TTP1
Lecture 17

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17 Photon emission in the electron scattering on an external potential

In the previous lecture, we have studied a process where electron scatters on an external electromagnetic field. Scattering implies a change in the three momentum of an electron over a certain (relatively short) time period. Momentum change is associated with the acceleration. In classical electrodynamics a charged particle that undergoes acceleration radiates electromagnetic waves. How should we describe such a process in quantum field theory?

Radiation implies the energy loss of a radiator; hence, in quantum field theory, it should correspond to an inelastic process where electron both scatters and emits an additional photon,

$$e(p_i) + A \to e(p_f) + \gamma(k), \tag{17.1}$$

where A indicates that external electromagnetic potential is present. The matrix element of this process reads

$$i\mathcal{M}_{fi} = -ieA_\mu \bar{u}(p_f)\left[\frac{\hat{\epsilon}(\hat{p}_f + \hat{k} + m)\gamma^\mu}{(p_f+k)^2 - m^2} + \frac{\gamma^\mu(\hat{p}_i - \hat{k} + m)\hat{\epsilon}}{(p_i-k)^2 - m^2}\right]u(p_i), \tag{17.2}$$

where $A^\mu = A^\mu(q)$ is the vector potential of the external electromagnetic field. In principle, the photon here is in the final state, so one should write ϵ^* instead of ϵ for its polarisation vector, but we will use ϵ for brevity.

We will eventually need this matrix element in a situation where photon has small but non-vanishing mass. Hence, as the first step we want to check the transversality of the matrix element in case $k^2 = \lambda^2 \neq 0$. Then, in Eq. (17.2) we replace $\epsilon \to k$ and find

$$\begin{aligned}
&\bar{u}(p_f)\left[\frac{\hat{k}(\hat{p}_f + \hat{k} + m)\gamma^\mu}{(p_f+k)^2 - m^2} + \frac{\gamma^\mu(\hat{p}_i - \hat{k} + m)\hat{k}}{(p_i-k)^2 - m^2}\right]u(p_i) \\
&= \bar{u}(p_f)\left[\frac{(2p_f k + k^2)\gamma^\mu}{2p_f k + k^2} + \frac{\gamma^\mu(2p_i k - k^2)}{-2p_i k + k^2}\right]u(p_i) = 0,
\end{aligned} \tag{17.3}$$

so the matrix element is transverse for arbitrary k^2. This means that we can use $-g_{\mu\nu}$ for the sum over photon's polarizations.

The next important step is to understand the dependence of the matrix element M_{fi} on the energy of the radiated photon. To this end, we use the

Dirac equation for the external spinors to rewrite Eq. (17.2) in such a way that the dependence on the photon energy becomes clear. We obtain

$$i\mathcal{M}_{fi} = -ieA_\mu(q)\bar{u}(p_f)\left[\left(\frac{2p_f\epsilon}{d_f} + \frac{2p_i\epsilon}{d_i}\right)\gamma^\mu + \frac{\hat{\epsilon}\hat{k}\gamma^\mu}{d_f} - \frac{\gamma^\mu\hat{k}\hat{\epsilon}}{d_f}\right]u(p_i), \tag{17.4}$$

where

$$d_f = 2p_f k + \lambda^2, \quad d_i = -2p_i k + \lambda^2. \tag{17.5}$$

It is clear from this expression that the dependence of the amplitude on the photon energy ω can be schematically described as follows

$$i\mathcal{M}_{fi} = \frac{M_{-1}}{\omega} + M_0 + ..., \tag{17.6}$$

where the first term is singular in the limit $\omega \sim \lambda \to 0$ and the second is regular.

To understand the implications of this (soft) singularity, we will first check if we need to care about it at all. To this end, we set the photon mass to zero and consider the cross section of inelastic process

$$\mathrm{d}\sigma_\gamma \sim |\mathcal{M}_{fi}|^2(2\pi)^4\delta^{(4)}(p_f + k - q - p_i)\frac{\mathrm{d}^3\vec{p}_f}{(2\pi)^3 2E_f}\,\frac{\mathrm{d}^3\vec{k}}{(2\pi)^3 2\omega}. \tag{17.7}$$

The photon phase space reads

$$\frac{\mathrm{d}^3\vec{k}}{(2\pi)^3 2\omega} = \frac{\mathrm{d}\Omega_\gamma\,\omega\mathrm{d}\omega}{2(2\pi)^3} \tag{17.8}$$

If ω is small, we can drop k in the energy-momentum conserving δ-function (physically, this means that if photon's energy is extremely small, photon's presence does not affect the kinematics of electron's scattering on the external vector potential). Then, we replace the amplitude with its singular component in the $\omega \to 0$ limit and find

It is now easy to see that the dependence of this cross section on the (small) photon mass is logarithmic. Indeed such dependence can only arise if the photon is soft $\omega \sim \lambda$. We then use the approximate expression for the amplitude as indicated above and find

$$\mathrm{d}\sigma_\gamma \sim |\mathcal{M}_{-1}|^2(2\pi)^4\delta^{(4)}(p_f - q - p_i)\frac{\mathrm{d}^3\vec{p}_f}{(2\pi)^3 2E_f}\int\limits_0^1 \frac{\mathrm{d}\omega}{\omega} = \infty. \tag{17.9}$$

Hence, we observe that, similar to the one-loop correction to the Dirac form factor, the real-emission cross section cannot be computed without an infra-red regulator.

In what follows, we will use the photon mass as a regulator, repeating what we did for the virtual corrections to the Dirac form factor. One interesting possibility is that the observed dependences on the infra-red regulators in real and virtual contributions is not accidental but a reflection of the fact that this splitting is somewhat unphysical and, if we combine these two quantities in a way that allows us to describe physical observables, the singular dependence on λ cancels out.

To check this hypothesis, we will try to combine the virtual (elastic) and the real-emission (inelastic) contributions. We will only pay attention to terms that develop logarithmic dependence on λ.

As the first step, we need to complete the calculation of the inelastic cross section and for this we need to determine the part of the singular part of the amplitude $\mathcal{M}_{-1}/\omega$ since this part of the amplitude determines the singular dependence of the cross section on λ. In Eq. (17.4), we can neglect λ^2 relative to $2p_{f,i}k \sim E_{f,i}\omega > E_{f,i}\lambda$ in $d_{i,f}$. Then,

$$\begin{aligned} \frac{\mathcal{M}_{-1}}{\omega} &= e\left[\frac{p_f\epsilon}{p_f k} - \frac{p_i\epsilon}{p_i k}\right]\,(-iA_\mu\;\bar{u}(p_f)\gamma^\mu u(p_i)) \\ &= e\,(\epsilon_\mu J^\mu_{\rm eik}(p_f,p_i,k))\;i\mathcal{M}_{\rm el}(p_f,p_i), \end{aligned} \tag{17.10}$$

where $\mathcal{M}_{\rm el}$ is the amplitude of the elastic process (i.e. of a process where the photon emission does not happen) and $J^\mu_{\rm eik}$

$$J^\mu_{\rm eik} = \frac{p_f^\mu}{p_f k} - \frac{p_i^\mu \epsilon}{p_i k}, \tag{17.11}$$

is the *eikonal* current responsible for the emission of soft photons.

It is now straightforward to compute the amplitude squared summed over photon polarizations and spins of the initial-state and final-state electrons. We find

$$\sum |\frac{M_{-1}}{\omega}|^2 = -J^\mu_{\rm eik}J_{\mu,{\rm eik}}\sum|\mathcal{M}_{\rm el}|^2 = e^2 Eik(p_f,p_i,k)\sum|\mathcal{M}_{\rm el}|^2, \tag{17.12}$$

where

$$Eik(p_f,p_i,k) = \frac{2p_f p_i}{(p_f k)(p_i k)} - \frac{m^2}{(p_f k)^2} - \frac{m^2}{(p_i k)^2}. \tag{17.13}$$

Using this expression for the amplitude squared in Eq. (17.12), we derive the following formula for the cross section of the inelastic process in the soft approximation

$$\mathrm{d}\sigma_\gamma \approx \mathrm{d}\sigma_{\mathrm{el}}(p_f, p_i)\, \frac{\alpha}{2\pi}\, \left[\frac{\mathrm{d}^3k}{(2\pi)\omega_k} Eik(p_f, p_i, k)\theta(\omega_{\max} - \omega)\right]. \qquad (17.14)$$

In Eq. (17.14) $\mathrm{d}\sigma_\gamma$ is the cross section of an inelastic process, $\mathrm{d}\sigma_{\mathrm{el}}$ is the cross section of an elastic process and we introduced a Heaviside function to ensure that the energy of the photon does not become large, i.e. $\lambda \gg \omega_{\max} \ll E_{f,i}$ and the soft approximation remains valid.

We now need to integrate the eikonal function over the photon energy and its emission angle. To do this, we will consider three terms in Eq. (17.13) separately. We will start with the last one and we emphasize that we are interested in contributions that behave like $\ln\lambda$. With this clarification in mind, we write

$$\int \frac{\mathrm{d}^3k}{(2\pi)\omega_k}\theta(\omega_{\max} - \omega)\frac{m^2}{(p_i k)^2} = \frac{2m^2}{E_i^2}\int\limits_{\lambda}^{\omega_{\max}} \frac{\beta_\lambda \mathrm{d}\omega}{\omega}\,\frac{\mathrm{d}\Omega_k}{\Omega_k}\frac{1}{(1-\beta_i\beta_\lambda\cos\theta_{ki})^2}$$
$$= \frac{m^2}{E_i^2}\ln\frac{\omega_{\max}}{\lambda}\int\limits_{-1}^{1}\frac{\mathrm{d}\cos\theta_{ki}}{(1-\beta_i\cos\theta_{ki})^2} = \frac{m^2}{E_i^2}\frac{2}{1-\beta_i^2}\,\ln\frac{\omega_{\max}}{\lambda} = 2\ln\frac{\omega_{\max}}{\lambda}. \qquad (17.15)$$

Note that we have set $\beta_\lambda = \sqrt{1-\lambda^2/\omega^2}$ to one in the intermediate stages of the calculation; this is justified because we are only interested in the logarithms of the photon mass.

Since the result in Eq. (17.15) does not depend on the momentum of the electron, we can immediately write down the result for the next integral

$$\int \frac{\mathrm{d}^3k}{(2\pi)\omega_k}\theta(\omega_{\max} - \omega)\frac{m^2}{(p_f k)^2} = 2\ln\frac{\omega_{\max}}{\lambda}. \qquad (17.16)$$

The last integral we need to compute reads

$$\int \frac{\mathrm{d}^3k}{(2\pi)\omega_k}\theta(\omega_{\max} - \omega)\frac{2p_f p_i}{(p_f k)(p_i k)}. \qquad (17.17)$$

Although it looks somewhat different as compared to the other two, we can make this calculation look similar to the previous ones by using Feynman parameters. We write

$$\frac{1}{(p_f k)(p_i k)} = \frac{1}{P_y^2} \int_0^1 \mathrm{d}y \, \frac{P_y^2}{(P_y k)^2}, \tag{17.18}$$

where $P_y = p_f y + p_i(1-y)$. We note that the integral is the same as what we previously considered provided that P_Y^2 is interpreted a a mass parameter squared. We find

$$\int \frac{\mathrm{d}^3 k}{(2\pi)\omega_k} \theta(\omega_{\max} - \omega) \frac{2p_f p_i}{(p_f k)(p_i k)} = 2 \ln \frac{\omega_{\max}}{\lambda} \int_0^1 \mathrm{d}y \, \frac{2p_f p_i}{P_y^2}. \tag{17.19}$$

Since $P_y^2 = m^2 - q^2 y(1-y)$, we find

$$\begin{aligned} &\int \frac{\mathrm{d}^3 k}{(2\pi)\omega_k} \theta(\omega_{\max} - \omega) Eik(p_f, p_i, k) \\ &= 2 \ln \frac{\omega_{\max}}{\lambda} \int_0^1 \mathrm{d}y \left[\frac{2p_f p_i}{m^2 - q^2 y(1-y)} - 2 \right] = \ln \frac{\omega_{\max}^2}{\lambda^2} \, Y(q^2), \end{aligned} \tag{17.20}$$

where we used Eq. (**??**) to identify the last integral with the function Y.

We now use Eq. (17.14) and write the cross section for an inelastic process as follows

$$\mathrm{d}\sigma_\gamma = \mathrm{d}\sigma_{\mathrm{el}}^{(0)} \, \frac{\alpha}{2\pi} \, \ln \frac{\omega_{\max}^2}{\lambda^2} \, Y(q^2). \tag{17.21}$$

In the previous lecture we discussed the calculation of radiative corrections to the elastic process and we have seen there that the Dirac form factor has a logarithmic dependence on the photon mass. Since

$$F_1(q^2) \approx 1 + \frac{\alpha}{2\pi} \ln \frac{\mu^2}{\lambda^2} Y(q^2), \tag{17.22}$$

we easily find that the elastic cross section with $\mathcal{O}(\alpha)$ accuracy reads

$$\mathrm{d}\sigma_{\mathrm{el}} = \mathrm{d}\sigma_{\mathrm{el}}^{(0)} \, \left(1 - \frac{\alpha}{\pi} \ln \frac{\mu^2}{\lambda^2} Y(q^2) \right). \tag{17.23}$$

Adding elastic and inelastic cross sections gives the so-called *inclusive* cross section; inclusive cross section, in the current case, gives us the probability for an electron to scatter.

It follows from our analysis that such an *inclusive* cross section does not contain infra-red divergent terms proportional to $\ln(\lambda)$. The result reads

$$\begin{aligned} \mathrm{d}\sigma_{\mathrm{el}} + \mathrm{d}\sigma_{\gamma} &= \mathrm{d}\sigma_{\mathrm{el}}^{(0)} \left(1 - \frac{\alpha}{\pi} \ln \frac{\mu^2}{\lambda^2}\, Y(q^2)\right) + \mathrm{d}\sigma_{\mathrm{el}}^{(0)} \frac{\alpha}{\pi}\, \ln \frac{\mu^2}{\lambda^2}\, Y(q^2) \\ &= \mathrm{d}\sigma_{\mathrm{el}}^{(0)}. \end{aligned} \tag{17.24}$$

Of course, the fact that we obtained $\mathrm{d}\sigma_{\mathrm{el}}^{(0)}$ for the inclusive cross section is a consequence of keeping only $\ln(\lambda)$ terms in the calculation. In reality, $\mathcal{O}(\alpha)$ corrections do appear on the right hand side of the above equation but they are not singular in the $\lambda \to 0$ limit and for this reason we do not show them.

Interpretation of the cancellation: operationally, it is impossible to define purely "elastic processes" since all detectors have finite energy resolution. All we can say is that the total energy carried by additional photons is smaller than or equal to it. If we interpret ω_{max} as a quantity that is equal to the detector resolution, the inclusive cross section that we computed corresponds to the elastic process where the energy of the electron is determined with $\omega_{\mathrm{max}}/E_i$ relative accuracy.