TTP1
Lecture 8

Kirill Melnikov
TTP KIT
25. Mai 2023

8 Perturbative expansion of the correlation functions

We consider a quantum field theory of a scalar field described by the following Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{8.1}$$

The additional $\mathcal{O}(\lambda)$ term makes this theory extremely complicated. For example, the equation of motion

$$\left(\partial_\mu \partial^\mu + m^2\right)\phi = -\frac{\lambda}{3!}\phi^3, \tag{8.2}$$

becomes non-linear and it is not known how to solve it exactly.

Nevertheless, we can quantize the theory following the same procedure as before, i.e. by calculating the canonical momentum $\pi = \delta\mathcal{L}/\delta(\partial_0\phi)$ and requiring that the standard commutation relation

$$[\pi(t,\vec{x}), \phi(t,\vec{y})] = -i\delta^{(3)}(\vec{x}-\vec{y}), \tag{8.3}$$

holds. We can also construct the Hamiltonian

$$H = \int \mathrm{d}^3\vec{x} \left[\frac{1}{2}\pi^2(t,\vec{x}) + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\right], \tag{8.4}$$

but, again, it will not be possible to find exact eigenvalues and eigenstates of this Hamiltonian. This is simply too complicated a problem.

All this is very similar to the case of an anharmonic oscillator in quantum mechanics which is also not amenable to an exact treatment. In that case, we develop perturbation theory for energies and wave functions. In quantum field theory, we will develop perturbation theory for Green's functions, i.e. the vacuum expectation values of products of fields ϕ.

We will discuss the simplest Green's function $\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle$ which involves just two fields. Here $|\Omega\rangle$ is the *exact* ground state of the theory with the Hamiltonian in Eq. (8.4). We expect that in the limit $\lambda \to 0$, the exact Green's function becomes the Green's function of a free theory

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle \to \langle 0|T\phi_0(x)\phi_0(y)|0\rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{i e^{-ip(x-y)}}{p^2 - m^2 + i0}. \tag{8.5}$$

To construct perturbation theory in λ, we write

$$H = H_0 + H_{\text{int}}, \tag{8.6}$$

where

$$H_0 = \int \mathrm{d}^3\vec{x} \left[\frac{1}{2}\pi^2(t,\vec{x}) + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{m^2}{2}\phi^2\right], \quad H_{\mathrm{int}} = \int \mathrm{d}^3\vec{x}\, \frac{\lambda}{4!}\phi^4. \tag{8.7}$$

To construct the perturbation theory, we imagine that at $t = t_0$, the field operator is fixed to $\phi(t_0,\vec{x})$, and that it can be written as a linear combination of creation and annihilation operators *of a free theory*

$$\phi(t_0,\vec{x}) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \left[a_{\vec{p}} e^{-ip_\mu x_0^\mu} + a_{\vec{p}}^+ e^{ip_\mu x_0^\mu}\right], \tag{8.8}$$

where $x_0 = (t_0,\vec{x})$. Eventually, we will see that the result for the Green's function will not depend on t_0. We can also imagine that t_0 is very large and negative and that the interaction term decouples at such values of t, so that we always start with a free theory.

The time evolution of the field ϕ is determined by the full Hamiltonian

$$\phi(t,\vec{x}) = e^{iH(t-t_0)}\phi(t_0,\vec{x})e^{-iH(t-t_0)}. \tag{8.9}$$

We rewrite this expression as follows

$$\phi(t,\vec{x}) = e^{iH(t-t_0)}e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}\phi(t_0,\vec{x})e^{-iH_0(t-t_0)}e^{+iH_0(t-t_0)}e^{-iH(t-t_0)}. \tag{8.10}$$

We have introduced H_0 into this formula and we imagine that, to compute H_0, we use creation and annihilation operators of a free theory. We then define

$$e^{iH_0(t-t_0)}\phi(t_0,\vec{x})e^{-iH_0(t-t_0)} = \phi_I(t,\vec{x}). \tag{8.11}$$

This field can be parameterized in terms of creation and annihilation operator of a free theory. We find

$$\phi_I(t,\vec{x}) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \left[a_{\vec{p}} e^{-ip_\mu x^\mu} + a_{\vec{p}}^+ e^{ip_\mu x^\mu}\right], \tag{8.12}$$

where $x = (t,\vec{x})$.

Next, we define the unitray time-evolution operator $U(t,t_0)$ as

$$U(t,t_0) = e^{+iH_0(t-t_0)}e^{-iH(t-t_0)}, \tag{8.13}$$

so that

$$\phi(t,\vec{x}) = U^{+}(t,t_0)\phi_I(t,\vec{x})U(t,t_0). \tag{8.14}$$

We will now derive a useful equation for the time-evolution operator. To this end, we write

$$\frac{\partial\phi(t,\vec{x})}{\partial t} = i[H,\phi(t,\vec{x})], \quad \frac{\partial\phi_I(t,\vec{x})}{\partial t} = i[H_0,\phi_I(t,\vec{x})]. \tag{8.15}$$

Also, given the relation between ϕ_I and ϕ, we can write

$$H(\phi,\pi) = U^{+}(t,t_0)H(\phi_I,\pi_I)U(t,t_0). \tag{8.16}$$

We now write

$$\begin{aligned}\frac{\partial\phi_I(t,\vec{x})}{\partial t} &= \frac{\partial}{\partial t}\left[U(t,t_0)\phi(t,\vec{x})U(t,t_0)^{+}\right] = \frac{\partial U(t,t_0)}{\partial t}\phi(t,\vec{x})U(t,t_0)^{+} \\ &+ U(t,t_0)\frac{\partial\phi(t,\vec{x})}{\partial t}U(t,t_0)^{+} + U(t,t_0)\phi(t,\vec{x})\frac{\partial U(t,t_0)^{+}}{\partial t}.\end{aligned} \tag{8.17}$$

We would like to turn ϕ-fields into ϕ_I-fields in this expression. To do that, we write

$$\begin{aligned}&\frac{\partial U}{\partial t}\phi(t,\vec{x})U^{+} + U\frac{\partial\phi(t,\vec{x})}{\partial t}U^{+} + U\phi(t,\vec{x})\frac{\partial U^{+}}{\partial t} \\ &= \frac{\partial U}{\partial t}U^{+}\left(U\phi(t,\vec{x})U^{+}\right) + iU[H(\phi,\pi),\phi]U^{+} + (U\phi(t,\vec{x})U^{+})U\frac{\partial U^{+}}{\partial t} \\ &= \frac{\partial U}{\partial t}U^{+}\,\phi_I + i[H(\phi_I,\pi_I),\phi_I] + \phi_I U\frac{\partial U^{+}}{\partial t}.\end{aligned} \tag{8.18}$$

To further simplify this expression, we make use of the fact that $UU^{+} = 1$, so that

$$\frac{\partial U}{\partial t}U^{+} + U\frac{\partial U^{+}}{\partial t} = 0. \tag{8.19}$$

Hence, combining this and the previous equation, we arrive at

$$\frac{\partial\phi_I(t,\vec{x})}{\partial t} = \left[\frac{\partial U}{\partial t}U^{+} + iH(\phi_I,\pi_I),\phi_I\right] = [iH_0(\phi_I,\pi_I),\phi_I]. \tag{8.20}$$

We conclude that

$$\frac{\partial U}{\partial t}U^{+} + iH(\phi_I,\pi_I) - iH_0(\phi_I,\pi_I) = 0. \tag{8.21}$$

Since

$$H = H_0 + H_{\rm int}, \tag{8.22}$$

we obtain

$$\frac{\partial U}{\partial t} U^{+} = -i H_{\rm int}(\phi_I, \pi_I). \tag{8.23}$$

Multiplying both sides of this equation with U and using the uniitarity of U-operator, we find

$$\frac{\partial U(t, t_0)}{\partial t} = -i H_{\rm int}\left(\phi_I, \pi_I\right) U(t, t_0). \tag{8.24}$$

The solution of this equation with the boundary condition $U(t_0, t_0) = 1$ reads

$$U(t, t_0) = T e^{-i \int\limits_{t_0}^{t} {\rm d}\tau \; H_{\rm int}(\phi_I(\tau), \pi_I(\tau))}, \tag{8.25}$$

where T is the time-ordering operator. This operator implies that when the exponential function is expanded in Taylor series and terms of these series are written as multiple integrals over respective times, $H_{\rm int}(\tau)$ with the largest τ should be at the left-most position etc.

Eventually, we will need this operator to set up perturbative expansion of Green's functions. However, it is easy to see that $U(t, t_0)$ needs to be generalized to enable that. Indeed, consider, a product of two fields $\phi(x)\phi(y)$. When we express them through fields $\phi_I(x)$ or $\phi_I(y)$ we find

$$\phi(x)\phi(y) = U^{+}(x_0, t_0)\phi_I(x) U(x_0, t_0) U^{+}(y_0, t_0)\phi_I(y) U(y_0, t_0). \tag{8.26}$$

We would like to introduce a new time evolution operator defined as follows

$$U(x_0, y_0) = U(x_0, t_0) U^{+}(y_0, t_0). \tag{8.27}$$

Using the definition of the U-operators, we find

$$U(x_0, y_0) = e^{i H_0 (x_0 - t_0)} e^{-i H (x_0 - y_0)} e^{-i H_0 (y_0 - t_0)}. \tag{8.28}$$

If $y_0 = t_0$, or $x_0 = t_0$, we either get the original U-operator or its conjugate version.

It is easy to convince oneself that

$$U(t_2, t_1) = T e^{-i \int\limits_{t_1}^{t_2} {\rm d}\tau \; H_{\rm int}(\phi_I(\tau), \pi_I(\tau))}, \tag{8.29}$$

so that if $t_1 < t_2 < t_3$

$$U(t_3, t_2)U(t_2, t_1) = U(t_3, t_1), \tag{8.30}$$

and that

$$U(t_3, t_2)U^+(t_3, t_2) = 1. \tag{8.31}$$

Next, we consider the two-point function $\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle$ and assume that $x_0 > y_0$, for definiteness. Then

$$\begin{aligned} &\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \langle\Omega|\phi(x)\phi(y)|\Omega\rangle \\ &= \langle\Omega|U^+(x_0, t_0)\phi_I(x)U(x_0, y_0)\phi_I(y)U(y_0, t_0)|\Omega\rangle. \end{aligned} \tag{8.32}$$

Parts of this expression involve $\phi_I(x)$ etc. and we understand what to do about them but there is also an exact vacuum state $|\Omega\rangle$ and we need to connect it with the ground state of the free theory.

There is a trick that we use to do that. We take a state $|0\rangle$ and consider its time evolution in the full theory

$$|\Psi(t)\rangle = e^{-iHt}|0\rangle. \tag{8.33}$$

Since $|0\rangle$ is not an eigenstate of H, $\Psi(t)\rangle$ is a linear combination of the exact eigenstates of the Hamiltonian H. We find

$$|\Psi(t)\rangle = \sum e^{-iE_n t}|n\rangle\langle n|0\rangle. \tag{8.34}$$

We now use this equation for $t = T + t_0$ and take the limit $T \to +\infty(1 - i\epsilon)$ with $\epsilon > 0$. This limit projects the sum on the ground state $|\Omega\rangle$.[1] We find

$$\lim_{t\to T+t_0} e^{-iHt}|0\rangle = e^{-iE_0(T+t_0)}|\Omega\rangle\langle\Omega|0\rangle. \tag{8.35}$$

Solving this equation for $|\Omega\rangle$, we obtain

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{e^{iE_0(T+t_0)}e^{-iH(T+t_0)}|0\rangle}{\langle\Omega|0\rangle}. \tag{8.36}$$

[1]It is important that $\epsilon > 0$ and we assume that there is a gap between the energy of the first excited state and the vacuum. This implies that the overlap between any excited state and $|0\rangle$ vanishes as $T \to \infty$ because $e^{-\epsilon(E_n-E_0)T} < e^{-\epsilon(E_1-E_0)T} \to 0$ as $T \to +\infty$.

We would like to rewrite this formula using operator $U(t_2, t_1)$. Since

$$U(t_2, t_1) = e^{iH_0(t_2-t_0)} e^{-iH(t_2-t_1)} e^{-iH_0(t_1-t_0)}, \tag{8.37}$$

and since $H_0|0\rangle = 0$, we can write

$$e^{-iH(T+t_0)}|0\rangle = U(t_0, -T)|0\rangle, \tag{8.38}$$

so that

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{e^{iE_0(T+t_0)} U(t_0, -T)|0\rangle}{\langle\Omega|0\rangle}. \tag{8.39}$$

We then perform a similar computation for $\langle\Omega|$. In this case, we take $t = t_0 - T$ and write

$$\langle\Omega| = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|e^{iH(t_0-T)}\ e^{iE_0(t_0-T)}}{\langle 0|\Omega\rangle}. \tag{8.40}$$

We can rewrite this formula as follows

$$\langle\Omega| = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|U(T, t_0)}{\langle 0|\Omega\rangle e^{-iE_0(T-t_0)}}. \tag{8.41}$$

We are now in position to assemble the Green's function. Assuming that $x_0 > y_0$, we write

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \\ \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|U(T, t_0)U^{+}(x_0, t_0)\phi_I(x)U(x_0, y_0)\phi_I(y)U(y_0, t_0)U(t_0, -T)|\Omega\rangle}{\langle 0|\Omega\rangle\langle\Omega|0\rangle e^{-iE_0 2T}}. \tag{8.42}$$

We use $U^{+}(x_0, t_0) = U(t_0, x_0)$ and find

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \\ \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|U(T, x_0)\phi_I(x)U(x_0, y_0)\phi_I(y)U(y_0, t_0)U(t_0, -T)|\Omega\rangle}{\langle 0|\Omega\rangle\langle\Omega|0\rangle e^{-iE_0 2T}}. \tag{8.43}$$

We then realize that the operators in the numerator of this expression can be written as

$$U(T, x_0\phi_I(x)U(x_0, y_0)\phi_I(y)U(y_0, t_0)U(t_0, -T) \\ = T\phi_I(x)\phi_I(y)U(T, -T) = T\phi_I(x)\phi_I(y)e^{-i\int\limits_{-T}^{T} \mathrm{d}\tau\ H_{\text{int}}(\phi_I(\tau), \pi_I(\tau))}. \tag{8.44}$$

To understand what should be done with the denominator in Eq. (8.43), we note that the vacuum state is supposed to be normalized. Then

$$1 = \langle \Omega | \Omega \rangle = \frac{\langle 0 | U(T, t_0) U(t_0, -T) | 0 \rangle}{\langle 0 | \Omega \rangle \langle \Omega | 0 \rangle e^{-iE_0 2T}}. \tag{8.45}$$

It follows that

$$\langle 0 | \Omega \rangle \langle \Omega | 0 \rangle e^{-iE_0 2T} = \langle 0 | U(T, t_0) U(t_0, -T) | 0 \rangle. \tag{8.46}$$

Hence, we find

$$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle = \lim_{T \to \infty(1-i\epsilon)} \frac{\langle 0 | T \phi_I(x) \phi_I(y) U(T, -T) | 0 \rangle}{\langle 0 | U(T, -T) | 0 \rangle}, \tag{8.47}$$

where the time evolution operator reads

$$U(T, -T) = e^{-i \int\limits_{-T}^{T} \mathrm{d}\tau \; H_{\rm int}(\phi_I(\tau), \pi_I(\tau))}. \tag{8.48}$$

Equation (8.47) is exact. However, its main virtue is that it allows us to construct an expansion in powers of the self-coupling λ that appears *only* in the interaction Hamiltonian $H_{\rm int}$. Hence, the perturbative expansion arises upon the expansion of the operator $U(T, -T)$ in series of λ.

We will discuss practical ways to visualize such an expansion in the next lecture. However, it is clear that important building blocks of such an expansion are Green's functions with a large number of $\phi_I(x)$ fields

$$\langle 0 | T \phi_I(x_1) \phi_I(x_2) \phi_I(x_3) ... \phi_I(x_N) | 0 \rangle. \tag{8.49}$$

We will have to understand how to compute such Green's functions. Since each of the fields $\phi_I(x)$ is a linear combination of creation and annihilation operators and since annihilation operators annihilate the vacuum $|0\rangle$, it is clear how to compute the above Green's function by a brute force – we simply need to move all annihilation operators to the right since if we manage to do that, all such contributions will vanish. An intelligent way to organize such calculations is known as the *Wick theorem* which we will now discuss.

Suppose we introduce positive-energy and negative-energy parts of the field $\phi_I(x)$

$$\phi(x) = \phi_I^+(x) + \phi_I^-(x), \tag{8.50}$$

where

$$\begin{aligned}\phi_I^+(x) &= \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\; a_{\vec{p}}\; e^{-ip_\mu x^\mu}\\ \phi_I^-(x) &= \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\; a_{\vec{p}}^+\; e^{ip_\mu x^\mu}.\end{aligned} \tag{8.51}$$

Thanks to the properties of creation and annihilation operators, we find

$$\langle 0|\phi_I^-(x) = 0, \quad \phi_I^+(x)|0\rangle = 0. \tag{8.52}$$

Let us consider the Green's function of two fields. We write

$$\begin{aligned}T\phi_I(x)\phi_I(y) &= \theta(x_0-y_0)\Big(\phi_I^+(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y)\\ &\quad + \phi_I^-(x)\phi_I^+(y) + \phi_I^+(x)\phi_I^-(y)\Big) + \ldots\\ &= \theta(x_0-y_0)\Big(\phi_I^+(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y)\\ &\quad + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(y)\phi_I^+(x) + [\phi_I^+(x), \phi_I^-(y)]\Big) + \ldots\end{aligned} \tag{8.53}$$

The only term that contributes to the Green's function is the commutator of the positive-energy and the negative-energy parts of the quantum fields. Hence, we find

$$\begin{aligned}\langle 0|T\phi_I(x)\phi_I(y)|0\rangle &= \langle 0|\theta(x_0-y_0)[\phi_I^+(x), \phi_I^-(y)]\\ &\quad + \theta(y_0-x_0)[\phi_I^+(y), \phi_I^-(x)]|0\rangle = D_F(x-y),\end{aligned} \tag{8.54}$$

the Feynman propagator. We also note that the commutators in the above formula are not operators but c-numbers; because of this

$$\langle 0|[\phi_I^+(y), \phi_I^-(x)]|0\rangle = [\phi_I^+(y), \phi_I^-(x)], \tag{8.55}$$

so that the vacuum expectation value does not need to be taken. We then define a contraction symbol

$$\begin{aligned}\overbracket{\phi_I(x)\phi_I}(y) &= \theta(x_0-y_0)[\phi_I^+(x), \phi_I^-(y)]\\ &\quad + \theta(y_0-x_0)[\phi_I^+(y), \phi_I^-(x)] = D_F(x-y),\end{aligned} \tag{8.56}$$

and write the T-product of the two fields as follows

$$T\phi_I(x)\phi_I(y) = N(\phi(x)\phi(y)) + \overbracket{\phi_I(x)\phi_I}(y). \tag{8.57}$$

The operator N stands for the "normal ordering operator" which places all creation operators to the left and all annihilation operators to the right. Since

$$\langle 0|N(\phi(x)\phi(y))|0\rangle = 0, \tag{8.58}$$

we conclude that

$$\langle 0|T\phi_I(x)\phi_I(y)|0\rangle = \overline{\phi_I(x)\phi_I}(y) = D_F(x-y). \tag{8.59}$$

The generalization of Eq. (8.57) to the case of many fields is called the Wick theorem. It reads

$$\begin{aligned} &T\phi_I(x_1)\phi_I(x_2)...\phi_I(x_N) = N\Big\{\phi_I(x_1)...\phi_I(x_N)+ \\ &\text{all possible contractions between the fields } \phi_I\Big\}. \end{aligned} \tag{8.60}$$

Note that "all possible contractions" includes terms where e.g. $\phi_I(x_1)$ is contracted with $\phi_I(x_2)$ and the rest of the fields are normal-ordered. Or, if all fields become contracted pair-wise, one can place such a term under the normal ordering sign since in this case there are no creation and annihilation operators left.

As an example, we re-write the time-ordered product of four fields through normal-ordered products. We find

$$\begin{aligned} &T\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4) = N\Big[\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4) \\ &+ \overline{\phi_I(x_1)\phi_I}(x_2)\phi_I(x_3)\phi_I(x_4) + \overline{\phi_I(x_1)\phi_I}(x_3)\phi_I(x_2)\phi_I(x_4) \\ &+ \overline{\phi_I(x_1)\phi_I}(x_4)\phi_I(x_2)\phi_I(x_3) + \overline{\phi_I(x_2)\phi_I}(x_3)\phi_I(x_1)\phi_I(x_4) \\ &+ \overline{\phi_I(x_2)\phi_I}(x_4)\phi_I(x_1)\phi_I(x_3) + \overline{\phi_I(x_3)\phi_I}(x_4)\phi_I(x_1)\phi_I(x_2) \\ &+ \overline{\phi_I(x_1)\phi_I}(x_2)\overline{\phi_I(x_3)\phi_I}(x_4) + \overline{\phi_I(x_1)\phi_I}(x_3)\overline{\phi_I(x_2)\phi_I}(x_4) \\ &+ \overline{\phi_I(x_1)\phi_I}(x_4)\overline{\phi_I(x_2)\phi_I}(x_3)\Big]. \end{aligned} \tag{8.61}$$

The beauty of this formula is that *only fully-contracted terms* contribute to the vacuum expectation value of T-ordered fields. Since each contraction is a Feynman propagator, we find

$$\begin{aligned} &\langle 0|T\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)|0\rangle = D_F(x_1-x_2)D_F(x_3-x_4) \\ &+ D_F(x_1-x_3)D_F(x_2-x_4) + D_F(x_1-x_4)D_F(x_2-x_3). \end{aligned} \tag{8.62}$$

The same result applies to the general case of N-fields where, of course, the number of contractions is larger.

We will now prove the Wick theorem. We have explicitly discussed it for $N = 2$, so it is natural to prove it by induction. We thus assume that the theorem is valid for $N-1$ fields and we use this information to prove that it is valid for N fields. We consider $T\phi_I(x_1)...\phi_I(x_N)$, assume for definiteness that x_1^0 is the largest time and write

$$\begin{aligned} &T\phi_I(x_1)...\phi_I(x_N) = \phi_I(x_1)T\phi_I(x_2)..\phi_I(x_N) \\ &= \phi_I(x_1)N[\phi_I(x_2)..\phi_I(x_N) + \text{contractions}]. \end{aligned} \tag{8.63}$$

We now write $\phi_I(x_1) = \phi_I^+(x_1) + \phi_I^-(x_1)$. The negative energy component we just leave where it is since this term is already normal-ordered

$$\begin{aligned} &\phi_I^-(x_1)N[\phi_I(x_2)..\phi_I(x_N) + \text{contractions}] \\ &= N[\phi_I^-(x_1)\phi_I(x_2)..\phi_I(x_N) + \text{contractions})]. \end{aligned} \tag{8.64}$$

On the contrary, we need to move $\phi_I^+(x_1)$ passed all $\phi_I^-(x_i)$ fields inside the normal ordering operator.

To see how this works, consider the term with no contractions. Then we write

$$\begin{aligned} \phi_I^+(x_1)\ N[\phi_I(x_2)..\phi_I(x_N)] &= N[\phi_I(x_2)..\phi_I(x_N)\phi_I^+(x_1)] \\ &+ \Big[\phi_I^+(x_1), N[\phi_I(x_2)..\phi_I(x_N)]\Big]. \end{aligned} \tag{8.65}$$

The first term here combines with a similar term in Eq. (8.64) and allows us to replace a sum of $\phi_I^-(x_1)$ and $\phi_I^+(x_1)$ with $\phi_I(x_1)$ under the normal ordering sign.

The second term in Eq. (8.65) produces $N-1$ contractions of $\phi_I(x_1)$ with the other fields. Indeed,

$$\begin{aligned} &\Big[\phi_I^+(x_1), N[\phi_I(x_2)..\phi_I(x_N)]\Big] = \sum_{i=2}^{N} N[\phi_I(x_2)...., [\phi_I^+(x_1), \phi_I^-(x_i)], ..\phi_I(x_N)] \\ &= \sum_{i=2}^{N} N[[\phi_I^+(x_1), \phi_I^-(x_i)]\ \phi_I(x_2)....\phi_I(x_{i-1})\ \phi_I(x_{i+1})..\phi_I(x_N)]. \end{aligned} \tag{8.66}$$

Since $x_1^0 > x_i^0$, we can replace the commutator with the contraction of the two fields

$$[\phi_I^+(x_1), \phi_I^-(x_i)] = \phi_I(x_1)\phi_I(x_i). \tag{8.67}$$

Identical arguments apply to terms in Eq. (8.63) which contain contractions between (some) of the fields $\phi(x_2), .., \phi(x_N)$. We therefore conclude that

$$T\phi_I(x_1)...\phi_I(x_N) = N[\phi_I(x_1)\phi_I(x_2)..\phi_I(x_N) + \text{contractions}], \tag{8.68}$$

as required by the Wick theorem.