TTP1
Lecture 10

Kirill Melnikov
TTP KIT
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10 S-matrix

A very important source of information about particle interactions is scattering experiments. These experiments work as follows: two beams of particles are assembled and sent towards each other. When these beams pass each other, particles in the beams interact and scatter. We observe scattered particles and study their numbers, their types, energies and spins, and their angular distributions. From this information, we try to understand as much as we can about their interactions. This, of course, is only possible if we know how to describe scattering in quantum field theory, connecting properties of scattered particles with the underlying Lagrangian.

To understand how to do this, we consider the following problem. We imagine that we have a scalar field theory described by the following Lagrangian

$$L = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - V(t,\phi), \tag{10.1}$$

where

$$V(t,\phi) = V(\phi)\theta(T_0, t). \tag{10.2}$$

The function $\theta(T_0, t)$ equals to one on the interval $-T_0 < t < T_0$ but adiabatically (very slowly) vanishes for smaller and larger values of t. Hence, there exists $T \gg T_0$ such that for $|t| > T$ our theory is, effectively, a free theory. We also assume that, because of the adiabatic switching of the interaction, the ground state of the theory does not change and equals to the exact one at $t = \pm\infty$.

Because of that, we can define the Hilbert space of the theory at $|t| > T$ *exactly*. To this end, we write

$$\begin{aligned} \phi(t > T, \vec{x}) &= \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3\sqrt{2E_k}} \left(a_{\vec{k}}(T) e^{-ik_\mu x^\mu} + a^+_{\vec{k}}(T) e^{ik_\mu x^\mu} \right), \\ \phi(t < -T, \vec{x}) &= \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3\sqrt{2E_k}} \left(a_{\vec{k}}(-T) e^{-ik_\mu x^\mu} + a^+_{\vec{k}}(-T) e^{ik_\mu x^\mu} \right). \end{aligned} \tag{10.3}$$

We now imagine that the scattering experiments can be described in the following way. At $t = -\infty$ we have a collection of free particles that are constructed using $a^+_{\vec{k}}(-T)$ operators acting on vacuum state $|0\rangle$. Then, as we let the time flow, scattering occurs and, by $t = \infty$, the initial state transforms into a collection of particles that fly in different directions. These particles can be described by a final state $|f\rangle$ defined using $a^+_{\vec{k}}(T)$ operators acting on the vacuum state.

To make this explicit, consider a typical (inelastic) scattering process where two particles with momenta $p_{1,2}$ produce n particles with momenta $p_{3,4,\ldots,n}$, i.e.

$$p_1 + p_2 \to p_3 + p_4 + \ldots + p_n. \tag{10.4}$$

We assume that $p_i \neq p_j$, for $i \neq j$ and that $p_i^2 = m^2$ for all i's. According to our discussion, to describe this process, we require the following matrix element

$$iT_{fi} = \langle f|i\rangle = \prod_{i=1}^{n} \sqrt{2E_i}\langle 0|a_{p_3}(T)\ldots a_{p_n}(T)\ a^+_{p_1}(-T)a^+_{p_2}(-T)|0\rangle. \tag{10.5}$$

There are two comments to make about this formula. First, the matrix element between initial and final states just described is usually called the S-matrix and is denoted by S_{fi}. We, on the other hand, wrote it as iT_{fi}. The matrix $\hat{T}$ is called the *transfer matrix*. Its relation to the S-matrix is as follows

$$S = \hat{1} + i\hat{T}. \tag{10.6}$$

The identity operator in the above formula describes processes *without scattering*; we explicitly exclude such processes by our assumption that none of the momenta in the scattering process are the same.

The second comment is about the prefactor $\prod \sqrt{2E_i}$ in Eq. (10.5). These are introduced to work with external states whose normalization is invariant under Lorentz transformations.

We would like to relate the matrix element T_{fi} to a quantity that depends on *exact* quantum fields ϕ rather than the creation and annihilation operators at very large or very small times. To do so, we consider the following integral, for $p^2 = m^2$,

$$\begin{aligned} I &= i\int \mathrm{d}^4x\ e^{ip_\mu x^\mu}\left(\partial^2 + m^2\right)\phi(x) \\ &= i\int \mathrm{d}^4x\ e^{ix_\mu p^\mu}\left(\partial_t^2 - \vec{\partial}^2 + m^2\right)\phi(x). \end{aligned} \tag{10.7}$$

We assume that $\phi(x)$ vanishes if $|\vec{x}| \to 0$ and integrate by parts in Eq.(10.7). Then

$$\int \mathrm{d}^4x\ e^{ix_\mu p^\mu}\ \vec{\partial}\cdot\vec{\partial}\ \phi(x) = -\int \mathrm{d}^4x\ e^{ix_\mu p^\mu}\vec{p}^2\ \phi(x). \tag{10.8}$$

Hence,

$$I = i\int \mathrm{d}^4x\; e^{ip_\mu x^\mu}\left(\partial^2 + m^2\right)\phi(x) = i\int \mathrm{d}^4x\; e^{ip_\mu x^\mu}\left(\partial_t^2 + E_{\vec{p}}^2\right)\phi(x), \quad (10.9)$$

where $E_{\vec{p}}^2 = \vec{p}^2 + m^2 = p_0^2$. To proceed further, we note that, if $p_0 = E_{\vec{p}}$, the following identity holds,

$$e^{ip_\mu x^\mu}\left(\partial_t^2 + E_{\vec{p}}^2\right)\phi(x) = -i\partial_t\left[e^{ip_\mu x^\mu}\left(i\partial_t + E_{\vec{p}}\right)\phi(x)\right]. \quad (10.10)$$

To check it, we compute the right-hand side explicitly. We find

$$\begin{aligned} -i\partial_t\left[e^{ip_\mu x^\mu}\left(i\partial_t + E_{\vec{p}}\right)\phi(x)\right] &= e^{ip_\mu x^\mu}\left(\partial_t^2 - iE_{\vec{p}}\partial_t\right)\phi \\ &+ e^{ip_\mu x^\mu}E_{\vec{p}}\left(i\partial_t + E_{\vec{p}}\right)\phi \\ &= e^{ip_\mu x^\mu}\left(\partial_t^2 + E_p^2\right)\phi. \end{aligned} \quad (10.11)$$

We use Eq.(10.10) in Eq.(10.9) and find

$$\begin{aligned} I &= i\int \mathrm{d}^4x(-i)\partial_t\left[e^{ip_\mu x^\mu}\left(i\partial_t + E_{\vec{p}}\right)\phi(x)\right] \\ &= \int \mathrm{d}^3x\; e^{ip_\mu x^\mu}\left(i\partial_t + E_{\vec{p}}\right)\phi(t,x)\,\Big|_{t=-\infty}^{t=+\infty}. \end{aligned} \quad (10.12)$$

At $t = \pm\infty$, $\phi(t,\vec{x})$ is written using its asymptotic form, Eq.(10.3). We find

$$\begin{aligned} &\lim_{t\to\pm\infty} e^{ip_\mu x^\mu}(i\partial_t + E_{\vec{p}})\phi(t,\vec{x}) = \lim_{t\to\pm\infty} e^{ip_\mu x^\mu}\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3\sqrt{2k_0}}\times \\ &\left\{a_{\vec{k}}(\pm T)(k_0 + E_{\vec{p}})e^{-ik_\mu x^\mu} + a_{\vec{k}}^+(\pm T)\left(E_{\vec{p}} - k_0\right)e^{ik_\mu x^\mu}\right\}. \end{aligned} \quad (10.13)$$

We use this equation in Eq.(10.12) and integrate over $\vec{x}$. We find

$$I = I_+ - I_-, \quad (10.14)$$

where

$$\begin{aligned} I_\pm = \frac{1}{\sqrt{2k_0}}\Bigg[&a_{\vec{p}}(\pm T)(k_0 + E_p)e^{i(p_0-k_0)x_0} \\ &+ a_{-\vec{p}}^+(\pm T)(E_{\vec{p}} - k_0)e^{i(p_0+k_0)x_0}\Bigg]_{k_0=E_{\vec{p}}=p_0,\,x_0=\pm\infty}, \end{aligned} \quad (10.15)$$

so that

$$I = \sqrt{2E_{\vec{p}}}\left(a_{\vec{p}}(T) - a_{\vec{p}}(-T)\right). \tag{10.16}$$

Hence,

$$i\int \mathrm{d}^4x\; e^{ip_\mu x^\mu}\left(\partial^2 + m^2\right)\phi(x) = \sqrt{2E_{\vec{p}}}\left(a_{\vec{p}}(T) - a_{\vec{p}}(-T)\right), \tag{10.17}$$

and similarly,

$$-i\int \mathrm{d}^4x\; e^{-ip_\mu x^\mu}\left(\partial^2 + m^2\right)\phi(x) = \sqrt{2E_{\vec{p}}}\left(a^+_{\vec{p}}(T) - a^+_{\vec{p}}(-T)\right). \tag{10.18}$$

We would like to use Eqs.(10.17,10.18) to construct the matrix element iT_{fi} in Eq.(10.5). For example, we can use the following equation

$$-i\int \mathrm{d}^4x\; e^{-ip_{1,\mu} x^\mu}\left(\partial^2 + m^2\right)\phi(x) = \sqrt{2E_{\vec{p}}}\left(a^+_{\vec{p}_1}(T) - a^+_{\vec{p}_1}(-T)\right), \tag{10.19}$$

to express $a^+_{\vec{p}_1}(-T)$ through an integral of ϕ. The problem is that upon doing that, we will also obtain $a^+_{\vec{p}_1}(T)$ in the relation between $a^+_{\vec{p}_1}(-T)$ and ϕ, and this is not what is needed in Eq.(10.5). A trick that is used to get rid of $a^+_{\vec{p}_1}(T)$ and $a_{\vec{p}_{3,...,N}}(-T)$ is to employ properties of the vacuum state $|0\rangle$ since $a_{\vec{p}}|0\rangle$ and $\langle 0|a^+_{\vec{p}}$ vanish. What we need to do is to ensure that all "unwanted" creation and annihilation) operators appear to the left (to the right) of all other operators in Eq.(10.5). To accomplish this, we introduce the time ordering into the definition of the S-matrix element and write

$$iT_{fi} = \sqrt{2E_1\, 2E_2\, ... 2E_n}\langle 0|T\left[a_{p_3}(T)...a_{p_n}(T)\; a^+_{p_1}(-T)a^+_{p_2}(-T)\right]|0\rangle. \tag{10.20}$$

The time ordering ensures that operators that depend on the largest time appear to the left of all other operators and operators that depend on the smallest time appear to the right of all other operators. Then, since $a_{\vec{p}}|0\rangle = 0$ and $\langle 0|a^+_{\vec{p}} = 0$, we can replace all the a and a^+ operators in the formula for S_{fi} with integrals over fields ϕ since additional terms $a^+_{\vec{p}}(T)$ and $a_{\vec{p}}(-T)$ provide vanishing contributions because of the T-product in Eq.(10.20). We therefore find

$$iT_{fi} = i^n\int\prod_{i=1}^{n}\mathrm{d}x_i\; e^{i\left(\sum\limits_{j=3}^{n}p_jx_j - p_1x_1 - p_2x_2\right)}\prod_{i=1}^{n}\left(\partial_i^2 + m^2\right)\langle 0|T\phi(x_1)...\phi(x_n)|0\rangle. \tag{10.21}$$

Note that in writing this formula we commuted the differential operators $\partial_i^2 + m^2$ with the time-ordering operator T. While this is not legitimate in general, it is allowed for computing the matrix element T_{fi} because additional terms will lead to disconnected contributions (see below).

The above equation relates the S-matrix elements with the Green's functions in the *exact theory*. Usually, it is not possible to find T_{fi} exactly but we have discussed how to construct exact Green's functions in perturbation theory when treating the strength of the interaction as a small parameter. If we combine this information with Eq. (10.21), we should be able to find rules to compute S_{fi} as a sum of Feynman diagrams.

We will start with considering two examples. The first one is the $2 \to 2$ scattering in a theory with the self-interaction $-\lambda/4!\phi^4$ To zeroth order in λ,

$$\begin{aligned} &\langle 0|T\phi(x_1)...\phi(x_4)|0\rangle = D_F(x_1 - x_2)D_F(x_3 - x_4) \\ &+ D_F(x_1 - x_3)D_F(x_2 - x_4) + D_F(x_1 - x_4)D_F(x_2 - x_3). \end{aligned} \tag{10.22}$$

To use this result in Eq. (10.21), we represent each propagator as

$$D_F(x - y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip_\mu (x-y)^\mu}. \tag{10.23}$$

Taking for definiteness the first term in Eq. (10.22), we write

$$\begin{aligned} &\prod_{i=1}^{4} \left(\partial_i^2 + m^2\right) D_F(x_1 - x_2)D_F(x_3 - x_4) \\ &= \int \frac{d^4 k_1}{(2\pi)^4} \frac{i(-k_1^2 + m^2)^2}{k_1^2 - m^2} e^{-ik_1(x_1 - x_2)} \int \frac{d^4 k_2}{(2\pi)^4} \frac{i(-k_2^2 + m^2)^2}{k_2^2 - m^2} e^{-ik_2(x_3 - x_4)}. \end{aligned} \tag{10.24}$$

Then, if we integrate such a term over x_1, x_2, x_3, x_4 with the exponential function $e^{ip_3x_3 + ip_4x_4 - ip_1x_1 - ip_2x_2}$, as required by Eq. (10.21), we find that the result is proportional to

$$\delta^{(4)}(p_1 + p_2)\delta^{(4)}(p_3 + p_4). \tag{10.25}$$

This term vanishes because for a scattering process $p_1 \neq -p_2$ and $p_3 \neq -p_4$. A similar analysis for all other term in Eq. (10.22) reveals that they are proportional to

$$\delta^{(4)}(p_1 - p_3)\delta^{(4)}(p_2 - p_4), \tag{10.26}$$

or

$$\delta^{(4)}(p_1 - p_4)\delta^{(4)}(p_2 - p_3). \tag{10.27}$$

Again, since we assumed that $p_3 \neq p_1$ and $p_2 \neq p_4$, such terms do not contribute to the matrix element T_{fi}. We therefore conclude that

$$T_{fi} \sim \mathcal{O}(\lambda), \tag{10.28}$$

which corresponds to our intuition that scattering requires interactions.

Hence, to find the first non-trivial term in T_{fi}, we need to expand the Green's function to first oder in λ; the result reads

$$-\frac{i\lambda}{4!}\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\int \mathrm{d}x\ \phi^4(x)|0\rangle. \tag{10.29}$$

We can write this Green's function as a sum of Feynman diagrams. Among these diagrams, there will be one, where all ϕ-fields are contracted with fields in the interaction Hamiltonian and many other diagrams where at least two fields out of the set $\phi(x_1), \phi(x_2), \phi(x_3), \phi(x_4)$ are contracted with each other. It is clear that, after integration over $x_{1,2,..,4}$ all latter terms will again produce a δ-function of the difference or the sum of two external momenta; as we already mentioned, all such contributions should be set to zero when evaluating the transfer matrix T_{fi}.

Hence, for computing the scattering matrix element, we should focus on the diagram where all "external" fields get contracted with the fields in the interaction Hamiltonian. It reads

$$-\frac{i\lambda}{4!}\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\int \mathrm{d}x\ \phi^4(x)|0\rangle \to -i\lambda\int \mathrm{d}^4x\ \prod D_F(x_i - x). \tag{10.30}$$

We then use this expression in Eq. (10.21), use momentum representation for each of the propagators, integrate over $x_{1,2,..,4}$ and x and find that the above expression evaluates to

$$iT_{fi} = -i\lambda(2\pi)^4\delta^{(4)}(p_3 + p_4 - p_1 - p_2). \tag{10.31}$$

From the point of view of Feynman diagrams, the above result is constructed by considering all *fully-connected* Feynman diagrams that contribute to the Green's function at this order of perturbation theory, discarding propagators that are related to their external legs (we refer to this as *amputated* diagrams) and then multiplying the result by a $(2\pi)^4$ and a δ-function that enforces the overall momentum conservation. Since the latter factors are present in the expression for a matrix element of *any* scattering process, it is customary to introduce a new scattering matrix element $\mathcal{M}_{fi}$ defined as

$$iT_{fi} = i\mathcal{M}_{fi}(2\pi)^4\delta^{(4)}(\sum_{i=3}^{n} p_i - p_1 - p_2). \tag{10.32}$$

Then, for the $2 \to 2$ scattering in λ^4 theory, we find

$$i\mathcal{M}_{fi} = -i\lambda + \mathcal{O}(\lambda^2). \tag{10.33}$$

We will now consider a $2 \to 2$ scattering in a theory with the self-interaction $g/3!\phi^3$. In this case scattering appears at *second* order in the expansion of perturbation theory in powers of g. The corresponding contribution to the Green's function is

$$\frac{1}{2!}\left(\frac{-ig}{3!}\right)^2 \langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\int \mathrm{d}^4z\ \phi(z)^3 \int \mathrm{d}^4y\phi(y)^3|0\rangle. \tag{10.34}$$

We need to contract external fields into interaction Hamiltonians and then contract the remaining fields in the interaction Hamiltonians between themselves. It is also clear that one has to contract two external fields into one vertex and two other fields into another vertex, to get a fully-connected diagram. There are three ways to do so and, as the result, three diagrams contribute.

Suppose we consider the case when $\phi(x_1)$ and $\phi(x_2)$ are contracted into one vertex and $\phi(x_3)$ and $\phi(x_4)$ into the other one. This contribution reads

$$(-ig)^2 \int \mathrm{d}^4z\mathrm{d}^4y\ D_F(x_1 - z)D_F(x_2 - z)\ D_F(x_3 - y)D_F(x_4 - y)D_F(z - y). \tag{10.35}$$

We use this result in Eq. (10.21), employ momentum representation for Feynman propagators, and obtain the following contribution to the scattering matrix

$$(-ig)^2 \frac{i}{(p_1 + p_2)^2 - m^2}(2\pi)^4\delta(p_1 + p_2 - p_3 - p_4). \tag{10.36}$$

When a different contraction is chosen, the result appears to be the same except that the remaining propagator differs. The full result for the $\mathcal{M}_{fi}$

amplitude reads

$$i\mathcal{M}_{fi} = -ig^2 \left[\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2} \right], \qquad (10.37)$$

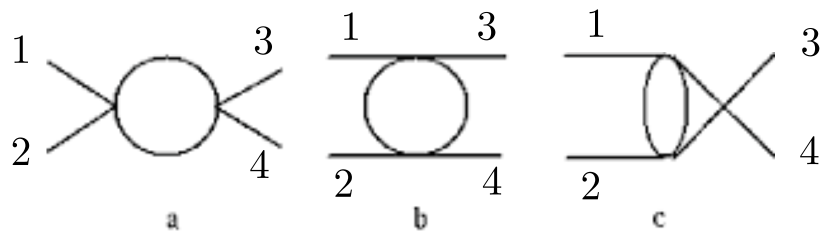
where we introduced the so-called Mandelstam variables s, t, u to describe the $2 \to 2$ scattering

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2. \qquad (10.38)$$

It should be clear from the above examples what happens in the general case. Here is a set of rules that allow us to write down mathematical expressions for scattering amplitudes $i\mathcal{M}_{fi}$ in a quantum field theory of a scalar field:

- For a given process, draw all Feynman diagrams that contribute to the relevant Green's function at the desired order in perturbation theory; assign relevant symmetry factors;
- Keep fully-connected diagrams, discard all other;
- Remove external lines (i.e. remove propagators and everything else that affects external lines only); assign relevant momenta for each incoming and outgoing line;
- Assign $(2\pi)^4\delta^{(4)}(\sum k_i)$ (in addition to the coupling constant) for each vertex, where $\{k_i\}$ is a set of incoming momenta;
- Assign $\mathrm{d}^4k/(2\pi)^4$ for each of the *internal* propagators;
- Perform as many trivial integrations over momenta assigned to internal propagators, removing momentum conserving δ-functions, as possible;
- Extract an overall four-momentum conserving δ-function $(2\pi)^4\delta(\sum_{i=3}^{n} p_i - p_1 - p_2)$ and discard it.

As an example, consider again the $2 \to 2$ scattering in $\lambda\phi^4/4!$ theory. We have earlier found that to first order in λ, the scattering amplitude $i\mathcal{M}_{fi}$ is



given by $-i\lambda$. What happens if we consider the scattering at the next order in perturbation theory?

Following the above rules, it is easy to convince oneself that we have to consider just three diagrams shown in the above figure. Consider the first diagram. The corresponding expression reads

$$\begin{aligned}&\frac{1}{2}\left(-i\lambda\right)^2\int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\;\frac{i}{k_1^2-m^2}\frac{i}{k_2^2-m^2}\times\\&(2\pi)^4\delta^{(4)}(p_1+p_2-k_1-k_2)\;(2\pi)^4\delta^{(4)}(k_1+k_2-p_3-p_4).\end{aligned}\qquad(10.39)$$

The factor 1/2 in front is the symmetry factor.

We can simplify the above expression by integrating over k_2 and removing the first δ-function. This means that in Eq. (10.39) we should replace

$$\frac{\mathrm{d}^4k_2}{(2\pi)^4}\;(2\pi)^4\delta^{(4)}(p_1+p_2-k_1-k_2)\to 1,\qquad(10.40)$$

and set

$$k_2\to p_1+p_2-k_1,\qquad(10.41)$$

in all remaining expressions. Upon doing that, we find

$$(2\pi)^4\delta^{(4)}(p_{12}-p_{34})\frac{(-i\lambda)^2}{2}\int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{i}{k_1^2-m^2}\frac{i}{(p_{12}-k_1)^2-m^2},\qquad(10.42)$$

where $p_{12}=p_1+p_2$ and $p_{34}=p_3+p_4$.

It should be clear now how the above calculation generalizes and, without further ado, we write down for the $2\to 2$ scattering amplitude through $\mathcal{O}(\lambda^2)$ in $\lambda\phi^4/4!$ theory. The result reads

$$i\mathcal{M}=-i\lambda+\frac{(-i\lambda)^2}{2}\sum_{j=2}^{4}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-m^2}\frac{i}{(k+p_{1j})^2-m^2}.\qquad(10.43)$$

where $p_{13}=p_1-p_3$ and $p_{14}=p_1-p_4$.