TTP1
Lecture 16

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16 The Dirac form factor and the various divergences

In the previous lecture we showed that interaction of electrons with the electromagnetic field is described by the two form factors and we discussed the one-loop calculation of the Pauli form factor at zero momentum transfer in some detail. We have also pointed out that the Dirac form factor suffers from a pathological behavior and in this lecture we will talk more about it.

An important idea that should allow us to investigate F_1 is that of the regularization. This means that we need to modify the calculation in such a way that we remove (or regulate) the apparent infinities and, once this is accomplished, we can discuss what happens when the regulator is lifted.

We begin with repeating the expression for the one-loop vertex function which we already saw in the previous lecture

$$\Lambda_1^\mu = \bar{u}(p_f)\Gamma_1^\mu u(p_i) = -e^2 \int \frac{d^4 k}{(2\pi)^4} \times \frac{\bar{u}(p_f)\gamma^\alpha(\hat{p}_f + \hat{k} + m)\gamma^\mu(\hat{p}_i + \hat{k} + m)\gamma_\alpha u(p_i)}{((k+p_f)^2 - m^2)((k+p_i)^2 - m^2)(k^2)}. \tag{16.1}$$

We have seen that Λ_1^μ can be described by two form factors. We have also seen that the expression for the Dirac form factor is pathological in that it contains divergences at $k \to \infty$, that we will refer to as ultraviolet, and at $k \to 0$, that we will refer to as infra-red.

To regularize these divergences, we will rewrite the photon propagator as

$$\frac{1}{k^2} \to \frac{1}{k^2 - \lambda^2} - \frac{1}{k^2 - M^2}. \tag{16.2}$$

We will consider M to be the largest parameter in the problem, i.e. significantly larger than electron energies and masses and λ to be the smallest parameter in the problem, i.e. much smaller than electron energies and masses. We will be interested in all terms which are not suppressed as powers of p/M and of λ/p. We will also focus exclusively on the calculation of the Dirac form factor.

As we already discussed previously, the way to proceed is to combine the propagators and shift the loop momentum. We find

$$\Lambda_1^\mu = -e^2\Gamma(3)\int [\mathrm{d}x]_3 \int \frac{\mathrm{d}^4 l}{(2\pi)^4}\left(\frac{1}{(l^2-P^2-\lambda^2 x_3)^3} - \frac{1}{(l^2-P^2-M^2x_3)^3}\right) \times \bar{u}(p_f)\gamma^\alpha(l+\hat{p}_f-P+m)\gamma^\mu(l+\hat{p}_i-P+m)\gamma_\alpha u(p_i). \tag{16.3}$$

In the numerator, we can drop terms that are linear in l and re-write the integrand as

$$\left(\frac{1}{(l^2-P^2-\lambda^2x_3)^3} - \frac{1}{(l^2-P^2-M^2x_3)^3}\right)\bar{u}(p_f)\gamma^\alpha\hat{l}\gamma^\mu\hat{l}\gamma_\alpha u(p_i) + \frac{\bar{u}(p_f)\gamma^\alpha(\hat{p}_f-P+m)\gamma^\mu(\hat{p}_i-P+m)\gamma_\alpha u(p_i)}{(l^2-P^2-\lambda^2x_3)^3}. \tag{16.4}$$

Note that we dropped $1/(l^2-P^2-M^2x_3)^3$ in the second term in the above equation because its contribution is suppressed as p^2/M^2.

We will now compute the contribution of the first term. Averaging over directions of l, we find

$$\left(\frac{1}{(l^2-P^2-\lambda^2x_3)^3} - \frac{1}{(l^2-P^2-M^2x_3)^3}\right)\bar{u}(p_f)\gamma^\alpha\hat{l}\gamma^\mu\hat{l}\gamma_\alpha u(p_i) \to \left(\frac{l^2}{(l^2-P^2-\lambda^2x_3)^3} - \frac{l^2}{(l^2-P^2-M^2x_3)^3}\right)\bar{u}(p_f)\gamma^\mu u(p_i). \tag{16.5}$$

To integrate this expression over l, we again perform the Wick rotation and find

$$\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\left(\frac{l^2}{(l^2-\Delta)^3} - \frac{l^2}{(l^2-\Delta_M)^3}\right) = \frac{i\Omega_4}{(2\pi)^4}\frac{1}{2}\int_0^\infty \mathrm{d}l_E^2\; l_E^4\left(\frac{1}{(l_E^2+\Delta)^3} - \frac{1}{(l_E^2+\Delta_M)^3}\right) \tag{16.6}$$

To compute the last integral, we introduce a function

$$G(\Delta,\Delta_M) = \int_0^\infty \mathrm{d}l_E^2\; l_E^4\left(\frac{1}{(l_E^2+\Delta)^3} - \frac{1}{(l_E^2+\Delta_M)^3}\right), \tag{16.7}$$

and compute

$$\partial_\Delta G(\Delta, \Delta_M) = -3 \int_0^\infty \mathrm{d} l_E^2 \; l_E^4 \frac{1}{(l_E^2 + \Delta)^4} = -\frac{1}{\Delta}. \tag{16.8}$$

Hence,

$$G(\Delta, \Delta_M) = -\ln \frac{\Delta}{\Delta_M}, \tag{16.9}$$

where we used the definition of the function G to deduce the boundary condition $G(\Delta_M, \Delta_M) = 0$. Hence, we find

$$\int \frac{\mathrm{d}^4 l}{(2\pi)^4} \left(\frac{l^2}{(l^2 - \Delta)^3} - \frac{l^2}{(l^2 - \Delta_M)^3} \right) = -\frac{i}{(4\pi)^2} \ln \frac{\Delta}{\Delta_M}. \tag{16.10}$$

As the next step we need to extract contribution to the Dirac form factor from the l-independent contribution to the numerator in Eq. (16.4). We contract Dirac indices and find

$$\bar{u}(p_f) \Big[-2(\hat{p}_i - \hat{P})\gamma^\mu(\hat{p}_f - \hat{P}) + 4m(p_f^\mu + p_i^\mu - 2P^\mu) - 2m^2\gamma^\mu \Big] u(p_i). \tag{16.11}$$

We now re-write the above expression neglecting linear terms in q since they will either cancel, or combine into the tensor structure $\sigma^{\mu\nu} q_\nu$. We find

$$\begin{aligned} &-2(\hat{p}_i - \hat{P})\gamma^\mu(\hat{p}_f - \hat{P}) \to \left(-2m^2 x_3^2 - 2q^2(1-x_2)(1-x_3)\right)\gamma^\mu, \\ &4m(p_f^\mu + p_i^\mu - 2P^\mu) \to 4m\pi^\mu x_3 \to 8m^2 x_3 \gamma^\mu, \end{aligned} \tag{16.12}$$

where in the last step we used the Gordon identity. The complete expression for the numerator becomes

$$\left(-2m^2(1 + x_3^2) + 8m^2 x_3 - 2q^2(1-x_1)(1-x_2)\right)\gamma^\mu. \tag{16.13}$$

The integration over l is the same as in the previous lecture, so that

$$[\Lambda_1^\mu]_{F_1} = -i\bar{u}(p_f)\gamma^\mu u(p_i) F_1^{(1)}(q^2), \tag{16.14}$$

where the one-loop contribution to the Dirac form factor reads

$$\begin{aligned} F_1^{(1)}(q^2) = \frac{\alpha}{4\pi} \int [\mathrm{d}x]_3 \Bigg[& 2 \ln \frac{\Delta_M}{\Delta} \\ & - \frac{(-2m^2(1 + x_3^2) + 8m^2 x_3 - 2q^2(1-x_1)(1-x_2))}{\Delta} \Bigg]. \end{aligned} \tag{16.15}$$

The quantities Δ and Δ_M are given by

$$\begin{aligned} \Delta \;\; &= m^2(x_1+x_2)^2 - q^2 x_1 x_2 + \lambda^2 x_3, \\ \Delta_M &= m^2(x_1+x_2)^2 - q^2 x_1 x_2 + M^2 x_3 \approx M^2 x_3. \end{aligned} \tag{16.16}$$

There are three points that we can learn from the above result. First, the dependence of $F_1^{(1)}(q^2)$ on the ultraviolet regulator M is logarithmic and it does not depend on the energies and momenta of the electrons,

$$F_1^{(1)}(q^2) \approx \frac{\alpha}{2\pi} \ln \frac{M^2}{m^2}. \tag{16.17}$$

In other words, this divergence is the same for $F_1^{(1)}(0)$ and for $F_1^{(1)}(q^2)$ for $q^2 \neq 0$. Second, $F_1^{(1)}(0) \neq 0$, in variance with our general argument that states that $F_1(0) = 1$. Finally, the infrared divergence is a non-trivial function of q, as we will now explain.

The infrared divergence implies that if we set $\lambda = 0$ in the expression for $F_1^{(1)}$, we will get an integrand that cannot be integrated. To see how this comes about, we set $\lambda \to 0$, change the integration variables $x_1 = xy, x_2 = x(1-y)$ and $x_3 = 1 - x$ and obtain

$$\Delta \to x^2 \left(m^2 - q^2 y(1-y) \right). \tag{16.18}$$

The integration measure becomes

$$\int [\mathrm{d}x]_3 = \int_0^1 x \mathrm{d}x \int_0^1 \mathrm{d}y. \tag{16.19}$$

Hence, we find

$$\int \frac{[\mathrm{d}x]_3}{\Delta} \to \int_0^1 \frac{\mathrm{d}x}{x} \int_0^1 \frac{\mathrm{d}y}{m^2 - q^2 y(1-y)} \tag{16.20}$$

We observe that the integral diverges at $x = 0$. Hence, unless the numerator of the second term in Eq. (16.15) vanishes, the integral needs to be regularized by keeping λ small but finite. At the same time, we also see that the only term in the numerator that needs to be regulated is the one that survives the

$x \to 0$ limit. This limit corresponds to $x_1 = 0$, $x_2 = 0$ and $x_3 = 1$, so that the numerator becomes

$$-4m^2 + 8m^2 - 2q^2 = 4(p_f p_i). \tag{16.21}$$

Hence,

$$F_1(q^2)|_{\text{ir. div}} = -\frac{\alpha}{2\pi} \int_0^1 x\mathrm{d}x\, \mathrm{d}y\, \frac{2p_f p_i}{x^2 f(y) + \lambda^2(1-x)}, \tag{16.22}$$

with

$$f(y) = m^2 - q^2 y(1-y). \tag{16.23}$$

To compute this integral, we split the integration region over x into two intervals

$$0 < x < \sigma, \quad \sigma < x < 1, \tag{16.24}$$

and we choose σ such that

$$\frac{\lambda}{\sqrt{f(y)}} \ll \sigma \ll 1. \tag{16.25}$$

This choice implies that, when integrating over the first interval, we can set $\lambda^2(1-x) \to \lambda^2$ and when we integrate over the second interval, we can set $\lambda \to 0$. Then

$$\begin{aligned} \int_0^\sigma \frac{x\mathrm{d}x}{x^2 f(y) + \lambda^2(1-x)} &\approx \int_0^\sigma \frac{x\mathrm{d}x}{x^2 f(y) + \lambda^2} = \frac{1}{2}\int_0^{\sigma^2} \frac{\mathrm{d}x^2}{x^2 f(y) + \lambda^2} \\ &\approx \frac{1}{2f(y)} \ln \frac{\sigma^2 f(y)}{\lambda^2}, \end{aligned} \tag{16.26}$$

and

$$\int_\sigma^1 \frac{x\mathrm{d}x}{x^2 f(y) + \lambda^2(1-x)} \approx \int_\sigma^1 \frac{x\mathrm{d}x}{x^2 f(y)} = \frac{1}{f(y)} \ln \frac{1}{\sigma}. \tag{16.27}$$

Adding the two contributions, we obtain

$$F_1(q^2)_{\text{ir. div}} = -\frac{\alpha}{2\pi} 2p_f p_i \int_0^1 \frac{\mathrm{d}y}{2f(y)} \ln \frac{f(y)}{\lambda^2}. \tag{16.28}$$

This result contains more than just the infrared divergence (the term that blows up if the limit $\lambda \to 0$ is attempted). We can separate this term by writing

$$F_1^{(1)}(q^2)_{\text{ir. div}} = -\frac{\alpha}{4\pi}\ \ln\frac{\mu^2}{\lambda^2}\ \int_0^1 \mathrm{d}y\ \frac{2p_f p_i}{f(q^2,y)} + ..., \tag{16.29}$$

where ellipses denote contributions that remain finite when the limit $\lambda \to 0$ is taken.

Although this integral can be computed, we will not need it. However, it is useful to summarize the result of our calculations so far by writing the following expression for the Dirac form factor

$$F_1(q^2) = 1 + \frac{\alpha}{2\pi}\ln\frac{M^2}{m^2} - \frac{\alpha}{4\pi}\ \ln\frac{\mu^2}{\lambda^2}\ \int_0^1 \mathrm{d}y\ \frac{2m^2-q^2}{f(q^2,y)} + \text{finite}. \tag{16.30}$$

This result is not satisfactory because it depends on the auxiliary parameters M and λ and the required limits $M \to \infty$ and $\lambda \to 0$ cannot be computed.

We will see that we need *two distinct* ideas to solve these two problems. The first idea addresses the $M \to \infty$ limit; it emphasizes the role of measured parameters in the construction of the theory and it is known under the name of *renormalization*. The second idea addresses $\lambda \to 0$ limit and emphasizes the need to re-think what we call "asymptotic states" for charged particles.

Let us begin with the $M \to \infty$ problem. Since we are able to compute $F_1(q^2)$ in a theory with the regulator, we can investigate the result of such a calculation. In particular, we can check whether our general conclusion that $F_1(0) = 1$ holds. We immediately find that it *does not*. The reason $F_1(0)$ should be equal to one is the definition of the electric charge; hence, if $F_1(0)$ is not one, it appears as if we do calculations in quantum field theory using *wrong electric charge of the electron*. Let us call this wrong charge e_0. Then, we have computed the following quantity

$$-ie_0F_1(q^2,e_0). \tag{16.31}$$

We know that at $q^2 = 0$, this quantity should be equal to $-ie$, where e is the *true* charge of the physical electron. Then

$$-ie_0F_1(q^2,e_0) = -ie_0F_1(0,e_0)\frac{F_1(q^2,e_0)}{F_1(0,e_0)} = -ie\frac{F_1(q^2,e_0)}{F_1(0,e_0)}. \qquad (16.32)$$

We can compute the ratio of form factors in perturbation theory. Since

$$F_1(q^2,e_0) = 1 + F_1^{(1)}(q^2,e_0) + ..., \qquad (16.33)$$

we find that we can write the true form factor as follows

$$F_1(q^2) = 1 + F_1^{(1,\mathrm{phys})} + \cdots , \qquad (16.34)$$

where

$$F_1^{(1),\mathrm{phys}}(q^2) = F^{(1)}(q^2,e_0) - F^{(1)}(0,e_0) \qquad (16.35)$$

The interpretation of the above result is that to obtain the one-loop correction to the Dirac form factor working with properly-defined electric charge of the electron, we need to compute the *difference* between $F^{(1)}(q^2)$ and $F_1^{(1)}(0)$. Since the ultraviolet-divergent contribution to $F^{(1)}$ is independent of q^2, *the difference of the two form factors is free of this problem* and admits $M \to \infty$ limit.

However, this subtraction affects the second problem since $F^{(1)}(0,e_0)$ contains additional infra-red divergences. We find

$$F_1^{(1),\mathrm{phys}} = -\frac{\alpha}{4\pi}\ln\frac{\mu^2}{\lambda^2}\,Y(q^2), \qquad (16.36)$$

where

$$Y(q^2) = \int_0^1 \mathrm{d}y \,\left[\frac{2p_fp_i}{m^2-q^2y(1-y)} - 2\right]. \qquad (16.37)$$

We will discuss how this term disappears in the next lecture.