eq. (7.4), we need to write them at the same point. Since

$$\mathcal{L}(x_1) = \mathcal{L}(x - \lambda a) = \mathcal{L}(x) - \lambda a^\mu \partial_\mu \mathcal{L}, \tag{7.29}$$

we conclude that

$$K^\mu = -a^\mu \partial_\mu \mathcal{L}. \tag{7.30}$$

Finally, we use Eq. (7.10) and find the conserved current

$$J^\mu = -a^\nu \frac{\delta \mathcal{L}}{\delta \partial^\mu \phi} \partial_\nu \phi + a^\mu \mathcal{L}. \tag{7.31}$$

Since a^μ is an arbitrary vector, we can introduce a rank-two tensor $T^{\mu\nu}$

$$T^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta \, \partial_\mu \phi} \, \partial^\nu \phi - g^{\mu\nu} \mathcal{L}. \tag{7.32}$$

This (symmetric) tensor is conserved

$$\partial_\mu T^{\mu\nu} = 0. \tag{7.33}$$

To understand the implications of the fact that a conserved current exists, we define the quantity that we will call the *charge*

$$Q(t) = \int \mathrm{d}^3\vec{x} \;\; J_0(t, \vec{x}). \tag{7.34}$$

In principle $Q(t)$ should be considered time-dependent. To check if this is indeed the case, we compute the time-derivative of $Q(t)$ and use Eq. (7.4). We then find

$$\frac{\mathrm{d}Q(t)}{\mathrm{d}t} = \int \mathrm{d}^3\vec{x} \, \frac{\partial J_0(t, \vec{x})}{\partial t}. = -\int \mathrm{d}^3\vec{x} \, \vec{\nabla} \cdot \vec{J} = -\oint\limits_{|\vec{x}|=\infty} \mathrm{d}^2\vec{S} \;\cdot \vec{J} = 0, \tag{7.35}$$

where we have assumed that $\vec{J}$ vanishes at spatial infinity which is a standard assumption. Hence, the "charge" $Q(t)$ is time-independent and remains constant for fields ϕ which satisfy equations of motion.

The above discussion applies to classical field theory and we need to understand how it changes in the quantum case. To make things simple, we consider the theory defined by Eq. (7.14). This theory is complicated since it contains arbitrary interactions and self-interactions of fields. However, the quantization of this theory proceeds in exactly the same way as for the free theory, namely, we compute the canonical momentum and require that an equal time commutator is canonical. We then find

$$\pi_1 = \partial_0 \phi_1, \quad \pi_2 = \partial_0 \phi_2, \tag{7.36}$$

and

$$[\pi_i(t,\vec{x}), \phi_j(t,\vec{y})] = -i\delta_{ij}\delta^{(3)}(\vec{x}-\vec{y}). \tag{7.37}$$

The charge Q is constructed from the time derivative of the Noether's current given in Eq. (7.18). We find

$$Q(t) = \int \mathrm{d}^3\vec{x}\left(\phi_1(t,\vec{x})\pi_2(t,\vec{x}) - \phi_2(t,\vec{x})\pi_1(t,\vec{x})\right). \tag{7.38}$$

To understand whether this operator depends on time or not, we need to compute its commutator with the Hamiltonian. The Hamiltonian reads

$$H = \int \mathrm{d}^3\vec{x}\left[\frac{\pi_1^2}{2} + \frac{\pi_2^2}{2} + (\vec{\nabla}\phi_1)^2 + (\vec{\nabla}\phi_2)^2 + V(\phi_1^2+\phi_2^2)\right]. \tag{7.39}$$

It is easy to see that H and Q commute so that Q is indeed time independent.

It is instructive to compute a commutator of Q and the fields $\phi_{1,2}$. Then,

$$\begin{aligned}
&[Q, \phi_1(t,\vec{x})] = -\int \mathrm{d}^3\vec{x}_1 \phi_2(t,\vec{x}_1)[\pi_1(t,\vec{x}_1), \phi_1(t,\vec{x})] = -i\phi_2(t,\vec{x}),\\
&[Q, \phi_2(t,\vec{x})] = i\phi_1(t,\vec{x}).
\end{aligned} \tag{7.40}$$

It is then easy to see that

$$\begin{aligned}
e^{iQ\theta}\phi_1 e^{-iQ\theta} &= \cos\theta\ \phi_1 + \sin\theta\ \phi_2,\\
e^{iQ\theta}\phi_2 e^{-iQ\theta} &= -\sin\theta\ \phi_1 + \cos\theta\ \phi_2,
\end{aligned} \tag{7.41}$$

Hence, the charge Q is the generator of the symmetry transformations of the theory that we consider.

To see this more clearly, consider a theory of N fields ϕ_i defined by the Lagrangian which is symmetric under a class of field transformations that represent a symmetry (Lie) group G. The group is defined by its Lie algebra which, in turn, is entirely determined by algebra generators and the structure constants

$$\left[T^a, T^b\right] = i f^{abc} T^c, \tag{7.42}$$

Group elements are constructed by exponentiating the generators

$$U = e^{iT_a \theta_a}. \tag{7.43}$$

We assume that the group is unitary $U^+ U = 1$; this implies that generators are Hermitian. We assume that fields in our theory transform in the fundamental representation of the group

$$\phi_i = U_{ij} \tilde{\phi}_j, \quad i, j \in \{1, .., N\}, \tag{7.44}$$

where N is the dimensionality of the representation.

To find conserved currents of this symmetry transformation, we consider its infinitesimal version and obtain

$$\phi_i = (1 + iT_a \theta_a)_{ij} \tilde{\phi}_j = \tilde{\phi}_i + i\theta_a T^a_{ij} \tilde{\phi}_j. \tag{7.45}$$

Independent symmetry transformations correspond to independent symmetry generators. The currents are then

$$J^a_\mu = \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_i} T^a_{ij} \phi_j = (\partial_\mu \phi_i) T^a_{ij} \phi_j. \tag{7.46}$$

We use these currents to define the time-independent charges

$$Q^a = \int \mathrm{d}^3\vec{x}\ \pi_i T^a_{ij} \phi_j. \tag{7.47}$$

The (equal-time) quantization condition is

$$[\pi_i(t, \vec{x}), \phi_j(t, \vec{y})] = -i\delta_{ij} \delta^{(3)}(\vec{x} - \vec{y}). \tag{7.48}$$

We now compute the commutator of the two charges

$$
\begin{aligned}
[Q^a, Q^b] &= \int \mathrm{d}^3\vec{x}\, \mathrm{d}^3\vec{y}\; [\pi_i(x) T^a_{ij} \phi_j(x), \pi_k(y) T^b_{km} \phi_m] \\
&= T^a_{ij} T^b_{km} \int \mathrm{d}^3\vec{x}\, \mathrm{d}^3\vec{y}\; \left(\pi_k(y)[\pi_i(x), \phi_m]\phi_j(x) + \pi_i(x)[\phi_j(x), \pi_k(y)]\phi_m \right) \\
&= -i T^a_{ij} T^b_{km} \int \mathrm{d}^3\vec{x}\, \mathrm{d}^3\vec{y}\; \delta^{(3)}(\vec{x} - \vec{y}) \left(\pi_k(y)\delta_{im}\phi_j(x) - \pi_i(x)\delta_{jk}\phi_m(y) \right) \\
&= i \int \mathrm{d}^3\vec{x}\; \pi_k(x)[T^a, T^b]_{ij}\phi_j(x) \\
&= -f^{abc} \int \mathrm{d}^3\vec{x}\; \pi_k(x) T^c \phi_j(x) = -f^{abc} Q^c.
\end{aligned}
\tag{7.49}
$$

Hence,

$$
[-iQ^a, -iQ^b] = i f^{abc}(-iQ^c). \tag{7.50}
$$

We conclude that operators $-iQ^a$ furnish a representation of the Lie algebra of the symmetry group G in the Hilbert space of the theory.

7.2 Discrete symmetries in Quantum Field Theory

We will now turn our attention to the so-called *discrete* symmetries. These symmetries are field transformations that do not have any continuous parameter that we called θ or λ in our previous examples. A good example is a symmetry under parity transformation. Parity changes vectors, e.g. $\vec{x}$, $\vec{p}$ etc. into $-\vec{x}$, $-\vec{p}$ and does not change *pseudo-vectors* such as angular momentum or magnetic field. There are three main types of discrete transformations: parity, charge and time-inversion. We will discuss these transformations using free Dirac theory.

In Quantum Field Theory, symmetry transformation are realized by unitary operators that act on creation and annihilation operators. Consider the parity transformation. We will define it by the following equations

$$
P a_{\vec{p},s} P^{-1} = \eta_a a_{-\vec{p},s}, \quad P b_{\vec{p},s} P^{-1} = \eta_a b_{-\vec{p},s}, \tag{7.51}
$$

The square of the parity transformation is an identity transformation $P^2 = 1$. This implies that $P^{-1} = P$ and, also, that $\eta_a^2 = \eta_b^2 = 1$.

We now use Eq. (7.51) to investigate how the fermion field ψ transforms under parity. We find

$$P\psi(t,\vec{x})P^{-1} = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3\,\sqrt{2E_{\vec{p}}}} \left[\eta_a u_{\vec{p},s} a_{-\vec{p},s} e^{-ip_\mu x^\mu} + \eta_b^* v_{\vec{p},s} b^+_{-\vec{p},s} e^{ip_\mu x^\mu}\right]. \tag{7.52}$$

We now change the integration variable $\vec{p} \to -\vec{p}$. This transformation does not affect $E_{\vec{p}}$ but transforms

$$p_\mu x^\mu \to p_\mu \tilde{x}^\mu, \tag{7.53}$$

where $\tilde{x}^\mu = (x^0, -\vec{x})$. We need to understand what happens to spinors. We find

$$u_{\vec{p},s} \to u_{-\vec{p},s} = \begin{pmatrix} \sqrt{\bar{\sigma}_\mu p^\mu}\xi_s \\ \sqrt{\sigma_\mu p^\mu}\xi_s \end{pmatrix} = \gamma_0 u_{\vec{p},s}. \tag{7.54}$$

The same applies to the negative energy solution except that

$$v_{-\vec{p},s} = -\gamma_0 v_{\vec{p},s}. \tag{7.55}$$

Hence, we need to choose $\eta_b^* = -\eta_a$ to have a simple transformation rule. We find

$$P\psi(t,\vec{x})P^{-1} = \eta_a\,\gamma_0\,\psi(t,-\vec{x}), \tag{7.56}$$

and we will use $\eta_a = 1$ in what follows. We will also need a transformation for the conjugate field $\bar{\psi}$. Using almost identical manipulations, we derive

$$P\bar{\psi}(t,\vec{x})P^{-1} = \bar{\psi}(t,-\vec{x})\gamma_0. \tag{7.57}$$

The next transformation is a charge-parity transformation. This transformation changes particles into anti-particles. We write

$$C a_{\vec{p},s} C^{-1} = \eta_a b_{\vec{p},s}, \quad C b_{\vec{p},s} C^{-1} = \eta_b a_{\vec{p},s} \tag{7.58}$$

Again, applying this transformation twice, we get the same quantity back, so $C^2 = 1$, $C^{-1} = C$ and $\eta_a^2 = \eta_b^2 = 1$.

We proceed with applying the charge-parity transformation to the fermion field

$$C\psi(x)C^{-1} = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3\,\sqrt{2E_{\vec{p}}}} \left[\eta_a u_{\vec{p},s} b_{\vec{p},s} e^{-ip_\mu x^\mu} + \eta_b^* v_{\vec{p},s} a^+_{\vec{p},s} e^{ip_\mu x^\mu}\right]. \tag{7.59}$$

Since the transformed field depends on b and a^+ it should be related to a complex conjugate field $\psi^*(t,x)$. We write

$$\psi^*(t,\vec{x}) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3\,\sqrt{2E_{\vec{p}}}} \left[v^*_{\vec{p},s} b_{\vec{p},s} e^{-ip_\mu x^\mu} + u^*_{\vec{p},s} a^+_{\vec{p},s} e^{ip_\mu x^\mu} \right]. \tag{7.60}$$

We now need to find a way to to make $u_{\vec{p},s}$ out of $v^*_{\vec{p},s}$ and $v_{\vec{p},s}$ out of $u^*_{\vec{p},s}$.

To understand how to do this, consider

$$v^*_{\vec{p},s} = \begin{pmatrix} \sqrt{p_\mu \sigma^*_\mu}\eta^*_s \\ -\sqrt{p_\mu \bar{\sigma}^{\mu *}}\eta^*_s. \end{pmatrix} \tag{7.61}$$

We then recall that among the Pauli matrices, only σ_2 is complex; therefore

$$\sigma^*_i = (-1)^{\delta_{i2}} \sigma_i. \tag{7.62}$$

Although this sign change seems to transform σ^μ and $\bar{\sigma}^\mu$ into something that we have not see before, this is not quite the case. Indeed, consider

$$\begin{aligned} -i\gamma_2 v^*_{\vec{p},s} &= -i \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p^\mu \sigma^*_\mu}\eta^*_s \\ -\sqrt{p^\mu \bar{\sigma}^*_\mu}\eta^*_s. \end{pmatrix} \\ &= -i \begin{pmatrix} -\sigma_2\sqrt{p_\mu \bar{\sigma}^{\mu *}}\eta^*_s. \\ -\sigma_2\sqrt{p_\mu \sigma^*_\mu}\eta^*_s \end{pmatrix} = \begin{pmatrix} -\sqrt{p_\mu \sigma^\mu}\,(-i\sigma_2)\eta^*_s. \\ -\sqrt{p_\mu \bar{\sigma}_\mu}\,(-i\sigma_2)\eta^*_s \end{pmatrix}. \end{aligned} \tag{7.63}$$

where in the last step we have used the fact that σ_2 anticommutes with $\sigma_{1,3}$ and commutes with itself. We choose the following basis spinors

$$\eta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \xi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{7.64}$$

Then, it follows from Eq. (7.63) that

$$-i\gamma_2 v^*_{\vec{p},s} = -u_{\vec{p},s}. \tag{7.65}$$

Applying complex conjugation to both sides, we find

$$-i\gamma_2 v_{\vec{p},s} = -u^*_{\vec{p},s}. \tag{7.66}$$

Using the fact that $\gamma_2^2 = -1$, we obtain

$$v_{\vec{p},s} = i\gamma_2\, u^*_{\vec{p},s}. \tag{7.67}$$

Hence, choosing $\eta_a = \eta_b = 1$, we conclude that

$$C\psi(x)C^{-1} = i\gamma_2\psi^*(x). \tag{7.68}$$

We can determine the transformation properties of the conjugate field following the steps above. We find

$$C\bar{\psi}(x)C^{-1} = -i\bar{\psi}^*(x)\gamma_2. \tag{7.69}$$

Another important symmetry transformation is the *time-reversal* symmetry, $t \to -t$. It follows from equations of motion in classical mechanics

$$m\frac{\mathrm{d}^2x}{\mathrm{d}^2t} = F(x), \tag{7.70}$$

that we can change $t \to -t$ without affecting the solutions. This means that if $x(t)$ is a solution, $x(-t)$ is the solution as well. In fact, $t \to -t$ means that $\vec{p} \to -\vec{p}$ and $\vec{L} = [\vec{r} \times \vec{p}] \to -\vec{L}$ as well.

The situation is more tricky in quantum mechanics and in Quantum Field Theory where the Schrödinger equation is *linear* in t. In Quantum Field Theory, we would like to have is an operator T that changes $\psi(t,\vec{x})$ into a quantity related to $\psi(-t,\vec{x})$

$$T\psi(t,\vec{x})T \sim \psi(-t,\vec{x}), \tag{7.71}$$

where we made use of the expectation that $T^2 = 1$. On the other hand, the time evolution of field operators can be made explicit

$$\psi(t,\vec{x}) = e^{iHt}\psi(\vec{x})e^{-iHt}, \tag{7.72}$$

which implies that

$$T\psi(t,\vec{x})T = Te^{iHt}T\ (T\psi(\vec{x})T)\ Te^{-iHt}T. \tag{7.73}$$

Hence, to determine action of T operator on the field, we need to ensure that

$$Te^{iHt}T = e^{-itH}. \tag{7.74}$$

There are two options to achieve this, either $\{T, H\} = 0$ or $[T, H] = 0$ but $Tc = c^*T$ *even* if c is an ordinary complex *number*. It turns out that the second option is the only consistent choice.

We now compute $T\psi(t,\vec{x})T$. We write

$$T\psi(t,\vec{x})T = \sum_s \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3\sqrt{2E_p}} \left(u^*_{\vec{p},s} T a_{\vec{p},s} T e^{ip_\mu x^\mu} + v^*_{\vec{p},s} T b^+_{\vec{p},s} T e^{-ip_\mu x^\mu} \right). \tag{7.75}$$

We now write

$$T a_{\vec{p},s} T = a_{-\vec{p},-s}, \quad T b^+_{\vec{p},s} T = b^+_{-\vec{p},-s}, \tag{7.76}$$

because three momentum and angular momentum change signs under the time inversion.

We obtain

$$T\psi(t,\vec{x})T = \sum_s \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3\sqrt{2E_p}} \left(u^*_{\vec{p},s} a_{-\vec{p},-s}, e^{ip_\mu x^\mu} + v^*_{\vec{p},s} b^+_{-\vec{p},-s} T e^{-ip_\mu x^\mu} \right). \tag{7.77}$$

We can now change $\vec{p} \to -\vec{p}$ and use the fact that

$$p_\mu x^\mu|_{\vec{p}\to-\vec{p}} = -p_\mu \tilde{x}^\mu, \tag{7.78}$$

where $\tilde{x}^\mu = (-t,\vec{x})$. It follows from the above formula that to relate $T\psi(t,\vec{x})T$ to $\psi(-t,\vec{x})$, we will need to find a way to map $u^*_{-\vec{p},s}$ onto $u_{\vec{p},s}$. One can show that, for a particular choice of the two-dimensional spinors that are used to define four-component spinors, the following equation holds

$$u^*_{-\vec{p},s} = \gamma^1\gamma^3 u_{\vec{p},-s}. \tag{7.79}$$

The same equation holds for $v^*_{-\vec{p},s}$. We conclude that the time-reversal transformation leads to

$$\begin{aligned} T\psi(t,\vec{x})T &= \gamma^1\gamma^3\psi(-t,\vec{x}), \\ T\bar{\psi}(t,\vec{x})T &= \bar{\psi}(-t,\vec{x})\gamma^3\gamma^1. \end{aligned} \tag{7.80}$$

The above symmetry transformations of fermion fields can be used to determine transformation properties of more complex quantities constructed out of fermion fields. These quantities are called *currents* and they appear when we construct field theories, both free and interacting. For example, consider the scalar current

$$J_s(x) = \bar{\psi}(x)\psi(x), \tag{7.81}$$

where the sum over spinor indices is assumed. To understand how $J_s(x)$ changes under parity, we write

$$\begin{aligned} PJ_s(x)P^{-1} &= P\bar{\psi}(x)P^{-1}\, P\psi(x)P^{-1} \\ &= \bar{\psi}(t,-\vec{x})\gamma_0\gamma_0\psi(t,-\vec{x}) = J_s(t,-\vec{x}), \end{aligned} \tag{7.82}$$

since $\gamma_0^2 = 1$.

Next, we consider how $J_s(x)$ changes under charge-parity transformation. We write[1]

$$\begin{aligned} CJ_s(x)C^{-1} &= C\bar{\psi}(x)C^{-1}\, C\psi(x)C^{-1} = \bar{\psi}^*(x)\gamma_2\gamma_2\psi^*(x) \\ &= -\psi^T(x)\gamma_0\psi^*(x) = \bar{\psi}(x)\psi(x) = J_s(x). \end{aligned} \tag{7.83}$$

Finally, we consider the time-reversal symmetry. We find

$$TJ_s(t,\vec{x})T^{-1} = J_s(-\vec{t},\vec{x}). \tag{7.84}$$

Finally, we study the result of applying joint CPT transformation to J_s. We find

$$[CPT]J_s(x)[CPT]^{-1} = J_s(-t,-\vec{x}). \tag{7.85}$$

Another quantity is a *vector current* defined as

$$J^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x). \tag{7.86}$$

For this object, we find

$$\begin{aligned} PJ^\mu(x)P^{-1} &= P\bar{\psi}(x)P^{-1}\gamma^\mu P\psi(x)P^{-1} \\ &= \bar{\psi}(t,-\vec{x})\gamma_0\gamma^\mu\gamma_0\psi(t,-\vec{x}) = \tilde{J}^\mu(t,-\vec{x}). \end{aligned} \tag{7.87}$$

The relationship between $\tilde{J}^\mu$ and J^μ follows from

$$\gamma^0\gamma^0\gamma^0 = \gamma^0, \qquad \gamma^0\vec{\gamma}\gamma^0 = -\vec{\gamma}. \tag{7.88}$$

Therefore,

$$\tilde{J}^\mu(t,-\vec{x}) = (J^0(t,-\vec{x}), -\vec{J}(t,-\vec{x})) \tag{7.89}$$

which is exactly how the four-vector (e.g. x^μ) transforms under parity transformations.

[1] Note that in the step before the last one, we used the fact that ψ-fields anticommute.

Next, consider the charge-parity transformation of the vector current. We find

$$\begin{aligned} CJ^{\mu}(x)C &= \bar{\psi}^{*}(x)\gamma_2\gamma^{\mu}\gamma_2\psi^{*}(x) \\ &= (-1)^{\delta_{\mu 2}}\bar{\psi}^{*}(x)\gamma^{\mu}\psi^{*}(x) = (-1)^{\delta_{\mu 2}}\psi^{T}(x)\gamma^{0}\gamma^{\mu}\psi^{*}(x) \\ &= (-1)^{\delta_{\mu 2}+1}\psi^{*}(x)\gamma^{\mu,T}\gamma^{0}\psi(x). \end{aligned} \tag{7.90}$$

Now, using

$$\gamma^{\mu,T} = (-1)^{1-\delta_{\mu,0}-\delta_{\mu,2}}\gamma^{\mu}, \tag{7.91}$$

we obtain

$$CJ^{\mu}(x)C = -J^{\mu}(x). \tag{7.92}$$

Finally,

$$\begin{aligned} TJ^{\mu}(t,\vec{x})T &= T\bar{\psi}(t,\vec{x})TT\gamma^{\mu}\psi(t,\vec{x})T \\ &= T\bar{\psi}(t,\vec{x})T(\gamma^{\mu})^{*}T\psi(t,\vec{x})T \\ &= (-1)^{\delta_{\mu 2}}\bar{\psi}(-t,\vec{x})\gamma^{3}\gamma^{1}\gamma^{\mu}\gamma^{1}\gamma^{3}\psi(-t,\vec{x}). \end{aligned} \tag{7.93}$$

Since $\gamma^{3}\gamma^{1}\gamma^{\mu}\gamma^{1}\gamma^{3} = (-1)^{\delta_{3\mu}+\delta_{1\mu}}$, we find

$$TJ^{\mu}(t,\vec{x})T = (-1)^{\delta_{\mu 1}+\delta_{\mu,2}+\delta_{\mu,3}}J^{\mu}(-t,\vec{x}) = -(-1)^{\delta_{\mu 0}}J^{\mu}(-t,\vec{x}). \tag{7.94}$$

Applying the combined CPT transformation, we find

$$[CPT]J^{\mu}(t,\vec{x})[CPT]^{-1} = -J^{\mu}(-t,-\vec{x}). \tag{7.95}$$

Next, consider the action of a free Dirac fermion

$$S = \int \mathrm{d}^4x\ \left(\bar{\psi}(t,\vec{x})i\partial_{\mu}\gamma^{\mu}\psi(t,x) - m\,\bar{\psi}(t,\vec{x})\psi(t,\vec{x})\right), \tag{7.96}$$

and apply the CPT transformation to it. We find

$$\begin{aligned} [CPT]S[CPT]^{-1} = \int \mathrm{d}^4x\ \Big(&[CPT]\bar{\psi}(t,\vec{x})i\partial_{\mu}\gamma^{\mu}\psi(t,x)[CPT]^{-1} \\ &- m\bar{\psi}(-t,-\vec{x})\psi(-t,-\vec{x})\Big). \end{aligned} \tag{7.97}$$

We now discuss the CPT-transformation for $\bar{\psi}i\partial_{\mu}\gamma^{\mu}\psi$. We can almost read of the result from the known transformation properties of the vector current J^μ but there are two point that require care. The first point is the explicit factor i in $\bar{\psi} i\partial_\mu \gamma^\mu \psi$ which changes sign under T. The second point is that, as a consequence of charge-parity, we have

$$C\bar{\psi}\gamma^\mu \partial_\mu \psi C \to \psi^T \gamma^\mu \partial_\mu \psi^*. \tag{7.98}$$

Since ∂_μ acts on ψ^* now (and not on ψ) and since we move ψ^* to the left of all γ matrices and other fields, the full transformation rule for the term $\bar{\psi}(t, \vec{x}) i\partial_\mu \gamma^\mu \psi(t, x)$ will look as follows

$$\begin{aligned} &[CPT]^{-1}\bar{\psi}(t, \vec{x})\partial_\mu \gamma^\mu \psi(t, x)[CPT]^{-1} \\ &= (-i)(-1)\left(\partial_\mu \bar{\psi}(-t, -\vec{x})\right)\gamma^\mu \psi(-t, -\vec{x}). \end{aligned} \tag{7.99}$$

Putting this term back into Eq. (7.97) and integrating by parts to put the partial derivative back on ψ, we find

$$\begin{aligned} [CPT]S[CPT]^{-1} = \int \mathrm{d}^4 x \Big(&- \bar{\psi}(-t, -\vec{x}) i\partial_\mu \gamma^\mu \psi(-t, -\vec{x}). \\ &- m\bar{\psi}(-t, -\vec{x})\psi(-t, -\vec{x})\Big). \end{aligned} \tag{7.100}$$

We now change $t \to -t, x \to -\vec{x}$ in Eq. (7.97), use the fact that $\partial_\mu \to -\partial_\mu$ after this transformation and obtain

$$[CPT]\ S\ [CPT]^{-1} = S, \tag{7.101}$$

so the Dirac action is invariant under the CPT transformation.

Let us consider a more complex theory where a fermion field couples to a boson field. A possible example is

$$S = S_D + S_s + S_{\text{int}}, \tag{7.102}$$

where S_D is the action of the Dirac field that we just explored,

$$S_s = \int \mathrm{d}^4 x\ \left(\frac{1}{2}(\partial_\mu \phi\ \partial^\mu \phi) - \frac{m^2}{2}\phi^2\right), \tag{7.103}$$

and

$$S_{\text{int}} = g\int \mathrm{d}^4 x\ \phi(x)\bar{\psi}(x)\psi(x). \tag{7.104}$$

One can easily convince oneself that for the scalar field ϕ the following transformation rules hold

$$P\phi(t,\vec{x})P = \phi(t,-\vec{x}), \quad C\phi(t,\vec{x})C = \phi(t,\vec{x}), \quad T\phi(t,\vec{x})T = \phi(-t,\vec{x}). \tag{7.105}$$

Using these results, as well as transformation rules for fermion fields, it is easy to see that S_s and S_{int} are invariant under the CPT-transformation. In fact, there is a very general statement known as the *CPT theorem* which states that a Quantum Field Theory which is Lorentz invariant and where usual relations between spin and statistics holds, is described by an action that is invariant under a simultaneous application of C, P and T transformations.