TTP1
Lecture 13

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13 Quantum electrodynamics

We will start discussing one of the most successful physical theories that is known to us today, the quantum electrodynamics. This is a theory that describes interactions of charged (strictly speaking elementary) particles, such as electrons, muons, τ-leptons etc. with the electromagnetic field. This theory is a prototype of more modern theories such as the theory of strong interactions (Quantum Chromodynamics) and the theory of weak interactions (the Standard Model).

Suppose we focus on a single fermion field (e.g. electron). Then, the Lagrangian of the theory reads

$$\mathcal{L}_{\rm QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \left(i\hat{\partial} - m - eA_\mu \gamma^\mu \right) \psi, \tag{13.1}$$

where e is the electron's charge, m is the electron's mass, A_μ is the vector potential of the electromagnetic field and $F^{\mu\nu}$ is the field-strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{13.2}$$

Recall that F^{0i} is the i-th component of the electric field and $1/2\epsilon_{ijk}F^{jk}$ gives the i-th component of the magnetic field.

The Lagrangian in Eq. (13.1) possesses an important feature known as the "gauge symmetry". This feature is very important as its generalizations are used to construct more complex theories which we mentioned at the beginning of this lecture. "Gauge symmetry" means the following. The Lagrangian in Eq. (13.1) is invariant under the following transformation of the fermion field

$$\psi(x) = e^{-i\theta}\psi'(x), \quad \bar{\psi}(x) = \bar{\psi}'(x)e^{i\theta}, \tag{13.3}$$

where θ is an arbitrary constant. This symmetry leads to the conservation of the fermion current $J^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$ which then implies conservation of the electric charge

Interestingly, the Lagrangian in Eq. (13.1) is invariant under a *stronger version* of the above transformation. Indeed, let us make the parameter θ in Eq. (13.3) an x-dependent function $\theta(x)$. Then

$$\psi(x) = e^{-i\theta(x)}\psi'(x), \quad \bar{\psi}(x) = \bar{\psi}'(x)e^{i\theta(x)}. \tag{13.4}$$

Substituting these expressions into the fermion part of the Lagrangian we find

$$\bar{\psi}\left(i\hat{\partial} - m - eA_\mu\gamma^\mu\right)\psi = \bar{\psi}'\left(i\hat{\partial} - m - e(A_\mu - e^{-1}\partial_\mu\theta(x))\gamma^\mu\right)\psi'. \quad (13.5)$$

We observe that we can remove the new term by redefining the vector potential A_μ. We write

$$A_\mu = A'_\mu + e^{-1}\partial_\mu\theta(x). \quad (13.6)$$

Then,

$$\bar{\psi}\left(i\hat{\partial} - m - eA_\mu\gamma^\mu\right)\psi = \bar{\psi}'\left(i\hat{\partial} - m - eA'_\mu\gamma^\mu\right)\psi'. \quad (13.7)$$

The field-stress tensor $F_{\mu\nu}$ is invariant under the transformation in Eq. (13.6) and we find

$$L(\bar{\psi}, \psi, A^\mu) = L(\bar{\psi}', \psi', A'^\mu). \quad (13.8)$$

Although we refer to this feature as "gauge symmetry", this is really not a symmetry in a sense that there are no conserved quantities that are associated with it. The "gauge symmetry" is a redundancy since, as it turns out, we employ too many degrees of freedom to describe physics that does not need all of them. We can see this already from Eq. (13.6) which basically means that by selecting functions θ with particular properties, we can impose certain conditions on the field A^μ that we want to work with. Obviously, this reduces the number of independent function that we need from four (i.e. four components of the vector $A^\mu(x)$) to a smaller number that we are about to find.

The overabundance of degrees of freedom in the Lagrangian Eq. (13.1) has consequences for quantization of QED that we will now discuss. Note that one can quantize QED in many different ways but the discussion below exposes physics behind complexities of QED quantization.

So, let us quantize QED. We already know how to quantize the Dirac field. We also identify the interaction term with $A_\mu\bar{\psi}\gamma^\mu\psi$. Hence, it remains to understand how to quantize the electromagnetic field. Although this can be done for a free field, for reasons that will become clear shortly, it is more convenient to start with the Lagrangian that includes the interaction term

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA_\mu J^\mu, \quad (13.9)$$

where J^μ is the electron current.

The vector potential A^μ has four components; if we can deal with them as if they are four independent scalar fields, quantization of QED would be straightforward. Let us try to do that. To this end, we need to write the Lagrangian separating A^μ and $\partial_0 A^\mu$. Upon doing that, we find

$$\mathcal{L} = \frac{1}{4}\sum_{i=1}^{3}(\partial_i A^0 + \partial_0 A^i)(\partial_i A^0 + \partial_0 A^i) - \frac{1}{4}\sum_{ij}\left(\partial_i A^j - \partial_j A^i\right)^2 - A_\mu J^\mu. \quad (13.10)$$

Although the above Lagrangian does not look too remarkable, it contains very important information, namely, that he canonical momentum of the field A^0 that we will refer to as π_0 vanishes. Indeed,

$$\pi_0 = \frac{\delta \mathcal{L}}{\delta \partial_0 A^0} = 0, \quad (13.11)$$

since L does not depend on $\partial_0 A^0$. We can compute three other canonical momenta without a problem. We find

$$\pi_i = \frac{\delta \mathcal{L}}{\delta \partial_0 A^i} = \frac{1}{2}(\partial_i A^0 + \partial_0 A^i). \quad (13.12)$$

The fact that π_0 vanishes has important quantities for the quantization since we cannot require that commutator of π_0 with A^0 is canonical. To see how to get around this problem, we compute the Hamiltonian (density) and find

$$\begin{aligned}\mathcal{H} = \sum_{i=1}^{3}\pi_i \partial_0 A^i - L = \frac{1}{2}\sum_{i=1}^{3}\pi_i^2 - \sum_{i=1}^{3}\pi_i \partial_i A^0 \\ + \frac{1}{2}\sum_{ij}(\partial_i A_j - \partial_j A_i)^2 + e(A_0 J_0 - \vec{A}\vec{J}).\end{aligned} \quad (13.13)$$

The classical Hamilton equation of motion is

$$\partial_0 \pi_0 = -\frac{\delta H}{\delta A_0} = -\partial_i \pi_i - eJ^0. \quad (13.14)$$

Under normal circumstances, this would be a dynamical equation. However, since $\pi_0 = 0$, we find

$$\partial_i \pi_i = -eJ^0. \quad (13.15)$$

This implies that not only $\pi_0 = 0$ but also that *three canonical momenta* π_i *are not independent and, therefore cannot be independently quantized*. This is the central issue with applying standard quantization rules to QED.

To overcome this problem, it is convenient to separate $\vec{\pi}$ into the transversal and the longitudinal components. We write

$$\vec{\pi} = \vec{p} + \vec{\nabla}\phi, \tag{13.16}$$

where $\vec{p}$ is chosen such that $\vec{\nabla} \cdot \vec{p} = 0$. Hence, the constraint Eq. (13.14) becomes

$$\vec{\nabla}^2 \phi = -eJ^0. \tag{13.17}$$

This equation is the Poisson equation familiar from electrodynamics. It can be solved explicitly and we find

$$\phi(t, \vec{x}) = \frac{e}{4\pi} \int \mathrm{d}^3\vec{y}\, \frac{J^0(t, \vec{y})}{|\vec{x} - \vec{y}|}. \tag{13.18}$$

To establish the relation between the auxiliary field ϕ and the potential A^μ, it is convenient to "fix the gauge", i.e. to choose function $\theta(x)$ such that A^μ satisfies certain constraints, c.f. Eq. (13.6). One of the things that one can require is that $\vec{\nabla} \cdot \vec{A}(x) = 0$ for all x. Then, since $\pi_i = \partial_0 A^i + \partial_i A^0$,

$$\vec{\nabla} \cdot \vec{\pi} = \vec{\nabla}^2 A^0 = \vec{\nabla}^2 \phi. \tag{13.19}$$

Hence, we can identify ϕ with A^0. Therefore,

$$A^0 = \frac{e}{4\pi} \int \mathrm{d}^3\vec{y}\, \frac{J^0(t, \vec{y})}{|\vec{x} - \vec{y}|}. \tag{13.20}$$

The Hamiltonian becomes

$$\begin{aligned} H = &\int \mathrm{d}^3\vec{x} \left[\frac{1}{2}\vec{p}^2 + \frac{1}{2}\sum_{i,j} (\partial_i A_j - \partial_j A_i)^2 - \vec{A} \cdot \vec{J} \right] \\ &+ \frac{e^2}{8\pi} \int \mathrm{d}^3\vec{x} \int \mathrm{d}^3\vec{y}\; \frac{J^0(t, \vec{x}) J^0(t, \vec{y})}{|\vec{x} - \vec{y}|}. \end{aligned} \tag{13.21}$$

This Hamiltonian looks peculiar, especially because of the last term, but it does not contain redundant degrees of freedom. However, this feature is not explicit as we still have three $\vec{A}$ fields and three canonical momenta $\vec{p}$ but they are not independent because they are transversal

$$\vec{\nabla}\cdot\vec{A}=0, \quad \vec{\nabla}\cdot\vec{p}=0. \tag{13.22}$$

Hence, both of these quantities are, effectively *two-dimensional* vectors.

Suppose we attempt to quantize the theory. Then in analogy with the quantization of the scalar field we write

$$\vec{A}(t,x)=\sum_{\lambda=1}^{2}\int\frac{\mathrm{d}^3\vec{k}}{(2\pi)^3\sqrt{2E_k}}\left(\vec{\epsilon}_{\lambda,\vec{k}}a_{\lambda,\vec{k}}e^{-ik_\mu x^\mu}+\vec{\epsilon}^{\,*}_{\lambda,\vec{k}}a^{+}_{\lambda,\vec{k}}e^{ik_\mu x^\mu}\right), \tag{13.23}$$

where $E_k=|\vec{k}|$ because photons are massless. Also, ϵ_λ are basis vectors and it is important that we sum over two λ's. This is manifestation of the fact that $\vec{A}$ is a two-dimensional vector given the transversality constraint. In momentum space, $\vec{\nabla}\cdot\vec{A}=0$ implies that $\vec{k}\cdot\vec{\epsilon}_\lambda=0$ so that vector $\vec{A}(\vec{k})$ exists in a two-dimensional plane which is orthogonal to $\vec{k}$. and vectors $\epsilon_{\lambda,\vec{k}}$ form an orthonormal vector basis in this plane Thus, the sum of polarizations reads

$$\sum_{\lambda=1}^{2}\epsilon^{i,*}_{\lambda,\vec{k}}\epsilon^{j}_{\lambda,\vec{k}}=\delta^{ij}-\frac{\vec{k}^i\vec{k}^j}{\vec{k}^2}. \tag{13.24}$$

Next, we need the canonical momentum $\vec{p}$. Using its definition and the fact that in the chosen (Coulomb) gauge $A_0=\phi$, we find

$$\vec{p}=\vec{\pi}-\vec{\nabla}\phi=\vec{\pi}-\vec{\nabla}A^0=\partial_0\vec{A}. \tag{13.25}$$

Computing the derivative, we find

$$\vec{p}(t,\vec{x})=-i\sum_{\lambda=1}^{2}\int\frac{\mathrm{d}^3\vec{k}}{(2\pi)^3\sqrt{2E_k}}\,E_k\left(\vec{\epsilon}_{\lambda,\vec{k}}a_{\lambda,\vec{k}}e^{-ik_\mu x^\mu}-\vec{\epsilon}^{\,*}_{\lambda,\vec{k}}a^{+}_{\lambda,\vec{k}}e^{ik_\mu x^\mu}\right). \tag{13.26}$$

We will assume the standard commutation relation for creation and annihilation operators

$$[a_{\lambda_1,\vec{k}_1},a^{+}_{\lambda_2,\vec{k}_2}]=\delta_{\lambda_1\lambda_2}(2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{k}_2). \tag{13.27}$$

It is straightforward to compute the commutation relation of $\vec{p}$ and $\vec{A}$. We find

$$[\vec{p}_i(t,\vec{x}), \vec{A}_j(t,\vec{y})] = -i \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\; e^{i\vec{k}(\vec{x}-\vec{y})}\; \sum_\lambda \vec{\epsilon}^{\,*}_{\lambda\vec{k}}\vec{\epsilon}_{\lambda\vec{k}}$$
$$= -i \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\; e^{i\vec{k}(\vec{x}-\vec{y})}\; \left(\delta_{ij} - \frac{\vec{k}_i\vec{k}_j}{\vec{k}^2}\right). \tag{13.28}$$

Performing Fourier transform, we find

$$[\vec{p}_i(t,\vec{x}), \vec{A}_j(t,\vec{y})] = -i\left(\delta_{ij} - \frac{\nabla^i\nabla^j}{\vec{\nabla}^2}\right)\delta^{(3)}(\vec{x}-\vec{y}), \tag{13.29}$$

which is the correct quantization condition given the constraints $\vec{\nabla}_i\vec{p}_i = \vec{\nabla}_i\vec{A}_i = 0$.

Having quantized QED, we can now develop a framework to compute arbitrary Green's functions and scattering amplitudes in this theory. Indeed, operator $a^+_{\lambda,\vec{k}}$ that appears in the expression for $\vec{A}$ in Eq. (13.23) creates a *photon* with momentum $\vec{k}$ and polarization vector $\epsilon_{\lambda,\vec{k}}$. There is a propagator for the photon field that one can compute directly from Eq. (13.23)

$$\langle 0|TA^i(x)A^j(y)|0\rangle = \int \frac{\mathrm{d}^4}{(2\pi)^4}\frac{i}{p^2+i0}\;\left(\delta_{ij} - \frac{\vec{p}^i\vec{p}^j}{\vec{p}^2}\right)e^{-ik(x-y)}. \tag{13.30}$$

For computing scattering amplitudes one needs a relation between creation and annihilation operators and the fields $\vec{A}$. One can easily show that e.g.

$$i\int \mathrm{d}^4x e^{ip_\mu x^\mu}\epsilon^*_{\lambda,\vec{p}}\partial^2\vec{A}(x) = \sqrt{2E_{\vec{p}}}\left(a_{\lambda,\vec{p}}(T) - a_{\lambda,\vec{p}}(-T)\right). \tag{13.31}$$

Hence, a final state photon with momentum $\vec{p}$ is described by a complex-conjugate polarization vector $\vec{\epsilon}_{\lambda,\vec{p}}$ and, similarly, a photon in the initial state with momentum $\vec{p}$ described by a polarization vector $\epsilon_{\lambda,\vec{p}}$; both of these vectors have to be multiplied into an amputated Green's function.

The interaction term in the Hamiltonian is

$$H_{\mathrm{int}} = -e\vec{A}\cdot\vec{J} + \frac{e^2}{8\pi}\int \mathrm{d}^3\vec{x}\;\int \mathrm{d}^3\vec{y}\;\;\frac{J^0(t,\vec{x})J^0(t,\vec{y})}{|\vec{x}-\vec{y}|}. \tag{13.32}$$

Using this interaction Hamiltonian, one can compute the interaction vertices between electrons and photons and use them to construct a perturbative expansion of the Green's functions.

Consider now the scattering of four fermions with momenta $e(p_1) + e(p_2) \to e(p_3) + e(p_4)$. There are two two distinct contributions to this amplitude – one, where the scattering occurs because a "photon" is exchanged between the two lines and another one where the second term in the interaction Hamiltonian contributes. We find

$$\begin{aligned} i\mathcal{M}_{fi} = & \frac{i(-ie)^2\left(\bar{u}(p_3)\gamma^0 u(p_1)\right)\left(\bar{u}(p_4)\gamma^0 u(p_2)\right)}{\vec{k}^2} \\ & + \frac{i(-ie)^2\left(\bar{u}(p_3)\gamma^i u(p_1)\right)\left(\bar{u}(p_4)\gamma^j u(p_2)\right)}{k^2 + i0}\left(\delta_{ij} - \frac{\vec{k}_k\vec{k}_j}{\vec{k}^2}\right) - (3 \leftrightarrow 4). \end{aligned} \tag{13.33}$$

We can formally write this result in a covariant form by introducing a propagator

$$i\mathcal{M}_{fi} = (-ie)^2 J^\mu_{31}\; J^\mu_{42}\; \bar{u}(p_4)\gamma^\nu(p_2) D_{\mu\nu}(k) - (3 \leftrightarrow 4), \tag{13.34}$$

where

$$J^\mu_{ab} = \bar{u}(p_a)\gamma^\mu u(p_b), \tag{13.35}$$

and

$$D^{\mu\nu}(k) = \begin{cases} \frac{i}{\vec{k}^2}, & \mu = 0, \nu = 0, \\ 0, & \mu = 0, \nu = 1,2,3 \\ 0, & \nu = 0, \mu = 1,2,3 \\ \frac{i}{k^2}\left(\delta_{ij} - \frac{\vec{k}_k\vec{k}_j}{\vec{k}^2}\right), & \nu = 1,2,3, \mu = 1,2,3 \end{cases} \tag{13.36}$$

An important feature of the current J^μ_{ab} is that it is conserved. The momentum-space version of that is

$$J^\mu_{ab} k_\mu = 0. \tag{13.37}$$

We can check this for our currents. Take J^μ_{31} for definiteness. Then

$$J^\mu_{31} k_\mu = \bar{u}(p_3)\hat{k}u(p_1) = \bar{u}(p_3)\left(\hat{p}_3 - \hat{p}_1\right)u(p_1) = \bar{u}(p_3)(m-m)u(p_1) = 0, \tag{13.38}$$

where we have used the Dirac equation for the spinors. It follows that if we modify the propagator $D^{\mu\nu}$ in the following way

$$D_{\mu\nu} \to D_{\mu\nu} + k_\mu \chi_\nu + k_\nu \chi_\mu + f k_\mu k_\nu, \tag{13.39}$$

where χ^μ and f are arbitrary functions of k since scattering amplitudes will not change. We will make use of this freedom to rewrite the Coulomb-gauge propagator in a covariant form.

To this end, we introduce a time-like four-vector $t^\mu = (1, 0, 0, 0)$. Then

$$\vec{k}^2 = (tk)^2 - k^2, \tag{13.40}$$

and we can also write

$$(0, \vec{k}) = k^\mu - t^\mu (tk). \tag{13.41}$$

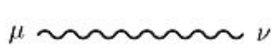
Then

$$\begin{aligned} \delta^{ij} - \frac{\vec{k}^i \vec{k}^j}{\vec{k}^2} &\to -g^{\mu\nu} + t^\mu t^\nu - \frac{(k^\mu - (tk)t^\mu)(k^\nu - (tk)t^\nu)}{(tk)^2 - k^2} \\ &= -g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2 - (tk)^2} + \frac{k^2}{k^2 - (tk)^2} t^\mu t^\nu - \frac{(tk)(t^\mu k^\nu + t^\nu k^\mu)}{k^2 - (tk)^2}. \end{aligned} \tag{13.42}$$

Hence, we find

$$\begin{aligned} D^{\mu\nu} &= \frac{i t^\mu t^\nu}{(tk)^2 - k^2} + \frac{i}{k^2} \Bigg(- g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2 - (tk)^2} \\ &+ \frac{k^2}{k^2 - (tk)^2} t^\mu t^\nu - \frac{(tk)(t^\mu k^\nu + t^\nu k^\mu)}{k^2 - (tk)^2} \Bigg) \\ &= \frac{-i g^{\mu\nu}}{k^2} + \text{terms with either } k^\mu \text{ or } k^\nu. \end{aligned} \tag{13.43}$$

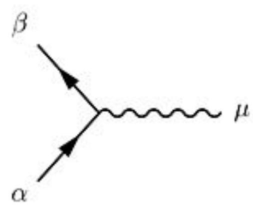
Thanks to the current conservation, we can drop k-dependent terms and use



$$= D^{\mu\nu}(k) = \frac{-i g^{\mu\nu}}{k^2}, \tag{13.44}$$

for the photon propagator in momentum space. The gauge choice that leads to this propagator is known as "Feynman gauge".

Feynman rules in the Feynman gauge become covariant. Since in this gauge there is no difference between the field and true propagating photons, we can describe photon electron interactions with a vertex



$$= -ie(\gamma^{\mu})_{\beta\alpha}, \tag{13.45}$$

and the overall energy-momentum conserving δ-function is not shown.

Finally, we note that we can replace the sum over physical polarizations

$$\sum_{\lambda=1}^{2} \epsilon_{\lambda}^{*,\mu} \epsilon_{\lambda}^{\nu} \tag{13.46}$$

with $-g^{\mu\nu}$ for *external* photons. The reason is the same as before. In principle, the sum over physical polarizations gives $\rho_{ij}(k)$. However, according to Eq. (13.42), for real ($k^2 = 0$) photons

$$\rho^{ij} = -g^{\mu\nu} + +\text{terms with either } k^{\mu} \text{ or } k^{\nu}. \tag{13.47}$$

Thanks to current conservation all the terms with k^{μ} can be dropped and $-g^{\mu\nu}$ can be used instead of the sum over physical polarizations.