TTP1
Lecture 15

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15 Electron form factors and the anomalous magnetic moment

Interaction of the electromagnetic field with leptons is described by the following term in the QED Lagrangian

$$L = -e \int \mathrm{d}^4x \; A_\mu(x) \; J^\mu(x). \tag{15.1}$$

We have discussed how to quantize QED in the previous lectures. In this lecture we will use these results to discuss the problem of interaction between a *given* electromagnetic field (we can call it external, classical, background etc.) and an electron. We do this by formally separating A^μ into two parts

$$A^\mu(x) \to A^\mu_{\rm ext}(x) + A^\mu(x), \tag{15.2}$$

and, when computing Green's functions we can consider Green's functions that contain fixed number of interactions between A^μ_{ext} and electrons.

The simplest quantity that one can study is the scattering of an electron on $A^\mu_{\rm ext}$. To describe it, we need to compute the following amplitude in the fully-interacting theory

$$T^\mu_{fi}(p_f, p_i, q) = -i \int \mathrm{d}^4x \; e^{-iqx} \; \langle e(p_f, s_f)|J^\mu(x)|e(p_i, s_i)\rangle, \tag{15.3}$$

where $J^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$.

Calculation of this quantity proceeds in exactly the same way as the calculation of ordinary scattering amplitude. We employ the LSZ reduction, and connect the above matrix element to the computation of the integral of the Green's function

$$\langle 0|T\psi(x_1)J^\mu(x)\bar{\psi}(x_2)|0\rangle, \tag{15.4}$$

which is then acted upon with Dirac operators multiplied with $\bar{u}$ for the outgoing and with u for the incoming fermion. Also, from the LSZ reduction process exponential functions e^{ip_fx} and e^{-ip_ix} will appear for the outgoing and incoming fermions. The integration over x in Eq. (15.3) will result in the energy-momentum conserving δ-function

$$(2\pi)^4\delta^{(4)}(p_f - p_i - q). \tag{15.5}$$

Hence, the rules for computing the quantity T^{μ}_{fi} are the same that we use to compute the scattering amplitudes for an electrons' scattering on a virtual "photon" except that this "photon" does not have the polarization vector or an associated Green's function and is characterized by a Lorentz index μ. Similar to the scattering amplitude, we remove the δ-function and call the rest the vertex function Γ^{μ}. In general,

$$\Gamma^{\mu}(p_f, p_i, q; s_f, s_i) = -i\bar{u}(p_f, s_f)\hat{O}^{\mu}u(p_i, s_f), \tag{15.6}$$

where $\hat{O}^{\mu}$ is a four-by-four matrix that depends on p_f, p_i and q.

It is straightforward to check that at leading order in the electron charge e $O^{\mu} = \gamma^{\mu}$, so that

$$\Gamma^{\mu}_0 = -i\bar{u}(p_f, s_f)\gamma^{\mu}u(p_i, s_f), \tag{15.7}$$

as, perhaps, to be expected.

We would like to discuss now what can be said about the vertex function Γ^{μ} beyond perturbation theory. We definitely know that Eq. (15.6) is an exact equation. We also know that an arbitrary four-by-four matrix O^{μ} can only depend on four-vectors p_1, p_2 and q and that an arbitrary four-by-four matrix can be written as a linear combination of 16 matrices

$$1, \quad \gamma_5, \quad \gamma^{\mu}, \quad \gamma_{\mu}\gamma_5, \quad \sigma_{\mu\nu}, \tag{15.8}$$

where $\sigma^{\mu\nu} = i/2[\gamma^{\mu}, \gamma^{\nu}]$. Then,

$$O^{\mu} = A^{\mu} + B^{\mu}\gamma_5 + C^{\mu}_{\alpha}\gamma^{\alpha} + D^{\mu}_{\alpha}\gamma^{\alpha}\gamma_5 + +E^{\mu}_{\alpha\beta}\sigma^{\alpha\beta}. \tag{15.9}$$

The various vectors and tensors must be composed of three vectors p_f, p_i, q, the metric tensor $g^{\mu\nu}$ and the Levi-Civita tensor $\epsilon^{\mu\nu\alpha\beta}$. Moreover, all these quantities can be functions of Lorentz-invariant combinations of scalar products of the three momenta p_f, p_i and q.

Although it appears that number of various tensors and vectors is high, we can reduce their number taking into account that i) some of these contributions (the ones with γ_5 and the Levi-Civita tensor) lead to parity-violating interactions whereas QED conserves parity; ii) we actually need a matrix element of O^{μ}, i.e. $\bar{u}(p_f, s_f)\hat{O}^{\mu}u(p_i, s_f)$, iii) spinors satisfy respective Dirac equations and iv) $p_f - p_i = q$. Choosing q^{μ} and $\pi^{\mu} = p^{\mu}_f + p^{\mu}_i$ as two independent vectors, we can write

$$A^{\mu} = A_1\pi^{\mu} + A_2 q^{\mu}, \quad B^{\mu} = 0, \quad D^{\alpha} = 0, \tag{15.10}$$

where the last two equation follow from the parity-conserving nature of QED.

Next, consider the tensor C^{μ}_{α}. Its most general form is

$$C^{\mu}_{\alpha} = C_1 g^{\mu}_{\alpha} + C_2 q^{\mu} q_{\alpha} + C_3 \pi^{\mu} \pi_{\alpha} + C_4 q^{\mu} \pi_{\alpha} + C_5 \pi^{\mu} q_{\alpha}. \tag{15.11}$$

It we contract this tensor with γ^{α} and compute the matrix element of the resulting matrix w.r.t $\bar{u}(p_f)$ and $u(p_i)$, the result simplifies because

$$\bar{u}(p_f)\hat{q}u(p_1) = 0, \quad \bar{u}(p_f)\hat{\pi}u(p_1) = 2m\bar{u}(p_f)u(p_1). \tag{15.12}$$

This means that some of the terms in Eq. (15.11) will vanish and some can be combined with A_1 and A_2. The only new term is the one with $C_1 g^{\mu}_{\alpha}$ which just returns the matrix element of γ^{μ}. A similar analysis of the tensor $E^{\mu}_{\alpha\beta}$ reveals that the only relevant contribution is $g^{\mu\alpha} q^{\beta}$.

Hence, we can write

$$\Gamma^{\mu} = -i\bar{u}(p_f) \left[A_1 \pi^{\mu} + A_2 q^{\mu} + C_1 \gamma^{\mu} + E_1 \sigma^{\mu\alpha} q_{\alpha} \right] u(p_i). \tag{15.13}$$

An additional constraint arises because Γ^{μ} represents a matrix element of the conserved current which implies that

$$q_{\mu} \Gamma^{\mu} = 0. \tag{15.14}$$

Since $\pi_{\mu} q^{\mu} = 0$ and $\bar{u}(p_f)\hat{q}u(p_i) = 0$, it follows that $A_2 = 0$. Finally, the number of independent structures can be further reduced by using the so-called Gordon identity. It reads

$$\bar{u}(p_f)\gamma^{\mu}u(p_i) = \bar{u}(p_f) \left[\frac{\pi^{\mu}}{2m} + \frac{i\sigma^{\mu\nu} q_{\nu}}{2m} \right] u(p_i). \tag{15.15}$$

This equation allows us to remove π^{μ} as an independent Lorentz structure in the computation of the vertex function. Hence, we finally find,

$$\Gamma^{\mu} = -i\bar{u}(p_f) \left[F_1(q^2)\gamma^{\mu} + F_2(q^2) \frac{i\sigma^{\mu\alpha} q_{\alpha}}{2m} \right] u(p_i), \tag{15.16}$$

where the two functions $F_{1,2}(q^2)$ are called Dirac and Pauli form factors, respectively.[1]

[1] Form factors are dimensionless quantities, so they are functions of q^2/m^2.

We will now discuss the meaning of these form factors. The idea is the following. Consider the non-relativistic limit of the electron scattering at the external electromagnetic field. In the non-relativistic limit $q^2/m^2 \ll 1$ and we can expand the form factors in series in q^2/m^2. The two first terms in the series $F_1(0)$ and $F_2(0)$ can be matched to the electron scattering in the electromagnetic field described by the Pauli Hamiltonian

$$H = \frac{(\vec{p} - e\vec{A})^2}{2m} + e\phi - \vec{\mu} \cdot \vec{B}, \tag{15.17}$$

where $\vec{\mu} = \mu\vec{\sigma}/2$ is the electron's magnetic moment.

This comparison leads to the conclusion that

$$F_1(0) = 1, \tag{15.18}$$

and

$$\vec{\mu} = \frac{e}{m}(F_1(0) + F_2(0))\frac{\vec{\sigma}}{2}. \tag{15.19}$$

In quantum mechanics, the magnetic moment of the electron is usually parameterized in terms of gyromagnetic factor g_e and Bohr magniton

$$\vec{\mu} = g_e \, \frac{e}{2m}\frac{\vec{\sigma}}{2}, \tag{15.20}$$

The two expressions match if

$$g_e = 2(F_1(0) + F_2(0)), \tag{15.21}$$

so that

$$F_2(0) = (g_e - 2)/2. \tag{15.22}$$

Note that the Dirac equation *predicts* that electron's g_e-factor equals to two. Hence, if $F_2(0)$ is different from zero, it would imply an effect that is not captured by the Dirac equation.[2]

We will now compute the two form factors in QED perturbation theory. At the second order of the perturbative expansion there is one diagram to consider. The expression reads

$$\begin{aligned} \Lambda_1^\mu = \bar{u}(p_f)\Gamma_1^\mu u(p_i) = -e^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \\ \times \frac{\bar{u}(p_f)\gamma^\alpha(\hat{p}_f + \hat{k} + m)\gamma^\mu(\hat{p}_i + \hat{k} + m)\gamma_\alpha u(p_i)}{((k+p_f)^2 - m^2)((k+p_i)^2 - m^2)(k^2)}. \end{aligned} \tag{15.23}$$

[2]For this reason $F_2(0)$ is called the *anomalous* magnetic moment of an electron.

Note that we should have added $+i0$ to all propagators; we do not display it for brevity.

As the next step, I would like to show that Λ_1^μ, as written, *cannot be computed*. To see this, consider the contribution that comes from very large values of the loop momentum k. Then, neglecting external momenta and masses in the integrand, we arrive at

$$\begin{aligned} &-e^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\bar{u}(p_f)\gamma^\alpha \hat{k}\gamma^\mu \hat{k}\gamma_\alpha u(p_i)}{(k^2)^3} \\ &= -e^2 \bar{u}(p_f)\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\rho\gamma_\alpha u(p_i) \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{k_\beta k_\rho}{(k^2)^3}. \end{aligned} \tag{15.24}$$

Since

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{k_\beta k_\rho}{(k^2)^3} = \frac{g_{\beta\rho}}{4} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^4}, \tag{15.25}$$

and

$$\gamma^\alpha\gamma^\beta\gamma^\mu\gamma_\beta\gamma_\alpha = 4\gamma^\mu, \tag{15.26}$$

we conclude that a contribution to Λ_1^μ from very large loop momenta reads

$$\Lambda_1^\mu \approx -e^2 \bar{u}(p_1)\gamma^\mu u(p_2) \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^4}. \tag{15.27}$$

The remaining integral is *infinite*. To see this, we imagine doing a radial integration in the four-dimensional space and find

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^4} \sim \int\limits_\mu^M \frac{\mathrm{d}k}{k} \sim \ln \frac{M}{\mu}. \tag{15.28}$$

Considering $M \to \infty$ limit, we conclude that Λ_1^μ indeed becomes infinite. One aspect to notice is that this divergence is proportional to leading order vertex function $\bar{u}(p_f)\gamma^\mu u(p_i)$ and, therefore, only contributes to the Dirac form factor F_1.

Next, we consider the opposite situation – the limit of the small loop momenta $k \to 0$. In this case, in the integrand in Eq. (15.23) we can neglect k in the *numerator*. Then we find

$$\begin{aligned} &\bar{u}(p_f)\gamma^\alpha(\hat{p}_f + \hat{k} + m)\gamma^\mu(\hat{p}_i + \hat{k} + m)\gamma_\alpha u(p_i) \\ &= \bar{u}(p_f)\gamma^\alpha(\hat{p}_f + m)\gamma^\mu(\hat{p}_i + m)\gamma_\alpha u(p_i) = (4p_f p_i)\bar{u}(p_f)\gamma^\mu u(p_i). \end{aligned} \tag{15.29}$$

The propagators can also be simplified, e.g.

$$(p_i + k)^2 - m^2 \approx 2p_i k, \quad (p_f + k)^2 - m^2 \approx 2p_f k. \tag{15.30}$$

Hence, in the limit of the small loop momentum, we find

$$\Lambda_1^\mu = -e^2(4p_i p_f)\bar{u}(p_f)\gamma^\mu u(p_i) \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{1}{(2p_i k)(2p_f k)(k^2)}. \tag{15.31}$$

This integral diverges logarithmically at small k and, again, this divergence only affects the Dirac form factor F_1 and not the Pauli form factor F_2.

Developing an understanding of how to properly deal with the divergences that we have just seen, both for F_1 at one loop but also in general, played a very important role in the development of quantum field theory. We will talk about this in the next lecture. At the same time, since all problems reside in F_1, we can attempt to compute $F_2(q^2)$ which should appear at the first oder in perturbation theory for the first time.

Calculation of loop integrals as in Eq. (15.23) proceed in several steps. The first step is to simplify the integrand and this is usually done by combining the propagators using Feynman's trick. To this end, we employ the formula

$$\begin{aligned}\frac{1}{A_1^{m_1}A_2^{m_2}...A_N^{m_N}} &= \frac{\Gamma(m_1+...+m_N)}{\Gamma(m_1)...\Gamma(m_N)} \\ &\times \int \prod_{i=1}^{N} \mathrm{d}x_i \; \delta(1-\sum_{i=1}^{N} x_i)\frac{\prod\limits_{i=1}^{N} x_i^{m_i-1}}{(A_1x_1 + A_2x_2 + ...A_Nx_N)^{\sum\limits_{i=1}^{N} m_i}},\end{aligned} \tag{15.32}$$

and write

$$\begin{aligned}&\frac{1}{((k+p_f)^2 - m^2)((k+p_i)^2-m^2)(k^2)} \\ &= \Gamma(3)\int \prod_{i=1}^{3} \mathrm{d}x_i \; \delta(1-x_1-x_2-x_3)\frac{1}{(k^2+2kP+i0)^3},\end{aligned} \tag{15.33}$$

where $P = p_f x_1 + p_i x_2$ and we restored infinitesimal imaginary part $i0$ in all progagators. In what follows I will use the notation

$$[\mathrm{d}x]_3 = \prod_{i=1}^{3} \mathrm{d}x_i \; \delta(1-x_1-x_2-x_3), \tag{15.34}$$

for brevity.

We use the representation shown in Eq. (15.33) in the integrand of Eq. (15.23), shift the loop momentum to $k = l - P$ and find

$$\begin{aligned}\Lambda_1^\mu = -e^2\Gamma(3)\int [\mathrm{d}x]_3 \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \\ \times \frac{\bar{u}(p_f)\gamma^\alpha(\hat{p}_f + \hat{l} - \hat{P} + m)\gamma^\mu(\hat{p}_i + \hat{l} - \hat{P} + m)\gamma_\alpha u(p_i)}{(l^2 - P^2 + i0)^3}.\end{aligned} \tag{15.35}$$

Since the denominator of the integrand depends on l^2, linear l-terms in the above equation can be dropped. It is also easy to convince oneself that the term quadratic in l contributes only to the Dirac form factor. Hence, if we are only interested in computing first non-vanishing contribution to the Pauli form factor, we can write

$$\begin{aligned}[\Lambda_1^\mu]_{F_2} = -e^2\Gamma(3)\int [\mathrm{d}x]_3 \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \\ \times \frac{\bar{u}(p_f)\gamma^\alpha(\hat{p}_f - \hat{P} + m)\gamma^\mu(\hat{p}_i - \hat{P} + m)\gamma_\alpha u(p_i)}{(l^2 - P^2 + i0)^3}.\end{aligned} \tag{15.36}$$

Then, using

$$\gamma_\alpha\gamma^\mu\gamma^\alpha = -2\gamma^\mu,\ \gamma_\alpha\gamma^\mu\gamma^\nu\gamma^\alpha = 4g^{\mu\nu},\ \ \gamma_\alpha\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\alpha = -2\gamma^\rho\gamma^\nu\gamma^\mu, \tag{15.37}$$

we rewrite the numerator in Eq. 15.36) as

$$\bar{u}(p_f)\Big[-2(\hat{p}_i - \hat{P})\gamma^\mu(\hat{p}_f - \hat{P}) + 4m(p_f^\mu + p_i^\mu - 2P^\mu) - 2m^2\gamma^\mu\Big]u(p_i) \tag{15.38}$$

We can drop the last term, as it contributes only to the Dirac form factor, and rewrite the second one by re-introducing γ^μ. We find

$$\bar{u}(p_f)\Big[-2(\hat{p}_i - \hat{P})\gamma^\mu(\hat{p}_f - \hat{P}) + 2m\left\{(\hat{p}_f + \hat{p}_i - 2\hat{P}), \gamma^\mu\right\}\Big]u(p_i). \tag{15.39}$$

We can now simplify this expression by expanding it to first order in q and systematically neglecting all the terms that only contribute to the Dirac form factor. We do this by expressing all momenta through p_f and q if they appear to the left of γ^μ and through p_i and q if they appear to the right. This allows us to use the Dirac equation for $\bar{u}(p_f)$ and $u(p_i)$ and we find

$$\begin{aligned}
&\bar{u}(p_f)(\hat{p}_f + \hat{p}_i - 2\hat{P})\gamma^\mu u(p_i) \to -(1-2x_2)\bar{u}(p_f)\hat{q}\gamma^\mu u(p_i),\\
&\bar{u}(p_f)\gamma^\mu(\hat{p}_f + \hat{p}_i - 2\hat{P})u(p_i) \to \ (1-2x_1)\bar{u}(p_f)\gamma^\mu\hat{q}u(p_i),\\
&\bar{u}(p_f)(\hat{p}_i - \hat{P})\gamma^\mu(\hat{p}_f - \hat{P})u(p_i) \to\\
&m\, x_3\bar{u}(p_f)(\gamma^\mu\hat{q}(1-x_1) - \hat{q}\gamma^\mu(1-x_2))u(p_i).
\end{aligned} \tag{15.40}$$

Finally, we collect all the terms, write

$$\gamma^\mu\hat{q} = q^\mu - i\sigma^{\mu\nu}q_\nu, \quad \hat{q}\gamma^\mu = q^\mu + i\sigma^{\mu\nu}q_\nu, \tag{15.41}$$

and discard all q_μ terms because they have to vanish. Finally, we obtain

$$\begin{aligned}
&[\Lambda_1^\mu]_{F_2} = e^2\bar{u}(p_f)\frac{i\sigma^{\mu\nu}q_\nu}{2m}u(p_i)\times\\
&\times (2m)^2 \int [\mathrm{d}x]_3\, x_3(1-x_3)\,\Gamma(3) \int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{1}{(l^2 - P^2 + i0)^3}.
\end{aligned} \tag{15.42}$$

We need to to integrate over the shifted loop momentum l. To do this, we consider a generalized integral

$$I(n) = \int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{1}{(l^2 - \Delta + i0)^n}. \tag{15.43}$$

Assuming $\Delta \geq 0$, we determine poles in the complex l_0 plane and find

$$l_0 = \pm\sqrt{\vec{l}^2 + \Delta} \mp i0. \tag{15.44}$$

The location of poles implies that we can deform the integration contour (the real axis) in such a way that we integrate over l_0 imaginary axis in the l_0 complex plane from $-i\infty$ to $i\infty$ without changing the result. Then, we write $l_0 = i\tilde{l}_0$ and find

$$I(n) = i(-1)^n \int \frac{\mathrm{d}^4 l_E}{(2\pi)^4}\frac{1}{(l_E^2 + \Delta)^n}, \tag{15.45}$$

where l_E is the Euclidean vector, so that $l_E^2 = \tilde{l}_0^2 + \vec{l}^2$. To compute the above integral, we use spherical coordinates in the four-dimensional space. Denoting the solid angle as Ω_4, we write

$$I(n) = i(-1)^n\frac{\Omega_4}{(2\pi)^4}\,\frac{1}{2}\int\limits_0^\infty \frac{l_E^2\mathrm{d}l_E^2}{(l_E^2 + \Delta)^n}. \tag{15.46}$$

Changing the integration variable $l_E^2 \to u$,

$$l_E^2 = \Delta \, \frac{u}{(1-u)}, \tag{15.47}$$

we find

$$\begin{aligned} I(n) &= i(-1)^n \frac{\Omega_4}{(2\pi)^4 \Delta^{n-2}} \frac{1}{2} \int_0^1 \mathrm{d}u \, \frac{u}{(1-u)^3} (1-u)^n \\ &= i(-1)^n \frac{\Omega_4}{(2\pi)^4 \Delta^{n-2}} \frac{1}{2} \frac{\Gamma(2)\Gamma(n-2)}{\Gamma(n)}. \end{aligned} \tag{15.48}$$

Finally, using

$$\Omega_4 = 2\pi^2 \tag{15.49}$$

we obtain

$$I(n) = \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{1}{(l^2 - \Delta + i0)^n} = \frac{i(-1)^n}{(4\pi)^2 \Delta^{n-2}} \frac{\Gamma(n-2)}{\Gamma(n)}. \tag{15.50}$$

In our case (cf. Eq. (15.42)), $n = 3$ and

$$\Delta = P^2 = m^2 (x_1 + x_2)^2 - q^2 x_1 x_2. \tag{15.51}$$

We obtain

$$[\Lambda_1^\mu]_{F_2} = -\frac{4ie^2}{(4\pi)^2} \, \bar{u}(p_f) \frac{i\sigma^{\mu\nu} q^\nu}{2m} u(p_i) \int \frac{[\mathrm{d}x]_3 x_3 (1-x_3)}{(x_1+x_2)^2 - q^2/m^2 x_1 x_2}. \tag{15.52}$$

Comparing this result with the general parameterization of the vertex function, we find

$$F_2(q^2) = \frac{\alpha}{\pi} \int \frac{[\mathrm{d}x]_3 x_3 (1-x_3)}{(x_1+x_2)^2 - q^2/m^2 x_1 x_2}, \tag{15.53}$$

where we introduced the fine structure constant $\alpha = e^2/(4\pi) \approx 1/137$.

To compute $F_2(0)$, we note that $x_1 + x_2 = (1 - x_3)$. Then, since

$$\int [\mathrm{d}x]_3 = \int_0^1 \mathrm{d}x_3 \int \mathrm{d}x_2 \, \theta(1 - x_3 - x_2) = \int_0^1 \mathrm{d}x_3 \, (1 - x_3), \tag{15.54}$$

the integral in Eq. (15.53) simplifies to

$$\int [\mathrm{d}x]_3 \, \frac{x_3(1-x_3)}{(1-x_3)^2} = \frac{1}{2}. \tag{15.55}$$

This implies

$$F_2(0) = \frac{\alpha}{2\pi}. \tag{15.56}$$

This is the result for the anomalous magnetic moment of the electron that was first obtained by J. Schwinger in 1947. Since $\alpha = 1/137$, our prediction is $a_e = 0.00116141$. The recent experimental measurement and significantly improved theoretical calculations give

$$\begin{aligned} a_e^{\mathrm{exp}} &= 0.00115965218073(28), \\ a_e^{\mathrm{the}} &= 0.00115965218161(23). \end{aligned} \tag{15.57}$$