TTP1
Lecture 12

Kirill Melnikov
TTP KIT
23. Juni 2023

12 Computing cross sections and decay widths

In the previous lectures, we have discussed how to compute scattering amplitudes in quantum field theories. Scattering amplitudes are probability amplitudes that a particular initial state $|i\rangle$ scatters or gets transformed into a particular final state $|f\rangle$. A quantity related to a probability of this to happen is the cross section. For a process where the initial state consists of two particles with momenta p_1 and p_2 and the final state of particles with momenta $p_3, ..p_N$, the expression for the cross section reads

$$\mathrm{d}\sigma_{fi} = \frac{1}{4\ J\ N_i} \frac{1}{S_f} \sum_{\lambda_f,\lambda_i} |\mathcal{M}_{fi}|^2 (2\pi)^4 \delta^{(4)}(\sum_{k=3}^{N} p_k - p_1 - p_2) \prod_{k=3}^{N} \frac{\mathrm{d}^3 \vec{p}_k}{(2\pi)^3 2E_k}. \quad (12.1)$$

In this formula, J is the flux factor

$$J = \sqrt{(p_1 p_2)^2 - m_1^2 m_2^2}, \quad (12.2)$$

N_i is the total number of different (internal) quantum degrees of freedom the initial state; we average over all possibilities which means that we do not assume any prior knowledge about the initial state except of particles' momenta. For example, for the collision of two scalars, $N_i = 1$, for the collision of a fermion and a scalar $N_i = 2$ because initial fermion has spin $1/2$; for a collision of two fermions $N_i = 2 \times 2 = 4$ etc.

The sum over $\lambda_{f,i}$ indicates that we have to sum the amplitude squared over internal quantum numbers (e.g. spins) of particles in the initial and in the final states (unless we would like to study production of polarized states or if we know exactly that particles in the initial state have particular polarization). We also have the energy-momentum conserving δ-function and relativistic phase-space elements for each of the final state particles. Finally, the factor $1/S_f$ is equal to $1/n!$ where n is the number of *identical* particles in the final state.

To visualize what you can do with the cross section of a particular reaction, imagine that we have to deal with collisions of two particles' beams at a typical accelerator (say the LHC). The colliding beams have particular geometries and particle densities, they fly towards each other with particular velocities etc. When beams collide, a process where two particles from the two beams get transformed to a final state f occurs; this process is described by a cross section $d\sigma_{fi}$. What we are interested in is how often this actually happens, say if a collider is run for a year. In addition to cross section, this depends on other aspects of beam collisions and this information is encapsulated in a quantity that is called *luminosity* L. The number of final states f produced per unit time in the collision of beams of particles i reads

$$\frac{dN_f}{dt} = d\sigma_{fi}\, L, \tag{12.3}$$

For CERN, where proton beams collide, $L \approx 2 \times 10^{34}\text{cm}^{-2}\text{sec}^{-1}$. There are 10^7 seconds in a year. Therefore, if you think that getting $\mathcal{O}(100)$ events per year is sufficient to study a particular process, CERN experiments can realistically study cross sections that are as small as $\sigma \sim 10^{-39}\text{cm}^2$.

In particle physics, the cross sections that we want to study are very small, so one uses a special unit which is called *barn*. It is defined as 1 bn $=$ 10^{-24} cm^2. Hence, according to the above discussion, the LHC experiments can study cross sections that are as small as 10^{-15} bn or *one femtobarn*. These are very small cross sections. For example, Higgs boson production cross section at the LHC is 40 pb, so each year LHC produces close to *ten million* Higgs bosons.

After this digression, we go back to the question of how cross sections are computed. We will consider the theory with $\lambda\phi^4/4!$ interaction. The process we are interested in is

$$\phi(p_1) + \phi(p_2) \to \phi(p_3) + \phi(p_4). \tag{12.4}$$

The amplitude for this process has been calculated in one of the previous lectures; we have found

$$\mathcal{M}_{fi} = -i\lambda, \tag{12.5}$$

so that $|\mathcal{M}_{fi}|^2 = \lambda^2$. Among the many factors that appear in the formula for the cross section, $N_i = 1$ since we deal with scalars and $S_f = 2! = 2$ since we have two identical particles in the final state. Also, we write

$$J = \sqrt{(p_1 p_2)^2 - m^4} = \frac{1}{2}\sqrt{s(s-4m^2)}, \tag{12.6}$$

where $s = (p_1 + p_2)^2$. Assembling all the factors, we find

$$d\sigma = \frac{\lambda^2}{4\sqrt{s(s-4m^2)}}(2\pi)^4\delta^{(4)}(p_3+p_4-p_1-p_2)\frac{d^3\vec{p}_3}{(2\pi)^3 2E_3}\frac{d^3\vec{p}_4}{(2\pi)^3 2E_4}. \tag{12.7}$$

The cross section is a Lorentz-invariant quantity, hence, we can compute it in *any* reference frame. We choose the center-of-mass frame where

$$p_1 = (E, \vec{p}), \quad p_2 = (E, -\vec{p}). \tag{12.8}$$

Then $s = (p_1 + p_2)^2 = 4E^2$. Using the above expressions for $p_{1,2}$, we write

$$\delta^{(4)}(p_1 + p_2 - p_3 - p_4) = \delta(2E - E_3 - E_4)\delta^{(3)}(\vec{p}_3 + \vec{p}_4). \tag{12.9}$$

Integrating over $\vec{p}_4$, we find $\vec{p}_4 = -\vec{p}_3$ and, since $E_3 = \sqrt{\vec{p}_3^2 + m^2}$ and $E_4 = \sqrt{\vec{p}_4^2 + m^2}$, we find $E_4 = E_3$. Hence,

$$\mathrm{d}\sigma = \frac{\lambda^2}{4\sqrt{s(s - 4m^2)}} \frac{1}{4E_3^2(2\pi)^2}\delta(2E_3 - 2E)\mathrm{d}^3\vec{p}_3. \tag{12.10}$$

We write

$$\mathrm{d}^3\vec{p}_3 = \mathrm{d}\Omega_3 p_3^2 \mathrm{d}p_3 = \mathrm{d}\Omega_3 p_3^2 \frac{\mathrm{d}p_3}{\mathrm{d}E_3}\mathrm{d}E_3 = \mathrm{d}\Omega_3 p_3 E_3 \mathrm{d}E_3, \tag{12.11}$$

where $\mathrm{d}\Omega_3$ is the infinitesimal solid angle which describes direction of $\vec{p}_3$.

Substituting this result into the formula for the cross section and integrating over E_3 (this leads to $E_3 = E$), we find

$$\mathrm{d}\sigma = \frac{\lambda^2}{4\sqrt{s(s - 4m^2)}} \frac{p_3}{8E_3(2\pi)^2}\mathrm{d}\Omega_3. \tag{12.12}$$

Since $p_3/E_3 = \beta_3$, where β_3 is the velocity of particle 3 and since $E_3 = E$, we write $\beta_3 = \beta = \sqrt{1 - m^2/E^2}$, $s - 4m^2 = 4E^2\beta^2$, we find

$$\mathrm{d}\sigma = \frac{\lambda^2}{64\pi s} \frac{\mathrm{d}\varphi}{2\pi} \, \mathrm{d}\cos\theta. \tag{12.13}$$

The total cross section is obtained upon integration over ϕ from 0 to 2π and over $\cos\theta$ from -1 to 1. We find

$$\sigma = \frac{\lambda^2}{32\pi s}. \tag{12.14}$$

Note also that Eq. (12.13) predicts an isotropic angular distribution of the produced particles in the center of mass frame of colliding particles.

Our second example is the annihilation of two fermions into two bosons in the Yukawa theory. The corresponding amplitude was computed in the previous lecture. It reads

$$i\mathcal{M}_{fi} = -g^2 \bar{v}(p_2) \left[\frac{i}{\hat{p}_1 - \hat{p}_3 - m} + \frac{i}{\hat{p}_1 - \hat{p}_4 - m} \right] u(p_1), \tag{12.15}$$

where m is the fermion mass.

To compute the cross section, we need to square the amplitude and sum the result over polarizations of the incoming fermions. To understand how this can be done, note that the above amplitude can be written in the following way

$$i\mathcal{M}_{fi} = -ig^2 \bar{v}_\alpha(s_2, p_2) \Gamma_{\alpha\beta} u_\beta(s_1, p_1), \tag{12.16}$$

where

$$\Gamma_{\alpha\beta} = \left[\frac{1}{\hat{p}_1 - \hat{p}_3 - m} + \frac{1}{\hat{p}_1 - \hat{p}_4 - m} \right]_{\alpha\beta}. \tag{12.17}$$

We require

$$\sum_{s_1,s_2} |\mathcal{M}_{fi}|^2 = g_s^2 \sum_{s_1,s_2} \bar{v}_\alpha(s_2, p_2) \Gamma_{\alpha\beta} u_\beta(s_1, p_1)\ \bar{v}^*_{\alpha_1}(s_2, p_2) \Gamma^*_{\alpha_1\beta_1} u^*_{\beta_1}(s_1, p_1). \tag{12.18}$$

To put this into a reasonable form, we write

$$\sum_{s_1,s_2} |\mathcal{M}_{fi}|^2 = \sum_{s_1,s_2} \bar{v}_\alpha(s_2, p_2) \Gamma_{\alpha\beta} u_\beta(s_1, p_1)\ u^*_{\beta_1}(s_1, p_1) \Gamma^+_{\beta_1\alpha_2} (\gamma_0)_{\alpha_2\alpha_1} v_{\alpha_1}(s_2, p_2). \tag{12.19}$$

Using $\gamma_0^2 = 1$, we rewrite this formula and find

$$\begin{aligned} \sum_{s_1,s_2} |\mathcal{M}_{fi}|^2 &= \sum_{s_1,s_2} \left(v(s_2, p_2)_{\alpha_1} \bar{v}(s_2, p_2)_\alpha \right) \Gamma_{\alpha\beta} \left(u_\beta(s_1, p_1)\ \bar{u}_{\beta_1}(s_1, p_1) \right) \left(\gamma_0 \Gamma^+ \gamma_0 \right)_{\beta_1\alpha_1} \\ &= \mathrm{Tr} \left[(\hat{p}_2 - m) \hat{\Gamma} (\hat{p}_1 + m) \left(\gamma_0 (\hat{\Gamma})^+ \gamma_0 \right) \right], \end{aligned} \tag{12.20}$$

where we have used

$$\begin{aligned} &\sum_{s_2} v_{\alpha_1}(s_2, p_2) \bar{v}(s_2, p_2)_\alpha = (\hat{p}_2 - m)_{\alpha_1\alpha}, \\ &\sum_{s_1} u_{\beta_1}(s_1, p_1)\ \bar{u}_\beta(s_1, p_1) = (\hat{p}_1 + m)_{\beta_1\beta}. \end{aligned} \tag{12.21}$$

To compute $\gamma^0\Gamma^+\gamma^0$, we use the fact that all γ-matrices satisfy the following equation

$$\gamma^\mu = \gamma^0\gamma^{\mu,+}\gamma^0. \tag{12.22}$$

Hence,

$$\gamma^0\left(\hat{a}\hat{b}\hat{c}\right)^+\gamma^0 = \hat{c}\hat{b}\hat{a}. \tag{12.23}$$

In our case, Γ is quite simple, so we obtain

$$\gamma_0\Gamma^+\gamma_0 = \Gamma. \tag{12.24}$$

Hence, the matrix element squared summed over spins of initial fermions that we need to compute reads

$$\begin{aligned}\sum_{s_1,s_2}|\mathcal{M}_{\rm fi}|^2 = g^4\mathrm{Tr}\Bigg[&(\hat{p}_2 - m)\left[\frac{\hat{p}_{13}+m}{t-m^2} + \frac{\hat{p}_{14}+m}{u-m^2}\right]\\ &(\hat{p}_1 + m)\left[\frac{\hat{p}_{13}+m}{t-m^2} + \frac{\hat{p}_{14}+m}{u-m^2}\right]\Bigg],\end{aligned} \tag{12.25}$$

where we introduced $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$ are the Mandelstam variables.

To complete the calculation of the cross section, it remains to compute a trace of the product of γ-matrices. This is not easy, especially, in cases when the number of γ-matrices is large but this is a mechanical procedure. Indeed, the following statements are true

- traces of *odd* number of γ matrices vanish;
- traces of even number of γ matrices can be computed recursively starting from
$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}. \tag{12.26}$$
- traces of γ_5 with other γ-matrices can be computed recursively starting from
$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\gamma^5] = 4i\epsilon^{\mu\nu\alpha\beta}, \tag{12.27}$$
where ϵ is a Levi-Civita tensor with $\epsilon^{0123} = 1$.

To show how the recursive procedure works, we compute the trace of four gamma matrices

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta]. \tag{12.28}$$

The idea is to move γ^β to the very left of the matrix chain by anti-commuting it with other γ-matrices. We then find

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta] = 2g^{\alpha\beta}\mathrm{Tr}[\gamma^\mu\gamma^\nu] - 2g^{\beta\nu}\mathrm{Tr}[\gamma^\mu\gamma^\alpha] + 2g^{\mu\beta}\mathrm{Tr}[\gamma^\nu\gamma^\alpha] - \mathrm{Tr}[\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\alpha]. \tag{12.29}$$

Because of the trace cyclic property

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta] = \mathrm{Tr}[\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\alpha]. \tag{12.30}$$

Hence,

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta] = 4\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\beta\nu} + g^{\mu\beta}g^{\nu\alpha}\right). \tag{12.31}$$

For simplicity, I will assume that masses of the final-state bosons vanish. Then, a calculation of the trace gives

$$\begin{aligned}\sum_{s_1,s_2}|\mathcal{M}_{\mathrm{fi}}|^2 = g^4\Bigg(&\frac{2u}{t-m^2} + \frac{2t}{u-m^2} - 4 - \frac{34m^2}{t-m^2} - \frac{34m^2}{u-m^2}\\ &- \frac{29m^4}{(t-m^2)^2} - \frac{29m^4}{(u-m^2)^2} - \frac{58m^4}{(t-m^2)(u-m^2)}\Bigg).\end{aligned} \tag{12.32}$$

This expression can be simplified further if we consider collisions in the center of mass frame and use the velocity of initial particles and the scattering angle to write the result for $|\mathcal{M}|^2$. Since

$$t = m^2 - \frac{s}{2}(1-\beta\cos\theta), \quad u = m^2 - \frac{s}{2}(1+\beta\cos\theta), \tag{12.33}$$

where θ is the angle of momentum $\vec{p}_3$ and the momentum $\vec{p}_1$, we find

$$\sum_{s_1,s_2}|\mathcal{M}_{\mathrm{fi}}|^2 = g^4\,\frac{3+\beta^2(26-24\cos^2\theta)+\beta^4(-29+32\cos^2\theta-8\cos^4\theta)}{(1-\beta^2c^2)^2}. \tag{12.34}$$

For small values of β, we find

$$\sum_{s_1,s_2}|\mathcal{M}_{\mathrm{fi}}|^2 \approx 3g^4. \tag{12.35}$$

In the opposite (ultra-relativistic limit) $\beta = 1$ and we find

$$\sum_{s_1,s_2} |\mathcal{M}_{\mathrm{fi}}|^2 = g^4 \, \frac{8\cos^2\theta}{\sin^2\theta}. \tag{12.36}$$

Hence, the non-relativistic angular distribution is mainly isotropic but in the ultra-relativistic limit particles are mostly produced in the forward $\theta = 0$ or in the backward $\theta = \pi$ direction.

To complete calculation of cross section, we need to compute the phase space. In this case a simple computation reads

$$\mathrm{d}\sigma = \frac{g^4}{128\pi s\beta} \, \frac{\mathrm{d}\phi \, \mathrm{d}\cos\theta}{4\pi} \sum_{s_1,s_2} |\mathcal{M}_{\mathrm{fi}}|^2 \tag{12.37}$$

There is another quantity that is computed quite often in particle physics, this is the decay width of a particle to a particular final state. The inverse of the (total) decay width gives us a lifetime of the particle. The formula for computing the width is very similar to the formula for computing the cross section except that the flux factor changes. The formula for a decay of a particle X to the final state with f reads

$$\mathrm{d}\Gamma_f = \frac{1}{2m_X N_i}\frac{1}{S_f} \sum_{\lambda_f,\lambda_i} |\mathcal{M}_{fi}|^2 (2\pi)^4 \delta^{(4)}(\sum_{k=1}^{N} p_k - p_X) \prod_{k=1}^{N} \frac{\mathrm{d}^3\vec{p}_k}{(2\pi)^3 2E_k}. \tag{12.38}$$

It is often convenient to compute the width in the rest frame of the decaying particle, where the four-momentum is just $p_X = (m_X, \vec{0})$.