TTP1
Lecture 11

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15. Juni 2023

11 Feynman rules for fermions

In this lecture we would like to discuss how Green's functions and scattering amplitudes can be computed in theories with Dirac fermion. For definiteness, consider a theory defined by the action

$$S = S_D + S_s + S_{\text{int}}, \tag{11.1}$$

where S_D is the action of the Dirac field

$$S_D = \int \mathrm{d}^4x \, \left(\bar{\psi}(x) i\partial_\mu \gamma^\mu \psi(x) - m\, \bar{\psi}(x)\psi(x) \right), \tag{11.2}$$

S_s is the action of the scalar field

$$S_s = \frac{1}{2}\int \mathrm{d}^4x \, \left(\partial_\mu \phi \, \partial^\mu \phi - m^2 \phi^2 \right), \tag{11.3}$$

and the last term is the interaction

$$S_{\text{int}} = \int \mathrm{d}^4x \, \left[-g\phi(x)\bar{\psi}(x)\psi(x) - \frac{\lambda}{4!}\phi(x)^4 \right]. \tag{11.4}$$

Similar to the case of a scalar field, we define an interaction Hamiltonian

$$H_{\text{int}} = \int \mathrm{d}^3\vec{x} \, \left[g\phi(t,\vec{x})\bar{\psi}(t,\vec{x})\psi(t,\vec{x}) + \frac{\lambda}{4!}\phi(x)^4 \right] \tag{11.5}$$

switch to the interaction representation for both ϕ and ψ fields, and obtain the following result for an arbitrary Green's function in an interaction theory

$$\begin{aligned} &\langle \Omega | T \psi_{\alpha_1}(x_1)...\psi_{\alpha_n}(x_n)..\bar{\psi}_{\beta_1}(y_1)..\bar{\psi}_{\beta_m}(y_m)...\phi(z_1)...\phi(z_l)|\Omega\rangle \\ &= \frac{\langle 0 | T \psi_{\alpha_1}(x_1)...\psi_{\alpha_n}(x_n)..\bar{\psi}_{\beta_1}(y_1)..\bar{\psi}_{\beta_m}(y_m)...\phi(z_1)...\phi(z_l) U(T,-T)|0\rangle}{\langle 0 | U(T,-T) | 0 \rangle}, \end{aligned} \tag{11.6}$$

where

$$U(T,-T) = e^{-i \int\limits_{-T}^{T} \mathrm{d}\tau \, H_{\text{int}}(\tau)}, \tag{11.7}$$

and on the right hand side of Eq. (11.6) all the fields are supposed to be treated as fields in the "interaction representation" (meaning that they are known functions of creation and annihilation operators).

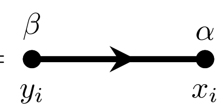
Computation of the Green's function in Eq. (11.6) follows the same rules as what has been discussed for the the scalar field theory. Namely, we expand the exponential function in $U(T,-T)$ in powers of the coupling constants g and λ, move all creation operators next to $\langle 0|$ and all annihilation operators next to $|0\rangle$, and make use of the fact that $\langle 0|a^+ = 0$ and $a|0\rangle = 0$.

We have discussed how to use the Wick theorem to do this efficiently for scalar fields. An extension of Wick theorem to the case of fermion is straightforward *but* we need to account for the fact that fermion fields *anticommute*. One can show that the Wick theorem for fermions reads

$$T(\psi_1 ... \psi_n) = N(\psi_1 ... \psi_n + \text{contractions})_{(-1)^{\text{perm}}}, \tag{11.8}$$

where ψ_i is a generic notation for ψ and/or $\bar{\psi}$ and the subscript in the above equation reminds us about the fact that there are relative signs between different terms in this expression that corresponds to (-1) raised to the power which equals to the number of permutations that are needed to get a particular term to match the ordering of the fermion fields on the left hand side. Contractions of fermion fields correspond to Feynman propagator

$$\overbracket{\psi_\alpha(x_i)\bar{\psi}_\beta}(y_i) = \langle 0|T\psi_\alpha(x_i)\bar{\psi}_\beta(y_i)|0\rangle =$$



$$= S_{\alpha\beta}(x_i - y_i) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \left(\frac{i}{\hat{p} - m + i0} \right)_{\alpha\beta} e^{-ip_\mu (x_i - y_i)^\mu}, \tag{11.9}$$

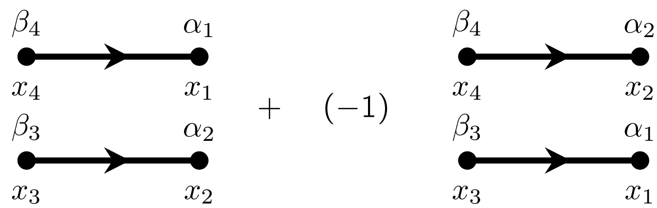
and the only possible contractions are those of ψ with $\bar{\psi}$.

As an illustration of the Wick theorem, we re-write the time-ordered product of four fermion fields using Wick theorem. We use the short-hand notation $\psi_i = \psi(x_i)$ and find

$$\begin{aligned} T\psi_1\psi_2\bar{\psi}_3\bar{\psi}_4 = N\Big(& \psi_1\psi_2\bar{\psi}_3\bar{\psi}_4 - \overbracket{\psi_1\bar{\psi}}_3\psi_2\bar{\psi}_4 + \overbracket{\psi_1\bar{\psi}}_4\psi_2\bar{\psi}_3 \\ & + \overbracket{\psi_2\bar{\psi}}_3\psi_1\bar{\psi}_4 - \overbracket{\psi_2\bar{\psi}}_4\psi_1\bar{\psi}_3 - \overbracket{\psi_1\bar{\psi}}_3\overbracket{\psi_2\bar{\psi}}_4 + \overbracket{\psi_1\bar{\psi}}_4\overbracket{\psi_2\bar{\psi}}_3 \Big). \end{aligned} \tag{11.10}$$

Note that if we take a vacuum expectation value of the above expression, all terms with uncontracted fields drop out and we are left with two fully contracted contributions. These are represented by two diagrams where either points

x_1, x_3 and x_2, x_4, or x_1, x_4 and x_2, x_3 are connected by fermion propagators



(11.11)

Note the relative *minus* sign between these diagrams. Note also that in variance with the scalar case, for fermions the initial point and the final point of a propagator are not the same since one of them corresponds to the field ψ and the other one to $\bar{\psi}$. Because of that, a fermion line has a *fermion flow direction* associated with it. This direction is such that, for a particular contraction, a line starts at the argument of $\bar{\psi}$ and ends at the argument of ψ.

To understand this better, we go back to the calculation of the *scattering matrix* and generalize what we did for scalar fields to fermions. We begin by considering fermions (as opposed to anti-fermions); then we need to find how to express the operators $a_{s,\vec{p}}$ and $a^{+}_{s,\vec{p}}$ through the fermion fields ψ and $\bar{\psi}$. Similar to the scalar field case, our starting point is the integral

$$I = \int \mathrm{d}^4x \; e^{ipx} \bar{u}_{\vec{p},s} \left(-i\hat{\partial} + m\right) \psi(x), \tag{11.12}$$

where $p^2 = m^2$. Writing $-i\hat{\partial} = -i\gamma^0\partial_t - i\vec{\gamma}\cdot\vec{\nabla}$ and integrating by parts over $\vec{x}$, we find

$$I = \int \mathrm{d}^4x \; e^{ipx} \bar{u}_{\vec{p},s} \left(-i\gamma_0\partial_t + \vec{\gamma}\cdot\vec{p} + m\right) \psi(x). \tag{11.13}$$

The spinor $\bar{u}_{s,\vec{p}}$ satisfies the Dirac equation $\bar{u}_{s,\vec{p}}(\hat{p} - m) = 0$. Therefore,

$$\bar{u}_{s,\vec{p}} \left(\vec{\gamma}\cdot\vec{p} + m\right) = \bar{u}_{s,\vec{p}} \, \gamma_0 p_0. \tag{11.14}$$

It follows that

$$\begin{aligned} I &= \int \mathrm{d}^4x \; e^{ipx} \bar{u}_{\vec{p},s} \left(-i\gamma_0\partial_t + \gamma_0 p_0\right) \psi(x) \\ &= \int \mathrm{d}^4x \; (-i\partial_t) e^{ipx} \bar{u}_{\vec{p},s}\gamma_0\psi(x) = -i \int \mathrm{d}^3\vec{x} e^{ip_\mu x^\mu} \bar{u}_{\vec{p},s}\gamma_0\psi(x) \Bigg|_{t=-\infty}^{t=+\infty}. \end{aligned} \tag{11.15}$$

In analogy with the scalar field case, at $t = \pm\infty$ the fermion field is represented by creation and annihilation operators of a free field theory. Therefore, we find

$$\begin{aligned} I = -i \int \mathrm{d}^3\vec{x} e^{ip_\mu x^\mu} \bar{u}_{\vec{p},s}\gamma_0 \psi(x) - i \int \mathrm{d}^3\vec{x} e^{ip_\mu x^\mu} \bar{u}_{\vec{p},s}\gamma_0 \times \\ \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3\sqrt{2E_k}} \sum_r \left[a_{r,\vec{k}} u_{r,\vec{k}}\; e^{-ik_\mu x^\mu} + b^+_{r,\vec{k}} v_{r,\vec{k}}\; e^{ik_\mu x^\mu} \right]. \end{aligned} \tag{11.16}$$

Note that in this formula x_0 is supposed to be very large and that creation and annihilation operators refer to asymptotic fields at $x_0 = \pm\infty$. Next, we integrate over $\vec{x}$ and $\vec{k}$; this leads to $\vec{k} = \vec{p}$ in terms with a and $\vec{k} = -\vec{p}$ in terms with b^+. The expression simplifies further since

$$\bar{u}_{s,\vec{p}}\gamma^0 u_{r,\vec{p}} = 2E_{\vec{p}}\delta_{rs}, \quad \bar{u}_{s,\vec{p}}\gamma^0 v_{r,-\vec{p}} = 0. \tag{11.17}$$

Therefore, we find

$$\begin{aligned} I &= \int \mathrm{d}^4x\; e^{ipx} \bar{u}_{\vec{p},s} \left(-i\hat{\partial} + m\right) \psi(x) \\ &= -i\sqrt{2E_p} \left(a_{s,\vec{p}}(T) - a_{s,\vec{p}}(-T)\right). \end{aligned} \tag{11.18}$$

A similar calculation gives

$$I = \int \mathrm{d}^4x\; \bar{\psi}(x) \left(i\overset{\leftarrow}{\hat{\partial}} + m \right) u_{s,\vec{p}} e^{-ipx} \tag{11.19}$$

$$= i\sqrt{2E_p} \left(a^+_{s,\vec{p}}(T) - a^+_{s,\vec{p}}(-T)\right). \tag{11.20}$$

Note that the differential operator $\hat{\partial}$ in the above formula acts on the field $\bar{\psi}(x)$, i.e. to the *left*, as indicated by an arrow on top of it.

We will also need to relate fields ψ and $\bar{\psi}$ to creation and annihilation operators for *anti-particles*. The calculation can be performed in exactly the same way as for a and a^+ and we obtain

$$\int \mathrm{d}^4x\; e^{-ip_\mu x^\mu} \bar{v}_{s,\vec{p}}(-i\hat{\partial}+m)\psi(x) = -i\sqrt{2E_{\vec{p}}} \left(b^+_{s,\vec{p}}(T) - b^+_{s,\vec{p}}(-T)\right), \tag{11.21}$$

and

$$\int \mathrm{d}^4x\; \bar{\psi}(x) \left(i\overset{\leftarrow}{\hat{\partial}} + m \right) v_{s,\vec{p}} e^{ip_\mu x^\mu} = -i\sqrt{2E_{\vec{p}}} \left(b_{s,\vec{p}}(T) - b_{s,\vec{p}}(-T)\right). \tag{11.22}$$

Let us explore the implications of these formulas by considering the four-fermion scattering. We are interested in the following matrix element

$$iT_{fi} = \prod_{i=1}^{4} \sqrt{2E_i} \langle \Omega | T a_{p_3,s_3}(T) a_{p_4,s_4}(T) a^+_{p_1,s_1}(-T) a^+_{p_2,s_2}(-T) | \Omega \rangle. \qquad (11.23)$$

We re-write it as

$$\begin{aligned} iT_{fi} = i^4 \int \prod_{i=1}^{4} \mathrm{d}x_i \; e^{\sum_{i=1}^{4} \eta_i p_{i,\mu} x_i^\mu} \\ \times \bar{u}_{\alpha_3}(s_3, p_3)(-i\hat{\partial}_3 + m)_{\alpha_3\beta_3} \; \bar{u}_{\beta_4}(s_4, p_4)(-i\hat{\partial}_4 + m)_{\alpha_4\beta_4} \\ \times \langle \Omega | T \psi_{\beta_3}(x_3) \psi_{\beta_4}(x_4) \bar{\psi}_{\beta_1}(x_1) \bar{\psi}_{\beta_2}(x_2) | \Omega \rangle \\ \times \left(i\overset{\leftarrow}{\hat{\partial}}_1 + m \right)_{\beta_1\alpha_1} \left(i\overset{\leftarrow}{\hat{\partial}}_1 + m \right)_{\beta_2\alpha_2} u_{\alpha_1}(p_1, s_1) u_{\alpha_2,s_2}(p_2, s_2). \end{aligned} \qquad (11.24)$$

The Green's function in the middle of this formula can be written through fields in the interaction representation and computed in perturbation theory. Similar to the scalar field theory, we only need to consider *fully-connected* diagrams since otherwise we will not capture the contribution to the transfer matrix T_{fi}.

We will now discuss the role of differential operators in Eq. (11.24). The Green's function in Eq. (11.24) contains fermion propagators associated with external points. Consider x_3, for definiteness. The dependence of the Green's function on x_3 is given by the fermion propagator

$$\langle 0 | T \psi_{\beta_3}(x_3) \bar{\psi}_\beta(y) | 0 \rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \left(\frac{i}{\hat{p} - m + i0} \right)_{\beta_3\beta} e^{-ip_\mu (x_3 - y)^\mu}. \qquad (11.25)$$

We require

$$\begin{aligned} \bar{u}_{\alpha_3}(s_3, p_3)(-i\hat{\partial}_3 + m)_{\alpha_3\beta_3} \langle 0 | T \psi_{\beta_3}(x_3) \bar{\psi}_\beta(y) | 0 \rangle \\ = -i \bar{u}_\beta(s_3, p_3) \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \; e^{-ip_\mu (x_3 - y)^\mu}. \end{aligned} \qquad (11.26)$$

We then integrated over x_3, obtain $\delta^{(4)}(p - p_3)$ and remove this δ-function by integrating over p. The net result is that there is a spinor $\bar{u}(p_3)$ that is associated with the final state outgoing fermion with momentum p_3 and this fermion is contracted to the rest of the Green's function via its spinor index. The factor $-i$ combines with one of the i's from i^4 in Eq. (11.24) and returns 1. For the incoming fermion, the calculation is similar; the result is that for an incoming fermion there is a spinor $u(p)$.

Such calculations can be generalized and a set of rules that we can use to construct scattering amplitudes in theories with fermions can be formulated. For definiteness, we will consider the interaction Lagrangian in Eq. (11.5) as an example but many of these rules are more general.

- To write down scattering amplitudes, we only consider fully-connected diagrams. For the scalar sector of the theory, we use Feynman rules described earlier. There are special rules for fermions that we point out below, but everything related to associating $\mathrm{d}^4p/(2\pi)^4$ for each internal line, integrating over as many internal momenta as possible and extracting and removing the overall energy-momentum conserving δ function remains valid.

- A fermion in the initial state with momentum p and spin s is described by a spinor $u(s,p)$. We associate an incoming fermion line with such a spinor.

- A fermion in the final state with momentum p and spin s is described by a spinor $\bar{u}(s,p)$. We associate an outgoing fermion line with such a spinor.

- An anti-fermion in the initial state with momentum p and spin s is described by a spinor $\bar{v}(s,p)$. We associate an *outgoing* fermion line with this spinor with the momentum $-p$.

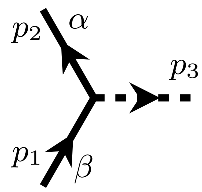
- An anti-fermion in the final state with momentum p and spin s is described by a spinor $v(s,p)$. We associate an *incoming* fermion line with this spinor with the momentum $-p$.

- For fermions, a propagator that carries momentum p reads

$$\beta \xrightarrow[p]{\qquad\qquad} \alpha = \left(\frac{i}{\hat{p}-m}\right)_{\alpha\beta} \tag{11.27}$$

  There is no separate propagator for fermions and anti-fermions.

- The interaction vertex of two fermions and a scalar boson reads



$$= -ig(2\pi)^4\delta^{(4)}(p_2 + p_3 - p_4)\mathbb{I}_{\alpha\beta} \qquad (11.28)$$

  Again, there is no separate vertex for fermions and anti-fermions. A vertex is attached to a fermion line that has a particular flow. This Feynman rule is particular to the Yukawa theory.

- Fermion flow can only originate/terminate on the external lines; this cannot happen "inside" a diagram. To write an expression for each fermion line, start at the "outgoing" end of the line and multiply spinor indices moving *against* the direction of the fermion flow.

- Diagrams which differ from each other by permutations of two identical fermions should have a relative ‘minus sign.

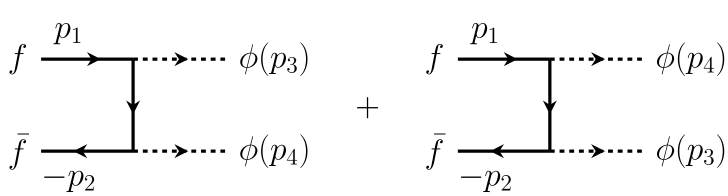
- Each closed fermion loop receives a minus sign.

- For each diagram with a continuous anti-particle line there is a minus sign (an "anti-particle" line means a fermion line whose beginning and end correspond to an anti-particle).

As an application of these rules, consider the process where a fermion with momentum p_1 and an anti-fermion with momentum p_2 annihilate into two scalars with momenta p_3 and p_4, i.e.

$$f(p_1) + \bar{f}(p_2) \to \phi(p_3) + \phi(p_4). \qquad (11.29)$$

There are two Feynman diagrams shown below



(11.30)

Note continuous fermion flow from incoming fermion to the incoming anti-fermion. According to the above rules, we start with an anti-fermion and work our way against the fermion flow line to an incoming fermion. The corresponding expression for the amplitude reads

$$i\mathcal{M}_{fi} = (-ig)^2 \bar{v}(p_2) \left[\frac{i}{\hat{p}_1 - \hat{p}_3 - m} + \frac{i}{\hat{p}_1 - \hat{p}_4 - m} \right] u(p_1). \qquad (11.31)$$

The expression in square brackets is a 4×4 matrix which is multiplied with two spinors from the right and from the left.