TTP1
Lecture 9

Kirill Melnikov
TTP KIT
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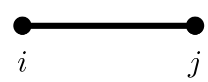
9 Feynman diagrams

As we have seen, the Wick theorem gives us a straightforward way to express any correlation function $\langle 0|T\phi(x_1)...\phi(x_N)|0\rangle$ in a non-interacting field theory as a sum of products of Feynman propagators. Consider for definiteness a four-point function $\langle 0|T\phi(x_1)...\phi(x_4)|0\rangle$. As discussed in the previous lecture, the following result holds

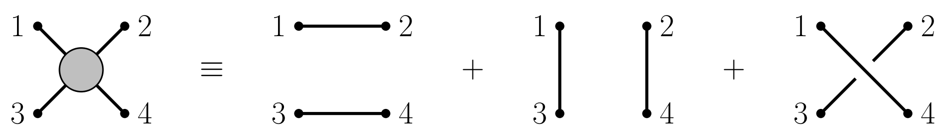
$$\begin{aligned}\langle 0|T\phi(x_1)...\phi(x_4)|0\rangle &= D_F(x_1-x_2)D_F(x_3-x_4)\\ &+ D_F(x_1-x_3)D_F(x_2-x_4)+D_F(x_1-x_4)D_F(x_2-x_3),\end{aligned} \tag{9.1}$$

where $D_F(x-y) = \langle 0|T\phi(x)\phi(y)|0\rangle$ is the Feynman propagator.

Let us develop a pictorial representation of this result. We will represent the four space-time points by dots and the Green's function as a whole by a blob. We will represent a Feynman propagator that depends on $x_i - x_j$ by a line that connects the points

$D_F(x_i - x_j) =$  (9.2)

Since the propagator is symmetric under the interchange of x_i and x_j, the line has no direction. Then Eq. (9.1) can be presented in a graphical form as follows

 (9.3)

The interpretation of this representation is straightforward: particles are created at two points, propagate to two other points and are absorbed there back into vacuum. All we need to do is to account for all possible ways this can happen and add the "probability amplitudes", as required by quantum mechanics.

The case of the four-point function is quite simple but it gets more complicated if we consider correlation functions with more than just one field produced at a given point. To see how this can happen, consider a two-point correlation function in a theory with self-interaction $\lambda\phi^4/4!$. In the previous lecture the following representation was derived for this Green's function

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\phi_I(x)\phi_I(y)U(T,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle}, \tag{9.4}$$

where the time evolution operator reads

$$U(T,-T) = e^{-i\int\limits_{-T}^{T} \mathrm{d}\tau\; H_{\rm int}(\phi_I(\tau),\pi_I(\tau))}, \tag{9.5}$$

and

$$H_{\rm int} = \frac{\lambda}{4!}\int \mathrm{d}^3\vec{x}\; \phi_I^4(\tau,\vec{x}). \tag{9.6}$$

Note that the time evolution operator is the *only* quantity in this formula that depends on λ.

We focus on the numerator of the ratio in Eq. (9.4) and expand it to first order in λ. We find[1]

$$\langle 0|T\phi_I(x)\phi_I(y)|0\rangle - \frac{i\lambda}{4!}\langle 0|T\phi_I(x)\phi_I(y)\int \mathrm{d}^4z\; \phi_I^4(z)|0\rangle. \tag{9.7}$$

The first term in Eq. (9.7) is a free Green's function; we know that it is described by the Feynman propagator $D_F(x-y)$. The second term in Eq. (9.7) can be written as a sum of products of all possible contractions of the fields ϕ_I. Let us describe all the possible contractions.

- We can contract $\phi_I(x)$ with $\phi_I(y)$. We then need to contract fields that appear in $\phi_I^4(z)$ between themselves. There are *three* distinct (but equivalent) ways to do that. Hence this contraction gives

$$-\frac{i\lambda}{4!}\,3\;D_F(x-y)\int \mathrm{d}^4z\; D_F(z-z)\;D_F(z-z). \tag{9.8}$$

- We contract $\phi_I(x)$ with one of $\phi_I(z)$'s and then contract $\phi_I(y)$ with one of the remaining $\phi_I(z)$'s. There are four ways to do the first contraction and three ways to do the second. Therefore, the corresponding contribution reads

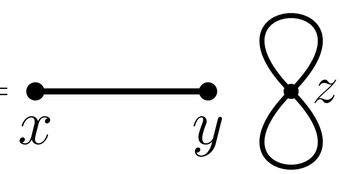
$$-\frac{i\lambda}{4!}\,4\cdot 3\int \mathrm{d}^4z\; D_F(x-z)D_F(y-z)\;D_F(z-z). \tag{9.9}$$

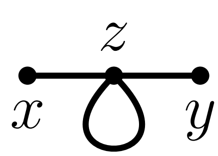
[1]For the time being we ignore the fact that we need to integrate over τ from $-T$ to T and take a limit $T\to\infty(1-i\epsilon)$.

There is nothing else that we can do so that the complete result reads

$$
\begin{aligned}
&-\frac{i\lambda}{4!}\langle 0|T\phi_I(x)\phi_I(y)\int \mathrm{d}^4z\ \phi_I^4(z)|0\rangle \\
&= -\frac{i\lambda}{4!}\ 3\ D_F(x-y)\int \mathrm{d}^4z\ D_F(z-z)\ D_F(z-z) \qquad (9.10)\\
&-\frac{i\lambda}{4!}\ 4\cdot 3\ \int \mathrm{d}^4z\ D_F(x-z)D_F(y-z)\ D_F(z-z).
\end{aligned}
$$

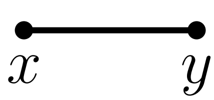
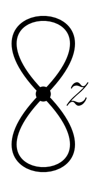
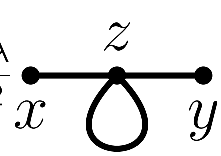
We can draw the various contributions that appear in the above expression using the rule that a line connects the points x_i and x_j in a Feynman propagator $D_F(x_i - x_j)$ and leaving aside the integration over z. The terms on the right hand side of Eq. (9.10) look as follows

$$D_F(x-y)\int \mathrm{d}^4z\ D_F(z-z)\ D_F(z-z) =$$  , (9.11)

$$\int \mathrm{d}^4z\ D_F(x-z)D_F(y-z)\ D_F(z-z) =$$  . (9.12)

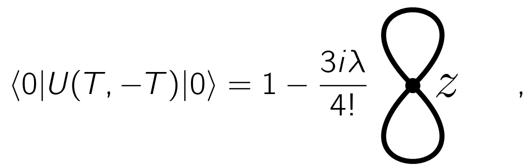
Hence, to first order in λ, we find

$$\lim_{T\to\infty(1-i\epsilon)}\langle 0|T\phi_I(x)\phi_I(y)U(T,-T)|0\rangle =$$

 $\left(1-\frac{i\lambda}{4!}\ 3\right.$  $\left.\right) - \frac{i\lambda}{2}$  , (9.13)

where integration over d^4z is assumed.

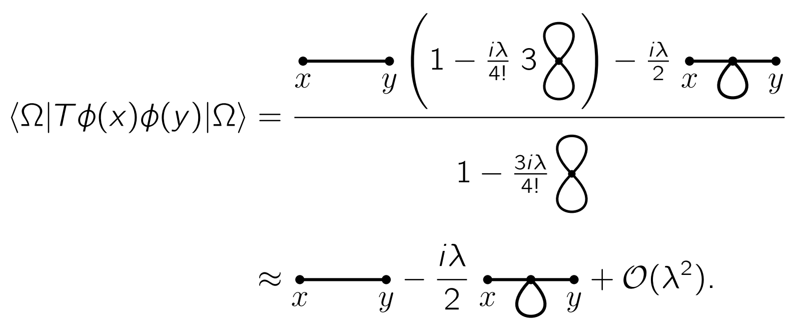
To find the Green's function in the full theory, we need to divide the above equation by the expectation value of the operator $U(T,-T)$. Expanding in λ, we find



(9.14)

where again one has to integrate over all possible values of z.[2]

Hence, to first order in λ, we obtain the following result for the Green's function in $\lambda\phi^4$ theory



(9.15)

An important feature of this formula is that all disconnected diagrams canceled out and the two-point function is described by a sum of *connected diagrams only*. This result is absolutely general; it applies to all Green's functions and all quantum field theories.

There is a certain nomenclature that is used to describe these pictures. The pictures themselves are called Feynman diagrams. Lines in these pictures are called (Feynman) propagators. Points, where several lines come together, are called vertices. More complicated diagrams can be constructed using larger number of external points, more vertices and more propagators.

All contributions to Green's functions can be analyzed following the discussion of the two-point function. For example, when computing the two-point Green's function through $\mathcal{O}(\lambda^3)$, three insertions of the interaction Hamilto-

[2] We will stop showing the coordinate of the point where the lines get together and we will stop mentioning that one has to integrate over this coordinate.

nian produce the following term[3]

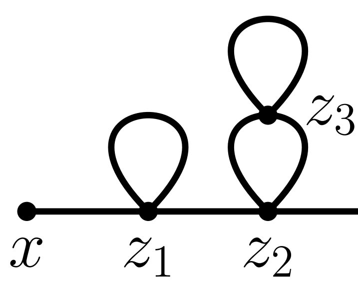
$$\frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3 \langle 0|T\phi(x)\phi(y)\prod_{i=1}^{3}\int \mathrm{d}^4 z_i\ \phi^4(z_i)|0\rangle. \tag{9.16}$$

The $1/3!$ arises from the expansion of the exponential with H_{int} in power series in λ.

To compute the contribution shown in Eq. (9.16) we will have to write it as a sum of all possible contractions of the fields ϕ. Consider the following possible contraction term

$$\begin{aligned}&\frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3 \int \prod_{i=1}^{3} \mathrm{d}z_i\ \langle 0|T\overbracket{\phi(x)\phi(z_1)}\overbracket{\phi(z_1)\phi(z_1)}\\ &\times \overbracket{\phi(z_1)\phi(z_2)}\overbracket{\phi(y)\phi(z_2)}\overbracket{\phi(z_2)\phi(z_3)}\overbracket{\phi(z_2)\phi(z_3)}\overbracket{\phi(z_3)\phi(z_3)}|0\rangle.\end{aligned} \tag{9.17}$$

The Feynman diagram that corresponds to this contraction looks as follows



(9.18)

It is clear that the contraction shown in Eq. (9.17) can be obtained in more than one way starting from the original expression. To calculate how many times this contraction appears, we should do the following. First, we need to contract $\phi(x)$ with one of the three interaction Hamiltonians and $\phi(y)$ with one of the remaining two. There are $3 \times 2 = 3!$ ways to do that. Lets call the coordinate of the the contracted with $\phi(x)$ z_1 and the coordinate of the vertex contracted with $\phi(y)$ z_2. There are four ways to select $\phi(z_1)$ from $\phi(z_1)^4$ to contract with $\phi(x)$ and four ways to select $\phi(z_2)$ to contract with $\phi(y)$. This gives 4×4 possibilities.

[3]To avoid cluttering formulas too much, from now on I will write ϕ instead of ϕ_I.

Furthermore, we need to contract one field with the argument z_1 and another field with the argument z_2. Since there are three fields left in each vertex, this can be done in 3×3 different ways. The remaining two fields in z_1 vertex need to be contracted with themselves, this does not give additional factors. The two fields in the z_2 vertex need to be contracted with two fields from the vertex with z_3 coordinate. This can be done in 4×3 different ways. Hence, we find that this contraction can be obtained in

$$3! \times 4 \times 4 \times 3 \times 3 \times 4 \times 3 = 10\ 368 \tag{9.19}$$

different ways.

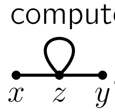
When we perform real calculations, we first draw diagrams and determine the relevant factors later. There are several factors, however, that always appear in the calculation. They are $1/n!$ from the Taylor expansion of $U(T, -T)$ in Taylor series in λ and $1/4!$ from each vertex. Effectively, $1/n!$ from the expansion of $U(T, -T)$ cancels against permutations of vertices that appear in a particular expansion term and each $1/4!$ cancels against $4!$ possible contractions of external fields into H_{int}. To account for these cancellations in an automatic way, we adopt the following rules for calculating Feynman diagrams

1. we will ignore the $1/n!$ factor from the expansion of $U(T, -T)$ in powers of λ;

2. we will assign a factor $-i\lambda$ to each vertex of a diagram *ignoring* the $1/4!$ factor in each of them.

According to these rules, the weight of the diagrams that we just considered is $(-i\lambda)^3$. However, the correct weight factor, that we can compute following earlier discussion, is $10368/3!/(4!)^3(-i\lambda)^3 = (-i\lambda)^3/8$.

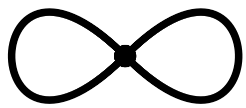
The additional factor (eight) by which we have to divide the naive weight to get the correct one is called the *symmetry factor*. It counts a number of ways to interchange components of a diagram[4] without changing it.

To understand how to compute the symmetry factors of individual diagrams, consider a diagram 

. Here, there is a line that starts and ends at the same point so that we can interchange the starting point and the final

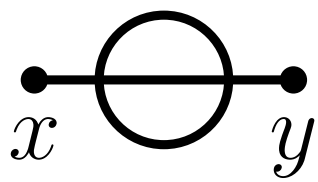
[4] Starting and ending points for each line as well as lines and vertices themselves.

point of the line without changing the diagram. Hence, a symmetry factor is 2 because there are two ways to assign the starting and ending points to this line that do not change the diagram. More complicated examples follow.



, $S = 8$

(9.20)



, $S = 3! = 6$

In the first case there are two ways assign the beginning and the end for each of the two closed lines and there is a possibility to interchange the left loop and the right loop. Altogether, the symmetry factor is 8. In the second case there are $3! = 6$ ways to interchange lines that connect the two vertices without changing the diagram.

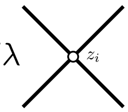
We are now ready to associate a set of rules that will allow us to associate mathematical expressions with each graph (Feynman diagram) that are needed to compute expressions of the type

$$\frac{\langle 0|T\phi(x_1)\phi(x_2)...\phi(x_m)\ e^{-i\lambda\int \mathrm{d}\tau\tau H_{\mathrm{int}}(\tau)}|0\rangle}{\langle 0|Te^{-i\lambda\int \mathrm{d}\tau\tau H_{\mathrm{int}}(\tau)}|0\rangle}. \tag{9.21}$$

To compute $\mathcal{O}(\lambda^N)$ contributions to this quantity in perturbation theory, we do the following.

- We draw external points x_1, x_2, ..., x_m;

- We introduce N vertices and associate with each vertex the factor $-i\lambda \mathrm{d}^4 z_i$

$-i\lambda$ 

; (9.22)

- We draw all *connected* diagrams that contribute to the expansion of Green's function to a desired order by connecting N vertices either among themselves or connecting them to external points $x_1, .., x_N$;

- We associate a Feynman propagator with each line of the diagram;

- We integrate over coordinate z_i of each vertex;

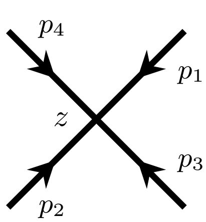
- For each diagram, we compute the symmetry factor by counting the number of symmetry transformations that do not change a diagram. We divide contributions of each diagram by its symmetry factor.

- We sum over all diagrams to obtain the final expression for the Green's function.

These rules allow us to compute the *position-space* Green's functions. As we will see in what follows quite often we require the momentum-space representation for Green's functions. To obtain it, we do the following. Imagine that we start with a position space expression and replace all Green's functions with

$$D_F(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i0} \, e^{-ip(x-y)}. \tag{9.23}$$

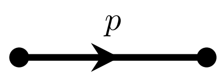
Once this is done, the integration over each vertex' coordinate becomes trivial. Indeed, in each vertex we have



$$\leftrightarrow \quad \int \mathrm{d}^4 z \, e^{-ip_1 z - ip_2 z - ip_3 z - ip_4 z} = (2\pi)^4 \delta^{(4)}(p_4 + p_1 + p_2 + p_1), \tag{9.24}$$

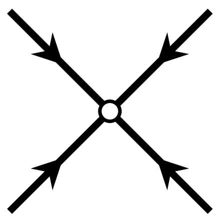
so that in each vertex the total four-momentum is conserved. Hence, to write expressions for Feynman diagrams that contribute to Green's functions, we can do the following

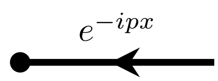
- associate the momentum space propagator with each line



$$= \frac{i}{p^2 - m^2 + i0}.$$

  Of course each line has to be given a different momentum and the direction of the momentum flow has to be chosen.

- associate  $= -i\lambda$ with each vertex;

- associate a factor 

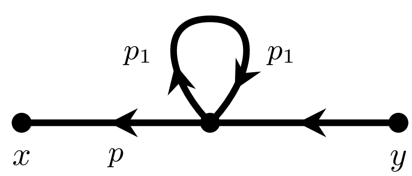
with each of the external points and the momentum that flows into the point.

- divide by the symmetry factor;

- assign momentum conserving δ-functions $((2\pi)^3\delta^{(4)}(\sum p_i))$ to each vertex; make sure that in-flowing and out-flowing momenta appear in the delta-function with different signs.

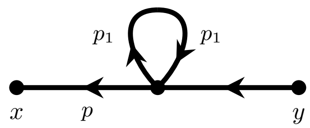
- integrate over all momenta.

As an example, we write a corresponding expression for the following Feynman diagram



. (9.25)

The symmetry factor is 1/2, so that



$$= \frac{-i\lambda}{2}\int \frac{d^4p}{(2\pi)^4}\,\frac{ie^{-ip(x-y)}}{(p^2-m^2+i0)^2} \times \int \frac{d^4p_1}{(2\pi)^4}\frac{i}{p_1^2-m^2+i0}. \tag{9.26}$$

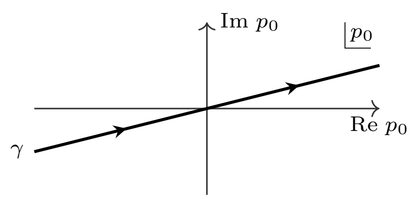
We will now go back to the issue that we mentioned earlier, namely that, for each vertex, we have to integrate over z^0, the time component of four-vector z, from $-T$ to T where $T = T_0(1-i\epsilon)$, and $T_0 \to \infty$. In each vertex we have $e^{-i(p_1+p_2+p_3+p_4)z}$ to integrate. Hence, a typical z_0 integral will look like

$$\int_{-T}^{T} dz_0 e^{-ip_0z_0}, \tag{9.27}$$

and we would like this integral to vanish for non-vanishing p_0-values. However, convergence of this integral depends on the sign of p_0 and z_0 since the real part of the argument of the exponential function is

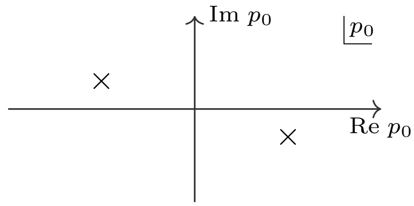
$$-i(\pm)T_0(-i\epsilon)p_0 = \mp T_0p_0\epsilon, \tag{9.28}$$

One way to enforce a uniform convergence is to make the phase pure imaginary; this can be achieved if p_0 will have an imaginary part that compensates the imaginary part in $T_0(1-i\epsilon)$. Then we should imagine that $p_0 \to p_0/(1-i\epsilon) \approx p_0(1+i\epsilon)$. This implies that points with $p_0 > 0$ receive small *positive* imaginary parts and points with $p_0 < 0$ receive small *negative* imaginary parts. Since, for computing the Green's function, we are supposed to integrate over all values of p_0, the convergence (and the limit $T_0 \to \infty$ is ensured if we integrate over the following contour γ



(9.29)

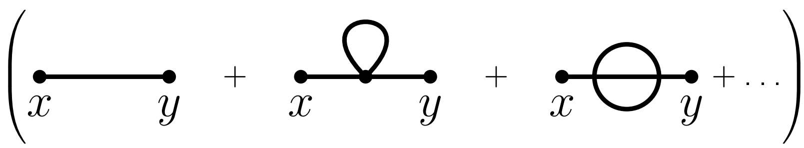
It is important to emphasize that this deformation can be done seamlessly *because* we employ *Feynman propagators* whose $i\epsilon$ prescription ensures that the location of the poles in p_0 complex plane is as shown below



. (9.30)

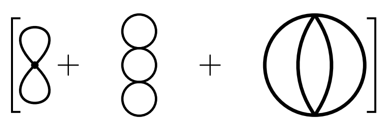
Next, I would like to mention the mechanism by which cancellation of disconnected diagrams happen. It is relatively straightforward to see that for the two-point function the sum of all disconnected diagrams exponentiates, which means that the following formula holds

$$\langle 0|T\phi(x)\phi(y)e^{-i\int \mathrm{d}\tau H_{\text{int}}(\tau)}|0\rangle =$$



(9.31)

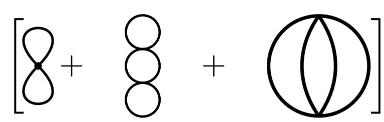
$\times \exp$



.

Since

$$\langle 0|Te^{-i\int \mathrm{d}\tau H_{\text{int}}(\tau)}|0\rangle = \exp$$

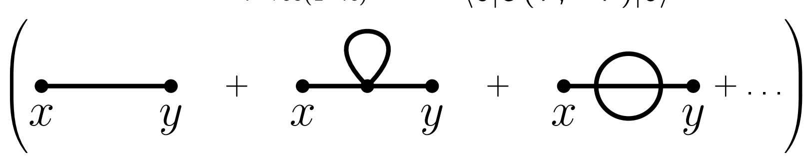


, (9.32)

we find

$$\langle \Omega|T\phi(x)\phi(y)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\phi_I(x)\phi_I(y)U(T,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle}$$

$=$

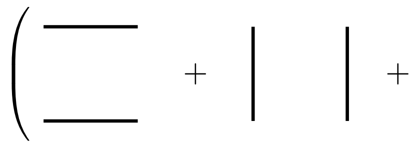


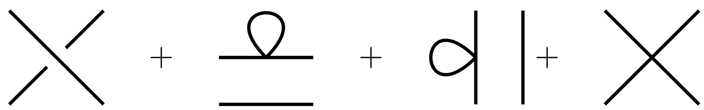
. (9.33)

In case of more complex correlation functions, the same formula holds. Note, however, that in this case "connected" does not mean "fully connected", as can be seen already by computing Green's function in a free theory.

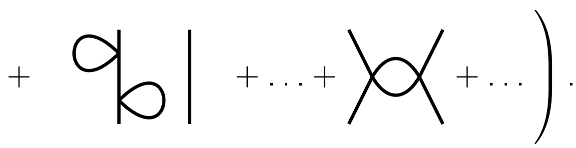
For example, the expansion of the four-point Green's function looks as follows

$$\langle\Omega|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|\Omega\rangle =$$





(9.34)



(9.35)