TTP1
Lecture 14

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July 7, 2023

14 Predictions of QED

I would like to discuss a few predictions that we can make using the theory that we constructed in the previous lecture. We will start with the discussion of the interactions between electrons and positrons. This interaction is described by two Feynman diagrams; one describes the scattering of an electron on a positron and the other one describes the annihilation. For our purposes only the scattering diagram is important. Using the Feynman rules discussed in the previous lecture, we can write

$$i\mathcal{M}_{fi} = \frac{-ie^2 J_{31}^{\mu} J_{\mu,24}}{q^2}, \tag{14.1}$$

where $q = p_1 - p_3$ and

$$J_{31}^{\mu} = \bar{u}(p_3)\gamma^{\mu}u(p_1), \tag{14.2}$$

is the electron current and

$$J_{42}^{\mu} = \bar{v}(p_2)\gamma^{\mu}v(p_4), \tag{14.3}$$

is the positron current and one factor (-1) is introduced to account for an anti-particle line.

We would like to understand what happens to this matrix element in the non-relativistic approximation, i.e. when

$$p_i^{\mu} = (E_i, \vec{p}_i), \quad i = 1, .., 4, \tag{14.4}$$

and $|\vec{p}_i| \ll m$ so that with

$$E_i = m + \frac{\vec{p}_i^{\,2}}{2m} + \mathcal{O}(\vec{p}^{\,4}). \tag{14.5}$$

We need to construct the expansion for of spinors and the currents in the non-relativistic limit. The γ-matrices in the Dirac representation read

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \vec{\sigma}_i \\ -\vec{\sigma}_i & 0 \end{pmatrix}. \tag{14.6}$$

The Dirac equations

$$(\hat{p} - m)\, u(p) = 0, \quad (\hat{p} + m)\, v(p) = 0, \tag{14.7}$$

have the following solutions

$$u(p,s) = \sqrt{E_p + m} \left(\begin{array}{c} \varphi(s) \\ \frac{\vec{\sigma}\vec{p}\varphi(s)}{E_p+m} \end{array} \right), \quad v(p,s) = \sqrt{E_p + m} \left(\begin{array}{c} \frac{\vec{\sigma}\vec{p}\varphi(s)}{E_p+m} \\ \varphi(s) \end{array} \right). \quad (14.8)$$

The non-relativistic limit of the currents depends on the component. To leading order in momenta $\vec{p}_1$, $\vec{p}_3$, $\vec{p}_2$, $\vec{p}_4$ the currents read

$$J^{\mu}_{31} = (2m)\delta^{\mu 0} \left(\varphi^{+}(s_3)\varphi(s_1)\right) = (2m)\delta^{\mu 0}\delta_{s_1 s_3}. \quad (14.9)$$

and

$$J^{\mu}_{42} = (2m)\delta^{\mu 0} \left(\varphi^{+}(s_2)\varphi(s_4)\right) = (2m)\delta^{\mu 0}\delta_{s_2 s_4}. \quad (14.10)$$

Finally,

$$q = p_1 - p_3 = (\frac{\vec{p}_1^2 - \vec{p}_3^2}{2m}, \vec{p}_1 - \vec{p}_3) \approx (0, \vec{q}), \quad (14.11)$$

where $\vec{q} = \vec{p}_1 - \vec{p}_3$.

Hence, the scattering amplitude in the non-relativistic limits becomes[1]

$$i\mathcal{M}_{fi} = \frac{ie^2(2m)^2}{\vec{q}^2}\, \delta_{s_1 s_3}\delta_{s_2 s_4}. \quad (14.12)$$

What can we learn from this amplitude? The first thing we can learn is that spin degrees of freedom play no role in the non-relativistic limit since spin cannot change in the scattering and we can simply ignore it. Second, we go back to quantum mechanics where we describe electron and positron as two different particles that interact with each other through a potential. The Hamiltonian reads

$$H = \frac{\vec{p}_{e-}^2}{2m} + \frac{\vec{p}_{e^+}^2}{2m} + U(\vec{r}_e, \vec{r}_{e^+}). \quad (14.13)$$

I would like to compute the process of electron-positron scattering in quantum mechanics. According to Fermi's golden rule formula, I need to compute the cross section

$$d\sigma = \frac{V}{2\beta}(2\pi)\delta(E_3 + E_4 - E_1 - E_2)|U_{fi}|^2 \frac{V d^3\vec{p}_3}{(2\pi)^3}\frac{V d^3\vec{p}_4}{(2\pi)^3}, \quad (14.14)$$

[1] As this point, it is worth pointing out that the annihilation diagram will give a contribution that will behave as $\mathcal{M}_{fi} \sim \mathcal{O}(1)$ in the non-relativistic limit and, for this reason, will be suppressed at $\vec{q}^2/m^2$.

where V is the space volume, β is the velocity, and

$$U_{fi} = \langle \Psi_f | U(\vec{r}_{e^-}, \vec{r}_{e^+} | \Psi_i \rangle. \tag{14.15}$$

The wave functions for the initial and final state are

$$|\Psi_i\rangle = \frac{e^{-i\vec{p}_1\vec{r}_{e^-}}}{\sqrt{V}} \frac{e^{-i\vec{p}_2\vec{r}_{e^+}}}{\sqrt{V}}, \qquad |\Psi_f\rangle = \frac{e^{-i\vec{p}_3\vec{r}_{e^-}}}{\sqrt{V}} \frac{e^{-i\vec{p}_4\vec{r}_{e^+}}}{\sqrt{V}}. \tag{14.16}$$

Then, we find

$$\begin{aligned} U_{fi} &= \frac{1}{V^2} \int \mathrm{d}^3 r_{e^-} \mathrm{d}^3 r_{e^+} e^{i\vec{p}_3\vec{r}_{e-} + i\vec{p}_4\vec{r}_{e+}} U(\vec{r}_e - \vec{r}_-) e^{-i\vec{p}_1\vec{r}_{e^-} - \vec{p}_2\vec{r}_{e^+}} \\ &= \frac{(2\pi)^3 \delta^{(3)}(\vec{p}_3 + \vec{p}_4 - \vec{p}_1 - \vec{p}_2)}{V^2} U(\vec{q}), \end{aligned} \tag{14.17}$$

where

$$U(\vec{q}) = \int \mathrm{d}^3\vec{r}\, e^{-i\vec{q}\vec{r}} U(\vec{r}), \tag{14.18}$$

is a Fourier transform of the potential. Since

$$|U_{fi}|^2 = \frac{1}{V^3} (2\pi)^3 \delta^{(3)}(\vec{p}_3 + \vec{p}_4 - \vec{p}_1 - \vec{p}_2) |U(\vec{q})|^2, \tag{14.19}$$

we find

$$\mathrm{d}\sigma_{fi} = \frac{1}{2\beta} (2\pi)^4 \delta^{(4)}(p_3 + p_4 - p_1 - p_2) |U(\vec{q})|^2 \frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^3} \frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^3}. \tag{14.20}$$

We can compute *the same* cross section in QED. We obtain

$$\mathrm{d}\sigma_{fi} = \frac{(2\pi)^4 \delta^{(4)}(p_3 + p_4 - p_1 - p_2)}{2\, s\, \beta} |\mathcal{M}_{fi}|^2 \frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^3 (2E_3)} \frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^3 (2E_4)}. \tag{14.21}$$

The two results should agree in the non-relativistic limit. This can only happen if

$$U(\vec{q}) = -\frac{\mathcal{M}_{fi}}{4m^2}, \tag{14.22}$$

where on the r.h.s. we should remove spin-conserving Kronecker symbols.

Hence, we find

$$U(\vec{q}) = -\frac{\mathcal{M}_{fi}}{4m^2} = -\frac{e^2}{\vec{q}^2}, \tag{14.23}$$

where spin-dependent parts of the amplitude have been omitted. Then,

$$U(\vec{r}) = -e^2 \int \frac{\mathrm{d}^3\vec{q}}{(2\pi)^3} \frac{e^2}{\vec{q}^2} \, e^{i\vec{q}\vec{r}} = -\frac{e^2}{4\pi r}, \tag{14.24}$$

which is an attractive Coulomb potential between electron and positron.

As the next step, we consider scattering of an electron on a heavy nuclei which we will represent by a fermion with a charge Z and mass M. The scattering matrix element reads

$$i\mathcal{M}_{fi} = \frac{ie^2 Z}{q^2} \, \bar{u}_3 \gamma^\mu u_1 \, \bar{u}_4 \gamma_\mu u_2, \tag{14.25}$$

where spinors $u_{2,4}$ describe a heavy nucleus, $u_{3,1}$ describe an electron, and $q = p_1 - p_3 = p_4 - p_2$.

We now square the amplitude and sum over spins of all spinors. We find

$$\sum |i\mathcal{M}_{fi}|^2 = \frac{e^4 Z^2}{(q^2)^2} \mathrm{Tr}\left((\hat{p}_3 + m)\gamma^\mu(\hat{p}_1 + m)\gamma^\nu\right) \mathrm{Tr}\left((\hat{p}_4 + M)\gamma_\mu(\hat{p}_2 + M)\gamma_\nu\right). \tag{14.26}$$

The two traces is easy to compute. We find

$$L^{\mu\nu} = \mathrm{Tr}\left((\hat{p}_3 + m)\gamma^\mu(\hat{p}_1 + m)\gamma^\nu\right) = 4\left(p_3^\mu p_1^\nu + p_3^\nu p_1^\mu - g^{\mu\nu}((p_3 p_1) - m^2)\right). \tag{14.27}$$

The trace for a heavy nucleus is obtained from $L^{\mu\nu}$ upon the replacement $p_3 \to p_4$, $p_1 \to p_2$ and $m \to M$. We find

$$H^{\mu\nu} = 4\left(p_2^\mu p_4^\nu + p_4^\nu p_2^\mu - g^{\mu\nu}((p_2 p_4) - M^2)\right) \tag{14.28}$$

The two tensors $L^{\mu\nu}$ and $H^{\mu\nu}$ have an important property

$$q_\mu L^{\mu\nu} = q_\mu H^{\mu\nu} = 0. \tag{14.29}$$

This allows us to simplify $H^{\mu\nu}$ by writing $p_4 = p_2 + q$ and then neglecting all terms with either q^μ or q^ν. We find

$$H^{\mu\nu} = 4\left(2p_2^\mu p_2^\nu - g^{\mu\nu}(p_2 p_4 - M^2) + \text{terms with } q^\mu \text{ or } q^\nu\right). \tag{14.30}$$

We will focus on the case when the nucleus is very heavy in comparison with electron's energy and mass $M \gg E_1, m$ and it is at rest, originally. From the energy-momentum conservation, we find that

$$p_4 \approx p_2 = (M, \vec{0}), \tag{14.31}$$

$E_3 = E_1$ and $|\vec{p}_3| = |\vec{p}_1|$ by the direction of $\vec{p}_3$ is arbitrary; the recoil is absorbed by the nucleus. Then $p_2 p_4 \sim M^2$ and

$$H^{\mu\nu} \approx 8 p_2^\mu p_2^\nu. \tag{14.32}$$

We then compute

$$\begin{aligned} L_{\mu\nu} H^{\mu\nu} &\approx 32 M^2 (E_3 E_1 + E_3 E_1 - (p_3 p_1) + m^2) \\ &= 64 M^2 E_1^2 (1 - \beta^2 \sin^2(\theta/2)). \end{aligned} \tag{14.33}$$

We compute the cross section and obtain ($4J = 4ME_1\beta$, $1/4$ because for two fermion we have to divide by 4)

$$\begin{aligned} \mathrm{d}\sigma = & \frac{Z^2 e^4}{16 M E_1 \beta} \frac{64 M^2 E_1^2 (1 - \beta^2 \sin^2(\theta/2)}{q^4} \\ & \times (2\pi)^4 \delta^4 (p_3 + p_4 - p_1 - p_2) \frac{\mathrm{d}^3 \vec{p}_4}{(2\pi)^3 2E_4} \frac{\mathrm{d}^3 \vec{p}_3}{(2\pi)^3 2E_3}. \end{aligned} \tag{14.34}$$

Since $E_4 \approx M$, we find

$$\mathrm{d}\sigma = \frac{4 Z^2 \alpha^2 E_1^2}{(q^2)^2} \left(1 - \beta^2 \sin^2(\theta/2)\right) \mathrm{d}\Omega, \tag{14.35}$$

where we have introduced the fine structure constant $\alpha = e^2/(4\pi) \approx 1/137$.

To understand the angular distribution, we note that because in case of the heavy nucleus electron's energy before and after the scattering remains unchanged, the momentum transfer is spatial

$$q = p_1 - p_3 = (0, \vec{p}_1 - \vec{p}_3). \tag{14.36}$$

Hence,

$$q^2 = -\vec{q}^2 = -4 E_1^2 \beta^2 \sin^2(\theta/2). \tag{14.37}$$

Therefore,

$$\mathrm{d}\sigma = \frac{Z^2 \alpha^2}{4 E_1^2 \beta^4} \frac{\left(1 - \beta^2 \sin^2(\theta/2)\right)}{\sin^4(\theta/2)} \mathrm{d}\Omega. \tag{14.38}$$

This is the differential cross section for the Rutherford scattering, i.e. scattering of an electron on a point-like nucleus. Note that the total cross section in this case, i.e.

$$\sigma = \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \mathrm{d}\Omega \tag{14.39}$$

is *infinite* because for small scattering angles the cross section is too singular.

Next, we consider a process of an electron-positron annihilation into a muon and anti-muon pair. Muon is an elementary particle which is very similar to an electron (a fermion, spin 1/2, same charge) but it is about 200 times heavier, $m_\mu/m_e \approx 200$. To produce a pair of muons, electron and positron should have energies higher than m_μ (in the center of mass frame) and, for this reason, we will neglect electron masses in what follows.

The process we consider is

$$e^-(p_1) + e^+(p_2) \to \mu^-(p_3) + \mu^+(p_4). \tag{14.40}$$

To compute the cross section, we require the matrix element

$$i\mathcal{M}_{fi} = \frac{ie^2}{Q^2}\left[\bar{v}_2\gamma^\mu u_1\right]\left[\bar{u}_3\gamma_\mu v_4\right], \tag{14.41}$$

where $Q = p_1 + p_2 = p_3 + p_4$. The amplitude squared summed over spins of all particles evaluates to

$$|\mathcal{M}_{fi}|^2 = \frac{e^4}{Q^4} L^{(e)}_{\mu\nu} L^{(\mu)\mu\nu}, \tag{14.42}$$

where

$$L^{(e)}_{\mu\nu} = \mathrm{Tr}\left[\hat{p}_2\gamma_\mu\hat{p}_1\gamma_\nu\right], \tag{14.43}$$

and

$$L^{(\mu)}_{\mu\nu} = \mathrm{Tr}\left[(\hat{p}_3 + m)\gamma_\mu(\hat{p}_4 - m)\gamma_\nu\right], \tag{14.44}$$

where m denotes the muon mass. Traces evaluate to

$$L^{(e),\mu\nu} = 4\left(p_2^\mu p_1^\nu + p_1^\mu p_2^\nu - g^{\mu\nu} p_1 p_2\right), \tag{14.45}$$

and

$$L^{(\mu),\mu\nu} = 4\left(p_3^\mu p_4^\nu + p_4^\mu p_3^\nu - g^{\mu\nu}(p_3 p_4 + m^2)\right), \tag{14.46}$$

Contracting the two tensors, we find

$$\begin{aligned} L^{(e),\mu\nu} L^{(\mu)}_{\mu\nu} &= 16\Big[2(p_2p_3)(p_1p_4) + 2(p_2p_4)(p_1p_3) - 2(p_1p_2)(p_3p_4 + m^2) \\ &\quad - 2(p_3p_4)p_1p_2 + 4p_1p_2(p_3p_4 + m^2)\Big] \\ &= 16\Big[2(p_2p_3)(p_1p_4) + 2(p_2p_4)(p_1p_3) + 2p_1p_2m^2\Big] \\ &= 8\Big[(m^2 - u)^2 + (m^2 - t)^2 + 2m^2 s\Big]. \end{aligned} \tag{14.47}$$

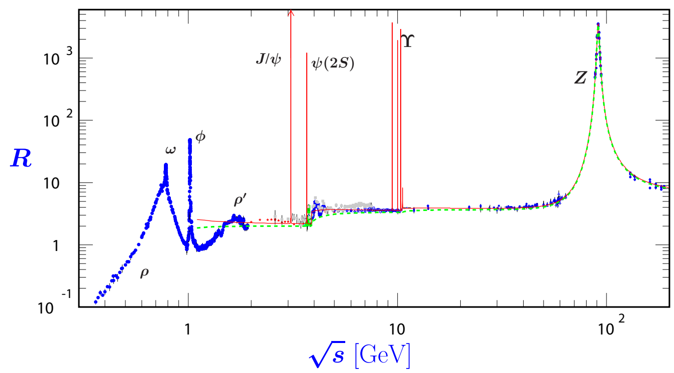

Figure 1: The ratio of $e^+e^- \to$ hadrons cross section to that of $e^+e^- \to \mu^+\mu^-$.

We then use Mandelstam variables to express scalar products

$$\begin{aligned} p_1p_2 &= s/2, \quad p_3p_4 = s/2 - m^2, \\ p_1p_3 &= p_2p_4 = m^2 - t, \quad p_1p_4 = p_2p_3 = m^2 - u. \end{aligned} \tag{14.48}$$

Then,

$$L^{(e),\mu\nu} L^{(\mu)}_{\mu\nu} = 4s^2 \left(2 - \beta^2 \sin^2\theta\right). \tag{14.49}$$

The cross section reads

$$\mathrm{d}\sigma = \frac{2\pi\alpha^2\beta}{s}\left(1 - \frac{\beta^2}{2}\sin^2\theta\right)\frac{\mathrm{d}\Omega}{4\pi}. \tag{14.50}$$

Integrating over angles, we find

$$\sigma = \frac{4\pi\alpha^2\beta}{3s}\left(1 + \frac{1-\beta^2}{3}\right). \tag{14.51}$$

At high energies, $\sqrt{s} \gg m_\mu$, $\beta \to 1$ and the cross section of $e^+e^- \to \mu^+\mu^-$ becomes

$$\sigma = \frac{4\pi\alpha^2}{3s}. \tag{14.52}$$

As a side remark, we notice that measurements of $e^+e^- \to$ hadrons, performed at e^+e^- colliders worldwide, revealed an interesting feature shown in Fig. 1.[2] What is shown there is the ratio R defined as

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e- \to \mu^+\mu^-)}, \qquad (14.53)$$

where the $e^+e^- \to \mu^+\mu^-$ cross section is taken from Eq. (14.52), i.e. in the massless approximation for both electrons and muons. Although one sees structures, there are large energy intervals where R behaves like a *constant*, e.g. $R \approx 2$ for 2 GeV $\leq \sqrt{s} \leq$ 3 GeV, $R \approx 3$ for 4 GeV $\leq \sqrt{s} \leq$ 10 GeV and $R \approx 4$ for 10.5 GeV $\leq \sqrt{s} \leq$ 90 GeV. One possible explanation is that $e^+e^- \to$ hadrons is, at its core, an incoherent sum of $e^+e^- \to$ elemenrary fermions with strong charges that we cannot observe otherwise. If this interpretation is correct (and it turns out to be correct) R counts the number of such fundamental fermions (quarks) that can be produced at a given energy.

[2]Hadrons are particles that primarily interact with each other by virtue of the strong force.