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Lecture 18

Kirill Melnikov
TTP KIT
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18 Proper treatment of the external legs

In quantum field theory Green's functions contain propagators associated with external legs; this happens both in the position space and in the momentum space. When we construct scattering amplitudes, we use the Lehmann-Symanzik-Zimmermann formula that requires us to act with a Klein-Gordon or Dirac operator on the field related to a particular external leg. Here is the formula for the transition matrix in a scalar field theory from lecture 10

$$iT_{fi} = i^n \int \prod_{i=1}^{n} \mathrm{d}x_i \; e^{i\left(\sum\limits_{j=3}^{n} p_j x_j - p_1 x_1 - p_2 x_2\right)} \prod_{i=1}^{n} \left(\partial_i^2 + m_0^2\right) \langle 0 | T\phi(x_1)...\phi(x_n)|0\rangle, \tag{18.1}$$

and, as we said earlier, Klein-Gordon operators $\partial_i^2 + m_0^2$ "amputate" external legs of Green's functions.[1]

However, in reality the situation is more subtle since, for a generic Green's function one can accumulate corrections that reside on the external legs that cannot be discarded because of the action of Klein-Gordon (or Dirac) operator. To see this, consider ϕ^4-theory as an example. In general, an external leg of any Green's function is a two-point Green's function itself. We have discussed the two-point Green's function in such a theory in Lecture 10. To one loop it reads

$$G(p) = \frac{i}{p^2 - m_0^2} + \frac{i}{p^2 - m_0^2}(-i\Sigma)\frac{i}{p^2 - m_0^2}, \tag{18.2}$$

where

$$-i\Sigma_1 = \frac{\lambda}{2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2 - m_0^2}. \tag{18.3}$$

To compute the contribution of the external-leg correction to the the scattering amplitude in ϕ^4 theory, we need to multiply $G(p)$ with $-p^2 + m_0^2$ and take the limit $p^2 \to m_0^2$. We immediately see that it is impossible to do so since the second term in Eq. (18.2) blows up.

To understand how to handle this situation, consider a two-point Green's function in some scalar theory in the momentum space. We can design a useful representation of the two-point Green's function by considering *one-*

[1] For the reasons that will become clear shortly, we denote the mass of the particle by m_0.

particle irreducible diagrams[2] and treating them as basic contributions to a two point function. Then in general

$$
\begin{aligned}
G(p, m_0) &= \frac{i}{p^2 - m_0^2} + \frac{i}{p^2 - m_0^2}(-i\Sigma(p^2, m_0^2))\frac{i}{p^2 - m_0^2} \\
&+ \frac{i}{p^2 - m_0^2}(-i\Sigma(p^2, m_0^2))\frac{i}{p^2 - m_0^2}(-i\Sigma(p^2, m_0^2))\frac{i}{p^2 - m_0^2} + ... \\
&= \frac{i}{p^2 - m_0^2 - \Sigma(p^2, m_0^2)},
\end{aligned}
\tag{18.4}
$$

where we have re-summed the geometric progression. The *self-energy function* $\Sigma(p^2, m_0^2)$ is composed of one-particle irreducible diagrams only and admits the standard loop expansion.

For ϕ^4 theory the one-loop contribution to Σ is shown in Eq. (18.3); it is independent of the four-momentum p. Then, it follows from Eq. (18.4) that the two-point function has a pole not at $p^2 = m_0^2$ as we originally thought but at $p^2 = m^2$ where

$$
m^2 = m_0^2 + \Sigma(m_0^2). \tag{18.5}
$$

The pole of the two-point function gives us the value of the mass of a free particle. The above result implies that the observed mass of the particle m differs from the mass parameter in the Lagrangian m_0 because of self-interactions. We will refer to the mass parameter in the Lagrangian as the *bare mass*.

The next point is quite obvious. In principle, we are not interested in the bare mass parameter since a proper theory should operate with observable quantities. This is true in general but in quantum field theory the significance of this issue is amplified because the relation between the bare mass and the physical mass usually involves divergent integrals that need to be regularized. For this reason, it becomes a bit more than just an inconvenience to keep the bare mass in the results.

Furthermore, when we match the theory with interactions to a free theory by switching interactions adiabatically as in the case of Lehmann-Symanzik-Zimmermann formula, we need to match to a free theory where particles have

[2] We say that a diagram is "one-particle irreducible" if it can not be turned into two disconnected pieces by cutting a single line.

correct masses since adiabatic switching of interactions is not supposed to change the theory spectrum. This implies that the mass parameter m_0 that appears in Klein-Gordon operator in Eq. (18.1) is supposed to be a physical mass m.

The thing is that we have to write *everything* in terms of the physical mass parameter and we need to assume that m_0^2 that appears in the the LSZ formula is *also* the physical mass parameter. In our ϕ^4-example at one-loop this change alone is sufficient to take care of the "external legs" problem since after that, the Klein-Gordon operator $\partial^2 + m^2$ will indeed remove the external Green's function $1/(p^2 - m^2)$ and no trace of the problem would remain. In addition, one will have to write amplitudes and cross sections using the physical mass m and not the bare mass m_0.

However, this is not the whole story since in general the self-energy function depends on p^2. To understand implications of this fact, consider again Eq. (18.4). For a generic function $\Sigma(p^2, m_0^2)$ the only assumption that we can make is that $G(p,m)$ should have a pole at the physical mass m^2 which, at this point we do not know. Expanding around the pole, we find

$$p^2 - m_0^2 - \Sigma(m^2, m_0^2) - \frac{\partial \Sigma(p^2, m_0^2)}{\partial p^2}(p^2 - m^2) + \mathcal{O}((p^2 - m^2)^2), \quad (18.6)$$

where the derivative is taken at $p^2 = m^2$. We can write this expression as

$$(p^2 - m^2) Z^{-1} + +\mathcal{O}((p^2 - m^2)^2), \quad (18.7)$$

where

$$Z^{-1} = 1 - \frac{\partial \Sigma}{\partial p^2}|_{p^2 = m^2}, \quad (18.8)$$

and

$$m^2 = m_0^2 + \Sigma(m^2, m_0^2). \quad (18.9)$$

One can compute the function $\Sigma(p^2, m_0^2)$ and calculate the relation between the physical mass and the bare mass, as well as the constant Z. We will do this below for the case of the electron in QED. However, before we do that we need to discuss important consequences of the obtained result.

Basically, what we have seen is that even after we express the two-point Green's function in terms of the physical mass, in the vicinity of the on-shell pole the function reads

$$G(p, m) \approx \frac{iZ}{p^2 - m^2}. \quad (18.10)$$

This expression for the two-point Green's function implies that upon acting with Klein-Gordon operator $\partial^2 + m^2$ on $G(p, m)$, we will still get a factor Z as a remnant. So it definitely seems that the external legs have re-appeared again and need to be dealt with.

The latter statement refers to the following point. Eq. (18.10) implies that at large times the interaction field *does not* smoothly go over to a "free field" described by regular creation and annihilation operators but rather to

$$\phi(t, \vec{x}) \to Z^{1/2} \phi_I(t, \vec{x}), \quad t \to T, \tag{18.11}$$

where

$$\phi_I(t, \vec{x}) \sim a_{\vec{k}} + a^+_{\vec{k}}, \tag{18.12}$$

and the creation and annihilation operators are properly normalized

$$[a_{\vec{k}}, a^+_{\vec{k}_1}] = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}_1). \tag{18.13}$$

The derivation of Lehmann-Symanzik-Zimmerman formula proceeds by relating an integral of the interacting field to its asymptotic value which then is expressed in terms of creation an annihilation operators. Thanks to Eq. (18.11), such relations will be modified; for example instead of Eq. (**??**), we will get

$$-i \int \mathrm{d}^4 x \, e^{-ip_\mu x^\mu} \left(\partial^2 + m^2\right) \phi(x) = \sqrt{2E_{\vec{p}}}\, Z^{1/2} \, \left(a^+_{\vec{p}}(T) - a^+_{\vec{p}}(-T)\right). \tag{18.14}$$

Since we need an expression that relates e.g. $\sqrt{2E_{\vec{p}}}\, a_{\vec{p}}$ and an integral of ϕ, we will have to divide Eq. (18.1) by $\sqrt{Z}$ for each leg. When this statement is combined with the fact that one gets additional factor of Z from each of the external lines when computing amputated Green's functions, previous rules for computing amplitudes from Green's functions (which boil down to the statement – ignore external legs) should be corrected to: start by ignoring external legs and doing exactly the same things as before, but at the end multiply the amplitude with $Z^{1/2}$ for each external leg, and re-write the result in terms of the physical mass, not the bare mass.

We will now apply the above discussion to Quantum Electrodynamics and compute electron's self-energy. We will denote it as $\hat{\Sigma}$. The one-loop expression reads

$$-i\hat{\Sigma}(p,m) = -e^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\gamma^\alpha(\hat{p}+\hat{k}+m)\gamma_\alpha}{((p+k)^2-m^2)(k^2-\lambda^2)}, \tag{18.15}$$

where we added the photon mass λ for the reasons to be discussed later. I also denote the mass of the electron as m but in principle at this point should be the bare mass m_0.

The self-energy diverges at large values of k. To take care of this problem, we again subtract a contribution of a hypothetical Pauli-Villars particle with the mass M. The regulated quantity reads

$$\hat{\Sigma}(p,m) = -ie^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\gamma^\alpha(\hat{p}+\hat{k}+m)\gamma_\alpha}{(p+k)^2-m^2} \left(\frac{1}{k^2-\lambda^2} - \frac{1}{k^2-M^2} \right). \tag{18.16}$$

We then introduce Feynman parameters, shift the loop momentum and find

$$\begin{aligned} \hat{\Sigma}(p,m) = -ie^2 \int [\mathrm{d}x]_2 \int \frac{\mathrm{d}^4 l}{(2\pi)^4}\, \gamma^\alpha(\hat{l}+\hat{p}(1-x_1)+m)\gamma_\alpha \\ \times \left(\frac{1}{(l^2-\Delta)^2} - \frac{1}{(l^2-\Delta_M)^2} \right), \end{aligned} \tag{18.17}$$

where

$$\Delta_\lambda = m^2 x_1 + \lambda^2 x_2 - p^2 x_1(1-x_1), \tag{18.18}$$

and Δ_M is the same as Δ_λ but with $\lambda \to M$. We note that the *linear* $\hat{l}$-term in Eq. (18.17) drops out because it is odd under $l \to -l$. The required l-integration therefore is

$$G(\Delta_\lambda, \Delta_M) = \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \left(\frac{1}{(l^2-\Delta_\lambda)^2} - \frac{1}{(l^2-\Delta_M)^2} \right). \tag{18.19}$$

Performing the Wick rotation, we find

$$G(\Delta_\lambda, \Delta_M) = \frac{i\Omega_4}{(2\pi)^4} \frac{1}{2} \int\limits_0^\infty l_E^2 \mathrm{d}l_E^2 \left(\frac{1}{(l_E^2+\Delta_\lambda)^2} - \frac{1}{(l_E^2+\Delta_M)^2} \right). \tag{18.20}$$

The differential equation is easy to derive

$$\frac{\partial G(\Delta_\lambda, \Delta_M)}{\partial \Delta_\lambda} = -\frac{i\Omega_4}{(2\pi)^4} \int\limits_0^\infty l_E^2 \mathrm{d}l_E^2\, \frac{1}{(l_E^2+\Delta_\lambda)^3} = -\frac{i\Omega_4}{(2\pi)^4} \frac{1}{2\Delta_\lambda}. \tag{18.21}$$

Integrating, we find

$$G(\Delta_\lambda, \Delta_M) = \frac{i}{(4\pi)^2}\ \ln\frac{\Delta_M}{\Delta_\lambda}, \tag{18.22}$$

where we have used the boundary condition $G(\Delta_\lambda, \Delta_\lambda) = 0$. Therefore,

$$\hat{\Sigma}(\hat{p}, m) = \frac{\alpha_s}{2\pi}\int [\mathrm{d}x]_2\ \left(-\hat{p}(1-x_1) + 2m\right)\ \ln\frac{\Delta_M}{\Delta_\lambda}. \tag{18.23}$$

It is clearly possible to perform the integration over the Feynman parameters; however, the exact result is not of interest to us. Rather, we need to understand the implications of this result for the mass renormalization and the wave function renormalization in QED.

In general, we write the one-loop expression for the electron self-energy as follows

$$\hat{\Sigma}(\hat{p}, m) = \Sigma_1(p^2, m^2)(\hat{p} - m) + m\,\Sigma_2(p^2, m^2), \tag{18.24}$$

where m is the pole mass. In principle, one can define $\Sigma_{1,2}$ as functions of the bare mass, but the required algebra is simpler if this representation is used.

We compute the electron two-point function by re-summing the geometric series and find

$$\hat{S}_e(p, m) = \frac{i}{\hat{p} - m_0 - \Sigma_1(p^2, m^2)(\hat{p} - m) - m\,\Sigma_2(p^2, m^2)}. \tag{18.25}$$

We now assume that the above expression has a pole at the physical electron mass m and derive an approximate expression for Eq. (18.25) in the vicinity of the pole. It is easy to understand that one can write $\hat{S}_e(p, m)^{-1}$ in the vicinity of the pole m as a particular (matrix) Taylor expansion

$$\hat{S}_e(p, m)^{-1} = Z^{-1}(\hat{p} - m) + X_2(\hat{p} - m)^2 + X_3(\hat{p} - m)^3 + \ldots \tag{18.26}$$

To find a relation between m and m_0, as well as Z^{-1}, we expand the denominator in Eq. (18.25) around $p^2 = m^2$, and use $p^2 - m^2 = 2m(\hat{p} - m) + (\hat{p} - m)^2$. The relation between masses can be found by simply setting $\hat{p} \to m$ and insisting that $\hat{S}_e(p, m)^{-1}$ vanishes. We find

$$m - m_0 - m\Sigma_2(m^2, m^2) = 0. \tag{18.27}$$

Defining the mass renormalization constant Z_m as

$$m_0 = Z_m m, \tag{18.28}$$

we find

$$Z_m = 1 - \Sigma_2(m^2, m^2). \tag{18.29}$$

To compute Z, we need to expand Σ_2 in Taylor series around m^2. Then

$$Z^{-1} = 1 - \Sigma_1(m^2, m^2) - 2m^2 \frac{\partial \Sigma_2}{\partial p^2}|_{p^2=m^2}. \tag{18.30}$$

We now identify $\Sigma_{1,2}$ in Eq. (18.23). As we said, the mass that appears in these equations is the bare mass m_0. However, since the difference between m_0 and m is formally $\mathcal{O}(\alpha)$ and since $\Sigma_{1,2}$ are also $\mathcal{O}(\alpha)$, we can simply use the pole mass m when computing these quantities to one loop. We find

$$\begin{aligned} \Sigma_1(p^2, m^2) &= -\frac{\alpha_s}{\pi} \int [\mathrm{d}x]_2 \ (1 - x_1) \ \ln \frac{\Delta_M}{\Delta_\lambda}, \\ \Sigma_2(p^2, m^2) &= \frac{\alpha_s}{\pi} \int [\mathrm{d}x]_2 \ (1 + x_1) \ \ln \frac{\Delta_M}{\Delta_\lambda}. \end{aligned} \tag{18.31}$$

We are now in position to compute Z_m and Z_2. To find Z_m, we set $p^2 = m^2$ in expression for $\Sigma_2(p^2, m^2)$ in Eq. (18.31) and find

$$Z_m = 1 - \frac{\alpha}{2\pi} \left[\frac{3}{2} \ln \frac{M^2}{m^2} + \frac{3}{4} \right] \tag{18.32}$$

For Z, we need to calculate the derivative with respect to p^2 and $p^2 = m^2$ and, as it is easy to see, this derivative becomes divergent in the infrared; hence, we need to keep $\lambda \neq 0$. We find

$$Z_2 = 1 - \frac{\alpha}{2\pi} \left(\frac{1}{2} \ln \frac{M^2}{m^2} - \ln \frac{m^2}{\lambda^2} + \frac{9}{4} \right). \tag{18.33}$$

It is now instructive to go back and compute the Dirac form factor $F_1(q^2)$ at $q^2 = 0$ using Eq. (**??**). The calculation is similar to what has been done above; we find

$$F_1^{(1)}(0) = \frac{\alpha}{2\pi} \left(\frac{1}{2} \ln \frac{M^2}{m^2} - \ln \frac{m^2}{\lambda^2} + \frac{9}{4} \right). \tag{18.34}$$

Comparing Eqs. (18.33,18.34), we derive the following relation between the Dirac form factor and the renormalization constant of the electron field Z_2

$$F_1|_{q^2=0} = Z_2^{-1}, \tag{18.35}$$

where $F_1 = 1 + F_1^{(1)}$.

This result has important implications. Recall that we were arguing that we have to ensure that $F_1(q^2 = 0)$ is zero since otherwise it will appear as if we work with the wrong electric charge. This is a "poor man's" way of getting the correct result. In reality, doing this is not necessary. In fact, according to the discussion in this lecture, we should be computing not the vertex function of the electron but the vertex function multiplied with $\sqrt{Z}$ per each leg. Since there are two legs, we find

$$Z\bar{u}(p_f)\Gamma^\mu u(p_i). \tag{18.36}$$

Expanding this formula through $\mathcal{O}(\alpha)$, we observe that the subtraction of the Dirac form factor at $q^2 = 0$ happens *automatically* because of the relation in Eq. (18.35).