TTP1
Lecture 19

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July 28, 2023

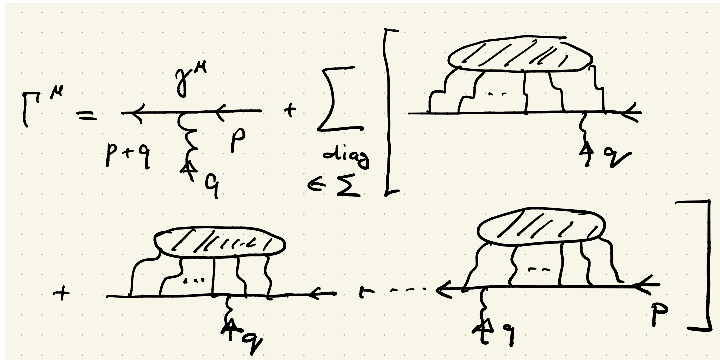

Figure 1: Relation between the vertex function and self-energy diagrams.

19 Ward-Takahashi identities

We have seen in the previous lecture that there is an interesting relation between the electron vertex function and the electron self-energy (or the renormalization constant related to it). In this lecture we will discuss the origin of this relation.

To this end, consider a single fermion line interacting with N off-shell photons. We will call the momentum of the incoming fermion p_0 and the incoming momenta of N photons q_1, q_2.. q_N. The interactions of these N photons with the electron line are ordered, i.e. p_0 first interacts with the photon with momentum q_1, then with the photon with momentum q_2, then with the photon with momentum q_3 and so on. We do not assume that any of the "particles" are on-shell; effectively, we work with a contributions to tree Green's functions. We will denote such an object as

$$E_1^N(q_N,..,q_1;p_0)\frac{1}{\hat{p}_N-m}\gamma^{\mu_N}\frac{1}{\hat{p}_{N-1}-m}\gamma^{\mu_{N-1}}....\frac{1}{\hat{p}_1-m}\gamma^{\mu_1}\frac{1}{\hat{p}_0-m}, \quad (19.1)$$

where

$$p_i = p_0 + \sum_{j=1}^{i} q_j, \quad (19.2)$$

and we do not indicate the dependence of the function E_1^N on photons' indices.

Next, we consider a photon with (incoming) momentum k that couples to the above function. The important difference between this k-photon and

N q-photons is that we account for all possible ways the photon k can couple to the fermion line. The interaction is described by the following function

$$G^\mu(q_N,..,q_1;p_0;k) = \sum_{i=0}^{N} E_{i+1}^{N}(q_N..q_{i+1};p_i+k)\gamma^\mu E_1^i(q_i,..,q_1;p_0) \quad (19.3)$$

where

$$E_1^0(..;p_0) = \frac{1}{\hat{p}_0 - m}, \quad E_{N+1}^{N}(..;p_N) = \frac{1}{\hat{p}_N - m}. \quad (19.4)$$

As the next step, we consider

$$G^\mu k_\mu = \sum_{i=0}^{N} E_{i+1}^{N}(q_N,..,q_{i+1};p_i+k)\hat{k}E_1^i(q_i,..,q_1;p_0). \quad (19.5)$$

For the i-th term in the sum, we write

$$\hat{k} = (\hat{p}_i + \hat{k} - m) - (\hat{p}_i - m), \quad (19.6)$$

and use the definition of E-functions to obtain

$$(\hat{p}_i - m)E_1^i(q_i,..q_1;p_0) = \gamma^{\mu_i}E_1^{i-1}(q_{i-1},..,q_1;p_0), \quad (19.7)$$

and

$$E_{i+1}^{N}(q_N..q_{i+1};p_i+k;)(\hat{p}_i+\hat{k}-m) = E_{i+2}^{N}(q_N,..,q_{i+2};p_{i+1}+k)\gamma^{\mu_{i+1}}. \quad (19.8)$$

Hence, we find

$$G^\mu k_\mu = \sum_{i=0}^{N}\Bigg[E_{i+2}^{N}(q_N,..,q_{i+2};p_{i+1}+k;)\gamma^{\mu_{i+1}}E_1^i(q_i,..q_1;p_0) \\ - E_{i+1}^{N}(q_N...,q_{i+1};p_i+k)\gamma^{\mu_i}E_1^{i-1}(q_{i-1},..,q_1;p_0)\Bigg]. \quad (19.9)$$

Next, we notice that if we shift the (dummy) summation index $i \to i+1$ in the second term in the sum the above formula, it becomes equal and opposite in sign to the first term in the sum. Hence, the two contributions to the sum cancel each other up to the boundary terms. We obtain the following result

$$k_\mu G^\mu(p_0;q_1..q_N;k) = E_1^N(q_N,..,q_1;p_0) - E_1^N(q_N,..,q_1;p_0+k). \quad (19.10)$$

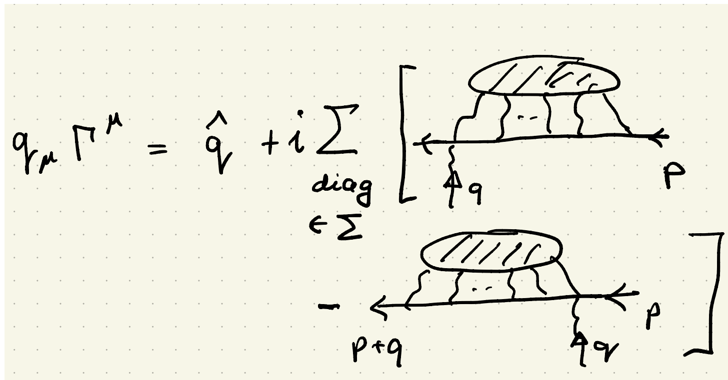

Figure 2: The illustration of the equation for $q_\mu \Gamma^\mu$.

To see what we can do with this result, consider electron self-energy function and take any diagram that contributes to it.[1] We can turn this diagram into a diagram that contributes to electron vertex function but this time defined as a Green's function with amputated external legs $\Gamma^\mu(p+q, p, q)$ and without assuming that $(p+q)^2 = m^2$ and $p^2 = m^2$. To do so, we need to consider all possible insertions of the external "photon" with momentum q into the electron line, see Fig. 2.

It is then easy to see (c.f. Fig.3), using Eq. (19.10), that the following equation holds

$$q_\mu \Gamma^\mu(p+q, p, q) = \hat{q} - \hat{\Sigma}(p+q) + \hat{\Sigma}(p) = iS_e^{-1}(p+q) - iS_e^{-1}(p). \quad (19.11)$$

As a next step, we consider the $q \to 0$ limit of the above equation. The left hand side is already small. We then write the right-hand side as

$$iS_e^{-1}(p_f) - iS_e^{-1}(p_i) = iS_e^{-1}(p_i+q) - iS_e^{-1}(p_i) \approx iq^\mu \frac{\partial S_e^{-1}(p_i+q)}{\partial p_i^\mu}. \quad (19.12)$$

Hence, we find

$$q_\mu \Gamma^\mu(p_i+q, p_i, q) \approx q_\mu \Gamma^\mu(p_i, p_i, 0) = iq^\mu \frac{\partial S_e^{-1}(p_i)}{\partial p_i^\mu}. \quad (19.13)$$

[1]We only consider diagrams that contain photons and a single electron line. In principle, there are also vacuum polarization diagrams where photons split into e^+e^- pairs and recombine back and there are light-by-light scattering diagrams etc. We do not discuss such contributions below although the result remains valid even if those are included.

Since this equation should hold for an arbitrary q, it follows that

$$\Gamma^\mu(p_i, p_i, 0) = i\frac{\partial S_e^{-1}(p_i)}{\partial p_i^\mu}. \tag{19.14}$$

We compute the derivative of the inverse electron propagator. We write

$$iS_e^{-1}(p) = (\hat{p} - m)\left(1 - \Sigma_1(p, m)\right) - m\Sigma_2(p, m), \tag{19.15}$$

so that

$$i\frac{\partial S_e^{-1}}{\partial p^\mu} = \gamma^\mu(1 - \Sigma_1(p, m)) - (\hat{p} - m)\frac{\partial \Sigma_1(p, m)}{\partial p^2}2p^\mu - \frac{\partial \Sigma_2(p, m)}{\partial p^2}2p^\mu. \tag{19.16}$$

We now consider the on-shell limit $p^2 \to m^2$ and consider

$$\bar{u}(p)i\frac{\partial S_e^{-1}}{\partial p^\mu}u(p). \tag{19.17}$$

We find

$$\begin{aligned}\bar{u}(p)i\frac{\partial S_e^{-1}}{\partial p^\mu}u(p) &= \bar{u}(p)\gamma^\mu u(p)\left[(1 - \Sigma_1) - 2m^2\frac{\partial \Sigma_2(p, m)}{\partial p^2}|_{p^2=m^2}\right] \\ &= Z^{-1}\bar{u}(p)\gamma^\mu u(p).\end{aligned} \tag{19.18}$$

where we have used the Gordon identity in the forward limit

$$\bar{u}(p)p_\mu u(p) = m\bar{u}(p)\gamma^\mu u(p). \tag{19.19}$$

It follows that

$$\bar{u}(p)\Gamma^\mu u(p) = F_1(0)\bar{u}(p)\gamma^\mu u(p) = Z^{-1}\bar{u}(p)\gamma^\mu u(p), \tag{19.20}$$

so that

$$ZF_1(0) = 1. \tag{19.21}$$

We have first seen this result at one-loop, after the explicit computation of Z and $F_1(0)$. The above derivation is valid to all orders in perturbation theory and in this sense this is exact result in QED. Again, the important implications of this result is that, since it is the combination $ZF_1(q^2)$ that appears as a "physical" form factor in the matrix elements that describes interaction of an electron with the external electromagnetic field, the physical form factor at $q^2 = 0$ is always equal to 1. As we said, this quantity is connected to the definition of the electric charge of a fermion. Hence, this result implies that the electric charge of a fermion is independent of e.g. the fermion mass and should be the same for e.g. muon and electron to all orders in α.