Theoretical Particle Physics I

Winter Term 2023/24

Karlsruhe Institute of Technology (KIT)

given by

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Contents

Chapter 1

Preliminary Remarks

1.1 Organisation

Lecture information available on ILIAS: script and exercise sheets, forum for questions, organisational details

Lecture: Mo, 9h45-11h15 (Kleiner Hörsaal B) and Thu, 9h45-11h15 (Kleiner Hörsaal A);
Exercises: to be determined

Exercise Group Leader: Dr. Christoph Borschensky
Tutors: M.Sc. Francisco Arco, M.Sc. Felix Egle

Criteria for obtaining the certificate of sucessful participation: see web page (ILIAS)

1.2 Literature

Textbooks:

[1] Bailin, David und Love, Alexander: *Introduction to gauge field theory*, Hilger

[2] Bjorken, James D. und Drell, Sidney D.: *Relativistische Quantenfeldtheorie*, BI-Wissenschaftsverlag.

[3] Böhm, M., Denner, A. und Joos, H.: *Gauge Theories of the Strong and Electroweak Interaction*, Teubner Verlag

[4] Cheng, Ta-Pei und Li, Ling-Fong: *Gauge theory of elementary partilce physics*, Oxford Science Publications

[5] Halzen, Francis und Martin, Alan D.: *Quarks and Leptons*, John Wiley & Sons, Inc.

[6] Itzykson, Claude und Zuber, Jean-Bernard: *Quantum Field Theory*, McGraw-Hill

[7] Kaku, Michio: *Quantum field theory*, Oxford University Press

[8] Kugo, Thaichiro: *Eichtheorie*, Springer

[9] Nachtmann, Otto: *Phänomene und Konzepte der Elementarteilchenphysik*, Vieweg

[10] Peskin, Michael E. und Schroeder, Daniel V.: *An introduction to quantum field theory*, Addison-Wesley

[11] Pokorski, Stefan: *Gauge field theories*, Cambridge University Press

[12] Ramond, Pierre: *Field theory*, Addison-Wesley

[13] Ryder, L.H.: *Quantum Field theory*, Cambridge University Press

[14] Sterman, George: *An Introduction to Quantum Field Theory*, Cambridge University Press

[15] Weinberg, Steven: *The quantum theory of fields*, Cambridge University Press

Web pages:

http://pdg.lbl.gov Particle Data Group

http://inspirehep.net Datenbank INSPIRE für Publikationen

http://arxiv.org Preprint-Archiv

http://www.cern.ch CERN

1.3 Preliminary Content

1. Preliminary Remarks
2. Introduction (conventions, Lorentz and Poincaré group)
3. Lagrange formalism for fields (equations of motion, Noether theorem, inner symmetries, group theory)
4. Quantisation of scalar fiels
5. Quantisation of spinor fields (Dirac field)
6. Quantisiation of spin-1 fields (vector fields)
7. Perturbation theory, Feynman rules, Feynman diagrams
8. Computation of cross sections
9. ...

Chapter 2

Introduction

Elementary particle physics means physics at smallest scales, respectively at highest (relativistic) energies. Look e.g. at the wave-particle duality and the de Broglie relation,

$$E = h\nu \rightsquigarrow E \uparrow \;\Leftrightarrow\; \nu \uparrow \;\Leftrightarrow\; \lambda \downarrow \quad \text{smallest scales} . \tag{2.1}$$

The basis of the description of high-energy physics is quantum field theory. It is the synthesis of quantum mechanics and special relativity. In quantum mechanics, we use wave equations. These cannot describe processes where the number or the type of the particles change. Moreover, relativistic wave equations exhibit inconsistencies (e.g. negative energy solutions). In quantum field theory we identify particles with modes of a field, and the field itself is quantised ("2nd quantisation"). This allows us to describe the creation and annihilation of particles.

An important concept are symmetries. In particular

(*i*) Space-time symmetries: They allow the Lorentz-/Poincaré-covariant formulation of the field theory. It leads to *mass* and *spin* as fundamental properties of the particles.

(*ii*) Internal symmetries: Particles are grouped into multiplets of symmetry groups. This leads to additional quantum numbers like isospin, hypercharge, colour, ... Lie groups play an important role in particle physics. They are used for the description of continuous symmetries.

(*iii*) Gauge symmetries: They are of particular importance. Local gauge symmetries allow to describe the interactions between fields dynamically. The fundamental interactions are described by gauge symmetries:

Electromagnetic interaction: $U(1)$

Weak interaction: $SU(2)$
unified to electroweak interactions: $SU(2) \times U(1)$

Strong interaction: $SU(3)$

Open problem: description of gravity as a gauge theory

Why do we do high-energy physics? - We want to find answers to our basic questions about the universe:

1. What is the universe made of?

2. How did the universe develop?

3. What are the fundamental building blocks of matter, and which forces hold them together?

What is the status of elementary particle physics today?

1. The known matter can be described by a few fundamental particles.

2. The diverse interactions are described by fundamental forces between the particles.

3. The physics laws can be described mathematically using a few fundamental principles (except for gravity).

We have a consistent model, whose particles all have been discovered: the *Standard Model* of particle physics. However, there are open questions, which cannot be answered within the Standard Model, like e.g. the nature of Dark Matter, why is there more matter than antimatter in the universe, ...

2.1 Conventions and Definitions

Natural units: In theoretical particle physics we use natural units (Planck units). We set the speed of light c and the Planck constant h equal to 1. The energy unit (which is not fixed by this choice) which is used, is the electron volt: $1\,\mathrm{eV} = 1.6 \cdot 10^{-19}\,\mathrm{J}$.

1. We set the speed of light c equal to 1:

$$c = 3 \cdot 10^8 \,\frac{\mathrm{m}}{\mathrm{s}} \equiv 1 \;\Rightarrow\; 1\ \mathrm{s} = 3 \cdot 10^8\ \mathrm{m} \tag{2.2}$$

2. The Planck constant is set equal to 1:

$$\hbar = \frac{h}{2\pi} = 6.6 \cdot 10^{-25}\,\mathrm{GeV\ s} \equiv 1 \;\Rightarrow\; 1\,\mathrm{s} = 1.5 \cdot 10^{24}\,\mathrm{GeV}^{-1}\ . \tag{2.3}$$

And

$$\hbar c = 1 \;\Rightarrow\; 1\,\mathrm{m} = 5.1 \cdot 10^{15}\,\mathrm{GeV}^{-1}\ . \tag{2.4}$$

Furthermore,

$$m = \frac{E_{\mathrm{rest}}}{c^2} = E_{\mathrm{rest}} \tag{2.5}$$

$$m = \frac{1\,\mathrm{eV}}{c^2} = \frac{1.6 \cdot 10^{-19}}{(3 \cdot 10^8)^2}\,\mathrm{kg} = 1.78 \cdot 10^{-36}\,\mathrm{kg} \stackrel{!}{=} 1\,\mathrm{eV} \;\Rightarrow\; 1\,\mathrm{kg} = 5.6 \cdot 10^{26}\,\mathrm{GeV} \tag{2.6}$$

3. The elementary electric charge $e > 0$ is given by the Sommerfeld fine-structure constant α:

$$\frac{e^2}{4\pi} = \alpha \approx \frac{1}{137....} \;\Rightarrow\; e = 0.3. \tag{2.7}$$

The charge e is dimensionless.

All physics units are hence given in terms of powers of energy. The exponent is the (mass) dimension. He therefore have

$$[\text{Length}] = [\text{Time}] = -1 \;, \quad [\text{Mass}] = 1 \;, \quad [e] = 0 \;. \tag{2.8}$$

Minkowski Metric A metric space is a vector space with a metric. We have the contravariant four-vector

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} t \\ \vec{x} \end{pmatrix} \quad \text{(contravariant)} \;. \tag{2.9}$$

The dual space of the vector space contains as elements the covariant four-vectors

$$x_\mu = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ -\vec{x} \end{pmatrix} \quad \text{(covariant)} \;. \tag{2.10}$$

The transition between contra- and covariant is mediated by the *Minkowski metric* $g_{\mu\nu}$,

$$x_\mu = g_{\mu\nu} x^\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} t \\ \vec{x} \end{pmatrix} = \begin{pmatrix} t \\ -\vec{x} \end{pmatrix} \;. \tag{2.11}$$

The scalar product (which is invariant under Lorentz transformations - see next section) is given by

$$x \cdot y = x_\mu y^\mu = x^\mu g_{\mu\nu} y^\nu = x^0 y^0 - \vec{x} \cdot \vec{y} \;. \tag{2.12}$$

For the length of a Lorentz vector,

$$x^2 = x_0^2 - \vec{x}^2 \;, \tag{2.13}$$

we have the classifications

$$\begin{array}{ll} x^2 > 0 & \text{time-like} \\ x^2 = 0 & \text{light-like} \\ x^2 < 0 & \text{space-like} \;. \end{array} \tag{2.14}$$

This means that (with $\Delta x = x_a - x_b$)

$$\begin{array}{ll} (\Delta x)^2 > 0 : & \text{Signals with } v < c \text{ can be sent from } x_a \text{ to } x_b. \\ (\Delta x)^2 = 0 : & \text{An event at } x_b \text{ is on the light-cone of } x_a. \\ (\Delta x)^2 < 0 : & \text{The events are not causally connected.} \end{array} \tag{2.15}$$

See also Fig. 2.1.

Differential operators in Minkowski space The covariant derivative is given by

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\tfrac{\partial}{\partial t}, \vec{\nabla} \right) \;, \tag{2.16}$$

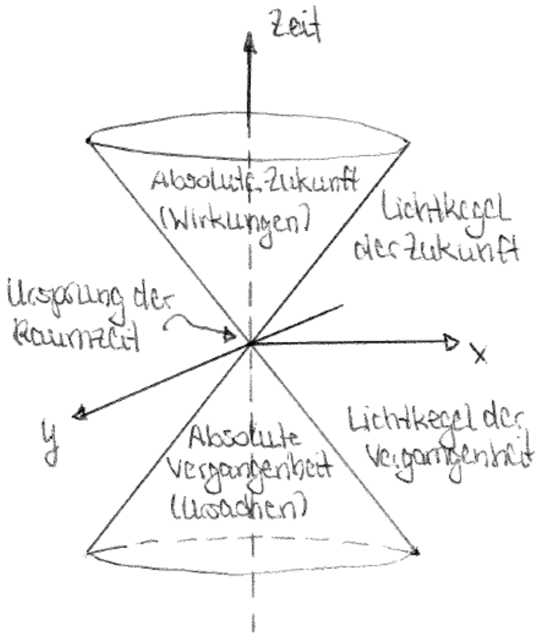

Figure 2.1: Light cone of the future and of the past.

and for the d'Alembert operator we have

$$\Box = \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \equiv \frac{\partial^2}{\partial t^2} - \Delta \ . \tag{2.17}$$

Tensors We have for the *metric tensor*

$$g_{\mu\nu} = g^{\mu\nu} \quad \text{und} \quad g_\mu^\nu = \delta_\mu^\nu \ . \tag{2.18}$$

The *Levi-Civita-tensor* is defined through

$$e^{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{for even permutations} \\ -1 & \text{for odd permutations} \\ 0 & \text{otherwise} \end{cases} \tag{2.19}$$

It is

$$\epsilon^{0123} = +1 \quad \Rightarrow \quad \epsilon_{0123} = g_{0\mu} g_{1\nu} g_{2\rho} g_{3\sigma} \epsilon^{\mu\nu\rho\sigma} = g_{00} g_{11} g_{22} g_{33} \epsilon^{0123} = -\epsilon^{0123} = -1 \ . \tag{2.20}$$

We also have

$$\epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2 \ , \quad \text{i.e. } \epsilon^{12} = 1 \ . \tag{2.21}$$

Einstein sum convention We sum over doubly appearing indices, i.e.

$$a_i b_i = \sum_i a_i b_i \ . \tag{2.22}$$

and

$$a_\mu b_\mu = \sum_{\mu=0}^{3} a_\mu b^\mu \ . \tag{2.23}$$

For four-vectors, the Greek indices run from 0 to 3 and the Latin indices run from 1 to 3.

2.2 Lorentz group and Poincaré group

2.2.1 The Lorentz transformation

In classical physics and special relativity the tensor concept plays a central role. According to the covariance principle, physics laws can be expressed through tensor equations:

$$\text{Physics laws} \;\Leftrightarrow\; \text{Tensor equations} \, . \tag{2.24}$$

Physics laws are invariant under coordinate transformations. A tensor equations relates vectors (tensors of rank 1) and tensors of higher rank. In quantum field theory we also deal with fermions. They have spin of half unit and are fundamentally different from bosons with unit spin. They are described through spinors. The covariance principle for fermions is

$$\text{Physics laws} \;\Leftrightarrow\; \text{Spinor equations} \, . \tag{2.25}$$

A typical example is the Dirac equation. Once the transformation properties of objects like tensors, spinors are known, we can construct invariant quantities, i.e. Lorentz invariants, from them. The Lagrangian density e.g. is a Lorentz-invariant quantity. From the Lagrangian density we can then derive the equations of motion.

All linear transformations in Minkowski space,

$$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu \tag{2.26}$$

$$\text{with } x'_\mu y'^\mu = x_\mu y^\mu \quad \text{for all } x, y \, , \tag{2.27}$$

are called Lorentz transformations. They form the Lorentz group. It corresponds to the pseudo-orthogonal group $O(3,1)$. This means for the 4×4 matrices that $\Lambda \in O(3,1)$. From (2.27) it follows that

$$g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu} \Lambda^\mu_{\ \rho} x^\rho \Lambda^\nu_{\ \sigma} x^\sigma = g_{\rho\sigma} x^\rho x^\sigma \quad \Rightarrow \tag{2.28}$$

$$g_{\rho\sigma} = g_{\mu\nu} \Lambda^\mu_{\ \rho} \Lambda^\nu_{\ \sigma} \, . \tag{2.29}$$

And hence

$$\Lambda^T g \Lambda = g \quad \Rightarrow \quad \det g = \det(\Lambda^T g \Lambda) \quad \Rightarrow \quad \det \Lambda = \pm 1 \, . \tag{2.30}$$

The inverse of Λ^μ_ν is given by $(\Lambda^{-1})^\mu_{\ \nu} = \Lambda_\nu^{\ \mu}$, as

$$\Lambda_\nu^{\ \mu} \Lambda^\nu_{\ \tau} = g_{\nu\rho} \Lambda^\rho_{\ \sigma} g^{\sigma\mu} \Lambda^\nu_{\ \tau} = (g_{\nu\rho} \Lambda^\rho_{\ \sigma} \Lambda^\nu_{\ \tau}) g^{\sigma\mu} \overset{\text{Eq. (2.29)}}{=} g_{\sigma\tau} g^{\sigma\mu} = g^\mu_\tau = \delta^\mu_\tau \text{ q.e.d.} \, . \tag{2.31}$$

2.2.2 Transformation properties

The covariant derivative and the d'Alembert operator transform as

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} = \Lambda_\mu^{\ \nu} \partial_\nu \tag{2.32}$$

and

$$\Box' = \Lambda_\mu^{\ \nu} \partial_\nu \Lambda^\mu_{\ \rho} \partial^\rho = g_\rho^{\ \nu} \partial_\nu \partial^\rho = \partial_\rho \partial^\rho = \Box \, . \tag{2.33}$$

The d'Alembert operator is hence Lorentz-invariant.

A contravariant tensor of rank n transforms as

$$T'^{\mu_1...\mu_n} = \Lambda^{\mu_1}_{\ \nu_1}...\Lambda^{\mu_n}_{\ \nu_n}T^{\nu_1...\nu_n} \ , \tag{2.34}$$

and analogous the covariant tensor $T_{\mu_1...\mu_n}$. A mixed tensor of rank (n, m) transforms as

$$T'^{\mu_1...\mu_n}_{\qquad \nu_1...\nu_m} = \Lambda^{\mu_1}_{\ \rho_1}...\Lambda^{\mu_n}_{\ \rho_n}\Lambda_{\nu_1}^{\ \sigma_1}...\Lambda_{\nu_m}^{\ \sigma_m}T^{\rho_1...\rho_n}_{\qquad \sigma_1...\sigma_m} \ . \tag{2.35}$$

The metric tensor $g_{\mu\nu}$ is invariant under the Lorentz transformation, i.e. $g'^{\mu\nu} = g^{\mu\nu}$, $g'_{\mu\nu} = g_{\mu\nu}$.

The transformation of the Levi-Civita tensor is given by

$$\epsilon'^{\mu\nu\rho\sigma} = \epsilon^{\mu\nu\rho\sigma}\mathrm{det}\Lambda = \pm\epsilon^{\mu\nu\rho\sigma} \ . \tag{2.36}$$

2.3 Lorentz group

The set of Lorentz transformations Λ forms a group g:

1. Unit element: $\mathbf{1} = \delta^\mu_\nu$.
2. Inverse: $(\Lambda^{-1})^\mu_{\ \nu} = \Lambda_\nu^{\ \mu}$.
3. Associativity: $\Lambda_1 \circ (\Lambda_2 \circ \Lambda_3) = (\Lambda_1 \circ \Lambda_2) \circ \Lambda_3$.
4. Closure: $\Lambda_1 \circ \Lambda_2 \in g$.

Special examples of Lorentz transformations are

- Space inversion (parity): $\Lambda_P = \mathrm{diag}(1, -1, -, 1-, 1)$, with

$$x'^\mu = (\Lambda_P)^\mu_{\ \nu}x^\nu = (x^0, -\vec{x})^T \ . \tag{2.37}$$

- Time reversal T: $\Lambda_T = \mathrm{diag}(-1, 1, 1, 1)$, with

$$x'^\mu = (\Lambda_T)^\mu_{\ \nu}x^\nu = (-x^0, \vec{x})^T \ . \tag{2.38}$$

Classification: The Lorentz group can be classified following two properties: the sign of the determinant, $\det\Lambda$, and the sign of $\Lambda^0_{\ 0}$. The Lorentz transformations

1. $L^\uparrow_+ = \{\Lambda \in L : \det\Lambda = +1, \ \Lambda^0_{\ 0} > 0\}$ are called proper orthochronous.
2. $L^\downarrow_+ = \{\Lambda \in L : \det\Lambda = +1, \ \Lambda^0_{\ 0} < 0\}$ are called proper non-orthochronous.
3. $L^\uparrow_- = \{\Lambda \in L : \det\Lambda = -1, \ \Lambda^0_{\ 0} > 0\}$ are called improper orthochronous.
4. $L^\downarrow_- = \{\Lambda \in L : \det\Lambda = -1, \ \Lambda^0_{\ 0} < 0\}$ are called improper non-orthochronous.

Examples for the four branches 1.–4. of the Lorentz group are the identity $\mathbf{1}$ =diag(1,1,1,1), the inversion $PT = -\mathbf{1}$, the space inversion Λ_P =diag(1,-1,-1,-1), the time reversal Λ_T = diag(-1,1,1,1). They form the group of discrete transformations.

2.4 The special Lorentz group and its decomposition

The group of transformations of a space with coordinates $(y_1, ..., y_m, x_1, ..., x_n)$ which leaves the quadratic form $(y_1^2 + ... + y_m^2) - (x_1^2 + ... x_n^2)$ invariant, is called the orthognal group $O(n, m)$. The Lorentz group is $O(3, 1)$, the $L_+^\uparrow$ is $SO(3, 1)$. The proper orthochronous Lorentz group

$$L_+^\uparrow = \{\Lambda \in SO(3,1) | \det\Lambda = 1,\ \Lambda^0_{\ 0} > 0\} \tag{2.39}$$

contains rotations and boosts. Every $\Lambda \in L_+^\uparrow$ can be written as product of a rotation R and a boost b,

$$\Lambda \in L_+^\uparrow \Rightarrow \Lambda = \Lambda_b \Lambda_R\ . \tag{2.40}$$

The rotations are given by

$$\Lambda(0, \vec{\varphi}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R(\vec{\varphi}) & \\ 0 & & & \end{pmatrix} \tag{2.41}$$

with the axis $\frac{\vec{\varphi}}{|\vec{\varphi}|}$ and the angle $\varphi = |\vec{\varphi}|$ and the rotation matrix elements $R(\vec{\varphi})_{ij}$.

A pure boost into a reference system which moves with a relative velocity v in the direction of the $x^i = x$-axis, is given by

$$\Lambda(\vec{\eta}, 0) = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{2.42}$$

Here we defined

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.43}$$

$$\beta = \frac{v}{c} \tag{2.44}$$

and

$$\eta = \mathrm{artanh} v \tag{2.45}$$

is the *rapidity*, with $-\infty < \eta < \infty$. In general, Λ can be parametrised by three angles and the three components of $\vec{v}$. The Lorentz group can be parametrised in a continuous and differentiable way by six parameters and forms a Lie group. It is non-compact because of $-\infty < \eta < \infty$.

2.5 Infinitesimal transformations, Lorentz algebra

We can expand Λ as

$$\Lambda^\mu_{\ \nu} = \delta^\mu_\nu + \delta\omega^\mu_{\ \nu} + \mathcal{O}((\delta\omega)^2) \ . \tag{2.46}$$

With

$$g_{\mu\nu}\Lambda^\mu_{\ \rho}\Lambda^\nu_{\ \sigma} = g_{\rho\sigma} = g_{\mu\nu}(\delta^\mu_\rho + \delta\omega^\mu_{\ \rho})(\delta^\nu_\sigma + \delta\omega^\nu_{\ \sigma}) + ... = g_{\rho\sigma} + \delta\omega_{\sigma\rho} + \delta\omega_{\rho\sigma} + ... \tag{2.47}$$

we find

$$\delta\omega_{\sigma\rho} = -\delta\omega_{\rho\sigma} \tag{2.48}$$

and hence that $\delta\omega$ is antisymmetric. It is hence defined by six independent parameters, which correspond to $\delta\varphi_i$ and $\delta\eta_i$. Denoting the *generators of the algebra* by $M^{\alpha\beta}$ we have

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}\right) \ . \tag{2.49}$$

With the infinitesimal representation

$$\Lambda^\mu_{\ \nu} = \delta^\mu_\nu + \delta\omega_{\alpha\beta}g^{\alpha\mu}\delta^\beta_\nu \stackrel{!}{=} \delta^\mu_\nu - \frac{i}{2}\delta\omega_{\alpha\beta}(M^{\alpha\beta})^\mu_{\ \nu} \tag{2.50}$$

we get

$$(M^{\alpha\beta})^\mu_{\ \nu} = i(g^{\alpha\mu}\delta^\beta_\nu - g^{\beta\mu}\delta^\alpha_\nu) \ . \tag{2.51}$$

We hence have

$$\Lambda^\mu_{\ \nu} = \left[\exp\left(-\frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}\right)\right]^\mu_{\ \nu} \quad \text{and} \quad (M^{\alpha\beta})^\mu_{\ \nu} = i(g^{\alpha\mu}\delta^\beta_\nu - g^{\beta\mu}\delta^\alpha_\nu) \tag{2.52}$$

The Lorentz algebra is

Lorentz algebra

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i(g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho} - g^{\nu\rho}M^{\mu\sigma}) \tag{2.53}$$

Special cases are

$$\begin{array}{ll} K_j = M_{0j} & \text{boosts in } x_j \text{ direction} \\ J_i = \frac{1}{2}\epsilon_{ijk}M_{jk} & \text{generators of infinitesimal rotations around the } x^i \text{ axis,} \\ \Rightarrow\ M_{kl} = \epsilon_{kln}J_n & \text{hence the angular momentum.} \end{array} \tag{2.54}$$

With the generators of the rotations, J_i $(i = 1, 2, 3)$, we hence have

$$\Lambda(0, \vec{\varphi}) \ = \ \exp(i\vec{\varphi}\cdot\vec{J}) \ . \tag{2.55}$$

And with the generators of the boosts K_j $(j = 1, 2, 3)$ and $\vec{\eta} = \eta\vec{v}/|\vec{v}|$ we have

$$\Lambda(\vec{\eta}, 0) = \exp(i\vec{\eta}\cdot\vec{K}) \ . \tag{2.56}$$

The generators fulfill the following algebra

Algebra

$$[J_k, J_l] = i\epsilon_{klm}J_m \quad \text{(angular momentum algebra)} \tag{2.57}$$

$$[K_j, K_n] = -i\epsilon_{jnk}J_k \quad \text{(two boosts involve a rotation)} \tag{2.58}$$

$$[J_k, K_l] = i\epsilon_{klm}K_m \tag{2.59}$$

Examples are

$$\Lambda_{R_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\delta\varphi & 0 \\ 0 & \delta\varphi & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \mathcal{O}(\delta\varphi)^2 = \mathbf{1} - i\delta\varphi J^3 + \mathcal{O}((\delta\varphi)^2) \ . \tag{2.60}$$

and

$$\Lambda_{b_3} = \begin{pmatrix} 1 & 0 & 0 & -\delta\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\delta\eta & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}((\delta\eta)^2) = \mathbf{1} + i\delta\eta K^3 + \mathcal{O}(\delta\eta)^2 \ . \tag{2.61}$$

For general rotations and boosts we have (cf. also Eqs. (2.55) and (2.56))

$$\Lambda_R = \mathbf{1} - i\delta\vec{\varphi}\vec{J} + \mathcal{O}((\delta\varphi)^2) \tag{2.62}$$

$$\Lambda_b = \mathbf{1} + i\delta\vec{\eta}\vec{K} + \mathcal{O}((\delta\eta)^2) \ , \tag{2.63}$$

with

$$J^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} , \ J^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} , \ J^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.64}$$

and

$$K^1 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \ K^2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \ K^3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} , \tag{2.65}$$

and

$$J^{i\dagger} = J^i \ \text{(hermitian)}, \qquad K^{i\dagger} = -K^i \ \text{(anti-hermitian)} \ . \tag{2.66}$$

2.6 The Poincaré group

Tensors or (relativistic) bosons are objects which transform according to the tensor representation of the Lorentz group. Spinors or (relativistic) fermions are objects which transform according to the spinor represenation of the Lorentz group. Hence, by studying the Lorentz group, we can distinguish between bosons and fermions and assign particles to one of the two categories. But to completely treat the world of elementary particles we need to study the Poincaré group.

The Poincaré group is the group of Lorentz transformations and translations in Minkowski space. It describes the structure of our space-time, and all its irreducible representations are characterised by mass and spin, hence by the fundamental properties of the elementary particles.

Poincaré transformations in Minkowski space are composed of a Lorentz transformation with Λ^{μ}_{ν} and a translation by a^{μ}. We hence have the translation group T and the Poincaré group P given by

$$\begin{aligned} T &= \{x^{\mu} \rightarrow x'^{\mu} = x^{\mu} + a^{\mu} \ : \ a^{\mu} \in \mathbb{R}^4\} && (2.67) \\ P &= \{x^{\mu} \rightarrow x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu} \ : \ \Lambda^{\mu}_{\nu} \in L, \ a^{\mu} \in \mathbb{R}^4\} \ . && (2.68) \end{aligned}$$

We have the following multiplication rule

$$(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2) \ . \qquad (2.69)$$

Hence P is a semi-direct product of L and T. The semi-direct product differs from the direct product, for which we have the multiplication rule $(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda_2\Lambda_1, a_1 + a_2)$.

The generators of the translation are given by

$$P_{\rho} = i\partial_{\rho} \ , \qquad (2.70)$$

as

$$f(x') = f(x + a) = \exp(ia^{\rho} P_{\rho}) f(x) \quad \text{(Taylor expansion)} \ . \qquad (2.71)$$

Together with the generators of the Lorentz transformation $M^{\mu\nu} = -M^{\nu\mu}$ we hence can write

$$\tilde{\Lambda} = \exp(-\frac{i}{2}\omega_{\alpha\beta} M^{\alpha\beta} + ia_{\mu} P^{\mu}) \ . \qquad (2.72)$$

The algebra is given by

Poincaré Algebra

$$\begin{aligned} [P^{\mu}, M^{\rho\sigma}] &= i(g^{\mu\rho} P^{\sigma} - g^{\mu\sigma} P^{\rho}) && (2.73) \\ [P^{\mu}, P^{\nu}] &= 0 && (2.74) \\ [M^{\mu\nu}, M^{\rho\sigma}] &= \text{see Lorentz algebra Eq. (2.53)} && (2.75) \end{aligned}$$

Infinitesimally, we have

$$(1 + \delta\omega, \delta a) = 1 - \frac{i}{2}\delta\omega_{\alpha\beta} M^{\alpha\beta} + i\delta a_{\mu} P^{\mu} + ... \qquad (2.76)$$

Let us check the relation Eq. (2.73). We have

$$\begin{aligned} & (1 + \delta\omega, 0)^{-1}(1, \delta a)(1 + \delta\omega, 0) \\ =& \ (1 + \frac{i}{2}\delta\omega_{\alpha\beta} M^{\alpha\beta})(1 + i\delta a_{\mu} P^{\mu})(1 - \frac{i}{2}\delta\omega_{\alpha\beta} M^{\alpha\beta}) + ... \\ =& \ 1 + i\delta a_{\mu} P^{\mu} - \frac{1}{2}\delta\omega_{\alpha\beta}\delta a_{\mu}[M^{\alpha\beta}, P^{\mu}] + ... \ , && (2.77) \end{aligned}$$

and we also have

$$
\begin{aligned}
(1+\delta\omega,0)^{-1}(1,\delta a)(1+\delta\omega,0) &= (1-\delta\omega,0)(1+\delta\omega,\delta a) = (1,\delta a^\mu - \delta\omega^{\mu\nu}\delta a_\nu) \\
&= 1 + i(\delta a^\mu - \delta\omega^{\mu\nu}\delta a_\nu)P_\mu + ... \\
&= 1 + i\delta a^\mu P_\mu - \frac{i}{2}\delta\omega^{\mu\nu}(\delta a_\nu P_\mu - \delta a_\mu P_\nu) + ... \qquad (2.78)
\end{aligned}
$$

The combination of Eqs. (2.77) and (2.78) yields Eq. (2.73).

2.6.1 Transformation of fields

Let us investigate the transformation of the fields. We have

$$
\phi(x) \stackrel{(\Lambda,a)}{\rightarrow} \phi'(x') = \phi'(\Lambda x + a) \ . \qquad (2.79)
$$

For the scalar field we have

$$
\phi'(x') = \phi(x) \qquad (2.80)
$$

and hence

$$
\phi'(x) \stackrel{!}{=} \phi(\Lambda^{-1}(x-a)) \ . \qquad (2.81)
$$

For infinitesimal translations this means

$$
\begin{aligned}
\phi'(x) &= \phi(x-\delta a) = \phi(x) - \delta a^\mu \partial_\mu \phi(x) + \mathcal{O}((\delta a)^2) = (1 + i\delta a^\mu(i\partial_\mu))\phi(x) + \mathcal{O}((\delta a)^2) \\
&= (1 + i\delta a^\mu P_\mu)\phi(x) + \mathcal{O}((\delta a)^2) \ , \qquad (2.82)
\end{aligned}
$$

from which follows (see also above)

$$
P^\mu = i\partial^\mu \ . \qquad (2.83)
$$

The momentum operator hence corresponds to the differential operator. For homgenous infinitesimal Lorentz transformations we get

$$
\begin{aligned}
\phi'(x) &\stackrel{!}{=} \phi(\Lambda^{-1}x) = \phi(x^\mu + \frac{i}{2}\delta\omega_{\alpha\beta}(M^{\alpha\beta})^\mu{}_\nu x^\nu + ...) \\
&= \phi(x) + \frac{i}{2}\delta\omega_{\alpha\beta}(M^{\alpha\beta})^\mu{}_\nu x^\nu \partial_\mu \phi(x) + ... \\
&= \phi(x) + \frac{i}{2}\delta\omega_{\alpha\beta} i(g^{\alpha\mu}\delta^\beta_\nu - g^{\beta\mu}\delta^\alpha_\nu)x^\nu \partial_\mu \phi(x) + ... \\
&= \phi(x) + \frac{i}{2}\delta\omega_{\alpha\beta} i(x^\beta\partial^\alpha - x^\alpha\partial^\beta)\phi(x) + ... \\
&= \phi(x) - \frac{i}{2}\delta\omega_{\alpha\beta} L^{\alpha\beta}\phi(x) + ... \ , \qquad (2.84)
\end{aligned}
$$

where we introduced the generalised angular momentum operator

$$
L^{\alpha\beta} = i(x^\alpha\partial^\beta - x^\beta\partial^\alpha) = x^\alpha P^\beta - x^\beta P^\alpha \ . \qquad (2.85)
$$

Non-infinitesimally, we have

$$
\phi'(x) = \phi(\Lambda^{-1}(x-a)) = \exp\left(-\frac{i}{2}\omega_{\alpha\beta}L^{\alpha\beta} + i a_\mu P^\mu\right)\phi(x) \ . \qquad (2.86)
$$

2.6.2 Groups

Be a pair $(G, *)$ with a set G and an inner binary connection/group multiplication. $* : G \times G \to G, (a, b) \mapsto a * b$ is called group if the following axioms are fulfilled

1. The group is *closed*. This means, if $g, h \,\epsilon\, G \Rightarrow g * h \,\epsilon\, G$.
2. *Associativity:* $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$.
3. $\exists$ *Identity element* e with the property $g * e = e * g = g \;\; \forall\, g \,\epsilon\, G$.
4. For each g there is an *inverse* g^{-1} with $g^{-1} * g = g * g^{-1} = e$.

Abelian group: A group is called *Abelian*, if $g * h = h * g$.

Continuous groups: They contain an infinite number of elements and are described by n parameters. The elements depend in a continuous and differentiable way on a set of real parameters θ^a, $a = 1, ..., n$, where n is the dimension of the group. For *Lie groups* n is finite. We chose $g(\theta = 0) = e$. All one-parameter Lie groups are Abelian. A typical example is $U(1)$ with the elements $e^{i\phi}$ and ϕ as parameter.
Examples of Lie groups:
(i) $O(N)$: orthogonal group, dimension $\frac{N(N-1)}{2}$. We have $MM^T = 1$ so that $\det M = \pm 1$. We have the $SO(N)$ for $\det M = 1$.
(ii) $U(N)$: unitary group, dimension N^2. We have $UU^\dagger = 1$. We have the $SU(N)$ for $\det U = 1$. Its dimension is $N^2 - 1$.
(iii) $SL(N, \mathbb{C})$: complex matrices A, $\det A = 1$, dimension $2N^2 - 2$. E.g. the symplectic group $\mathrm{Sp}(2n, \mathbb{C})$.

2.7 Representations

A representation R of a group is an operation assigning a linear operator $D_R(g)$ to a group element g:

$$g \to D_R(g) \; , \tag{2.87}$$

with

1. $D_R(e) = \mathbf{1}$ – Identity operator
2. $D_R(g_1)D_R(g_2) = D_R(g_1 \circ g_2)$ – The mapping preserves the group structure.

The space on which the operators D_R act, is called the basis for the representation R. A typical example is the matrix representation. Here the basis is a vector space of dimension n, and the group element is represented by an $(n \times n)$ matrix.

$$(D_R(g))^i_j, \quad i, j = 1, ..., n \; . \tag{2.88}$$

We call n the dimension of the representation R. It is equal to the dimension of the basis space on which it acts.

Let us look at an example and choose $(\phi^1, ..., \phi^n)$ as element of a basis vector space, on which we apply a group transformation induced by the group element g,

$$\phi^i \to (D_R(g))^i_j \phi^j \; . \tag{2.89}$$

Choosing a representation hence allows us to interpret g as a transformation on a certain space.

A subspace of a representation is invariant if the action of $D_R(g)$ on a vector of this subspace results in another vector in the same subspace. A representation R is called *irreducible* if the only invariant subspaces are the zero space and the basis space itself. Or, in other words, an irreducible representation is a non-zero representation that has no proper nontrivial subrepresentation.

Two representations R and R' are called *equivalent* if there is a matrix S such that

$$D_R(g) = S^{-1} D_{R'}(g) S \; . \tag{2.90}$$

An example is the change of the basis in the vector space spanned by the ϕ^i. Note that in general the dimension of the matrix $D_R(g)$ changes with the representation R.

The Lie algebra is independent of the representation. The Lie group has a smooth dependence on the parameter θ^a, and for infinitesimal θ^a we have

$$D_R(\theta) = 1 + i\theta^a T_R^a + \mathcal{O}(\theta^2) \; , \tag{2.91}$$

with

$$T_R^a = -i \left| \frac{\partial D_R(\theta)}{\partial \theta^a} \right|_{\theta=0} \tag{2.92}$$

The T_R^a are called the *generators* of the group in the representation R. A group element $g(\theta)$ can be represented as

$$D_R(g(\theta)) = e^{i\theta^a T_R^a} \; . \tag{2.93}$$

Note that if the T_R^a are hermitian, the $D_R(g(\theta))$ are unitary and we hence have a unitary representation.

Let us look at the group multiplication. We have

$$e^{i\alpha_a T_R^a} e^{i\beta_a T_R^a} = e^{i\theta_a T_R^a} \tag{2.94}$$

for some α, β, θ. But in general, $e^A e^B \neq e^{A+B}$. The relation defines the *Lie bracket* (*Lie algebra*), given by

$$[T^a, T^b] = i f^{abc} T^c \; . \tag{2.95}$$

We call the f^{abc} *structure constants*. They are independent of the represenation R. The *Jacobi identity* is given by

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0 \; . \tag{2.96}$$

For Abelian groups we have $f^{abc} = 0$. All irreducible representations of Abelian groups are one-dimensional. E.g. we have

$$U(1) \text{ and its representation } D(g(\theta)) = e^{i\theta} \; . \tag{2.97}$$

Important representations: We have the following important representations:

Fundamental representation: It is the non-trivial representation with the smallest dimension. E.g. the generators of $SO(3,1)$ are given by (4×4) matrices and those of the $SU(N)$ by $(N\times N)$ matrices.

Adjoint representation: The representation has the same dimension as the number of generators. For Lie groups they can be written in terms of the structure constants, namely $(T^a_{\rm ad})_{bc} = -if^{abc}$.

Casimir operators C: These are operators which commute with all generators T^a, i.e.

$$[C, T^a] = 0 \ . \tag{2.98}$$

They are proportional to **1** in all irreducible representations. Let us look as example at the algebra of the angular momentum, given by

$$[J^i, J^j] = i\epsilon^{ijk} J^k \ . \tag{2.99}$$

The Casimir operator is

$$\vec{J}^2 = \sum_i J^i J^i \ , \tag{2.100}$$

hence

$$[\vec{J}^2, J^i] = 0 \ . \tag{2.101}$$

This allows to e.g. diagonalise $\vec{J}^2$ and J^3 simultaneously. The irreducible representations $D^{(j)}$ for fxed values of j are given by the eigenvalues of the Casimir operator, namely

$$\vec{J}^2|j,m> \;=\; j(j+1)|j,m> \tag{2.102}$$
$$J^3|j,m> \;=\; m|j,m> \ , \quad m = -j, -j+1, ..., j \ . \tag{2.103}$$

The $\{|j,m>\}$ correspond to a $(2j+1)$-dimensional multiplet for each fixed j.

Casimir operators of the Poincaré group
We have introduced before the Poincaré algebra, in particular

$$[P^\mu, P^\nu] = 0 \ , \quad [J^i, P^j] = i\epsilon^{ijk} P^k \quad (i,j \in \{1,2,3\}) \ , \quad [J^i, P^0] = 0 \ , \tag{2.104}$$
$$[K^i, P^j] = iP^0 \delta_{ij} \ , \quad [K^i, P^0] = iP^i \ . \tag{2.105}$$

The operator P^0 is the Hamilton operator, and we can infer that J^i, P^i are conserved, but K^i is not conserved. The Casimir operator is given by $P_\mu P^\mu$. Let us proof this:

$$[K^i, P_\mu P^\mu] = [K^i, P_0^2] - [K^i, \vec{P}^2] = 2iP^i P^0 - 2i\delta_{ij} P^j P^0 = 0 \ , \tag{2.106}$$

and analogous for J. Its action on a one-particle state $|p,s>$ with momentum p^μ and $p^2 = m^2$ is

$$P^\mu P_\mu |p,s> = m^2 |p,s> \ . \tag{2.107}$$

The mass is hence the eigenvalue of the Casimir operator.

There is another Casimir operator given in terms of the *Pauli-Lubanski vector*, which is defined as

$$W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} M_{\nu\rho} P_\sigma \ . \tag{2.108}$$

The Casimir operator is $W_\mu W^\mu$. We give a sketch of the proof: $W_\mu W^\mu$ is Lorentz-invariant and hence commutes with $M^{\nu\sigma}$. We also have $[W^\mu, P^\nu] = 0$ so that $[W^\mu W_\mu, P^\nu] = 0$.

For non-zero mass m, we have in the rest frame $p^\mu = (m, \vec{0})$ that $W^0 = 0$. And we have

$$W^i = \frac{m}{2}\epsilon^{0ijk}M_{jk} = \frac{m}{2}\epsilon^{ijk}M_{jk} = mJ^i \tag{2.109}$$

so that

$$-W_\mu W^\mu |p, s>= m^2 s(s+1)|p, s> \ . \tag{2.110}$$

The $|p, s>$ corresponds hence to massive particles of spin s with $2s+1$ spin degrees of freedom.

For $m = 0$, the p^μ can be written as $p^\mu = (\omega, 0, 0, \omega)$, leading to

$$-W_\mu W^\mu = \omega^2[(K^2 - J^1)^2 + (K^1 - J^2)^2] \ . \tag{2.111}$$

It can be shown that $W^\mu = hP^\mu$, where h is the *helicity* given by

$$h = \frac{\vec{p}}{|\vec{p}|}\vec{J} \tag{2.112}$$

and the eigenvalues are $h = \pm s$. For the photon we have $h = \pm 1$ and for a massless fermion we have $h = \pm\frac{1}{2}$. For the graviton we have $h = \pm 2$. We hence have 2 spin degrees of freedom.

Chapter 3

The Lagrangian Formalism for Fields

3.1 Quantum Field Theory

In classical physics, we know particles and fields like e.g. the electromagnetic field. A particle is fixed by its space-time-coordinates. A field on the other hand has an infinite number of degrees of freedom. Particles are described in quantum mechanics through a wave function. Fields appear as external fields, e.g. the Maxwell field. Through quantisation of the electromagnetic field, photons are introduced and thereby the wave-particle dualism of light. From the point of view of theoretical physics, elementary particles, the smallest building blocks of matter, are the lowest excitation levels of certain fields.

In order to describe elementary particles and their interactions we use **quantum field theory (qfth)**. In this theory, the principles of classical field theory and quantum mechanics are joined into one theory. The theory goes beyond quantum mechanics, by treating particles and fields uniformly. Not only observables like e.g. energy are quantised, but also the **interacting** (particle) fields themselves. The quantisation of the fields is also called **second quantisation**. It allows to explicitly take into account **creation** and **annihilation** of pairs of particles. **Relativistic quantum field theories** take into account **special relativity** and are applied in elementary particle physics.[1]

The application of quantum field theory hence allows to solve a fundamental problem of quantum mechanics: Its inability to describe systems with varying particle number. As known from relativistic quantum mechanics, there are, according to the relativistic **Klein-Gordon equation** and the **Dirac equation**, solutions of negative energy, which are interpreted as antiparticles. Thereby particle-antiparticle pairs can be created if there is sufficient energy.[2] This is impossible in a system with constant particle number.

The first step towards a quantum field theory consists in finding the **Lagrangian density** for the quantum fields. These deliver via the **Euler-Lagrange equation** the differential equation for the fields. The differential equations are for the **scalar field** the Klein-Gordon equation, for the **spinor field** the Dirac equation, and for the photon the **Maxwell equations**. These are the equations of motion for the free fields, which do not interact. They have been derived from Lagrangian densities for free fields. In order to introduce interactions

[1] Non-relativistic quantum field theories are relevant e.g. in solid state physics.

[2] In the Dirac picture this corresponds to lifting a particle from the Dirac lake of states with negative energies to a state with positive energy. Thereby a hole is generated in the Dirac lake, which is interpreted as a positron, and leads to an electron with a positive energy, so that we get an electron-positron pair.

between the fields, the Lagrangian densities have to be extended by additional terms.

3.2 Transition from the Discrete to the Continuum System

We investigate the transition from the discrete to the continuum system by taking the example of a chain consisting of mass points. The mass points of mass m shall be connected with each other through springs with spring constant k. This is a system of coupled harmonic oscillators. Be a the average distance between two mass points and q_i the deviation of the ith mass point from its rest/equilibrium position. We then have the kinetic energy T

$$T = \sum_i \frac{1}{2} m \dot{q}_i^2 \; . \tag{3.1}$$

The potential energy V is given by

$$V = \sum_i \frac{1}{2} k (q_{i+1} - q_i)^2 \; . \tag{3.2}$$

With this we get the equation of motion for the ith mass point,

$$m\ddot{q}_i = -\frac{\partial V}{\partial q_i} = k(q_{i+1} - q_i) - k(q_i - q_{i-1}) \; . \tag{3.3}$$

On the other hand the equation of motion can also be deduced from the Lagrangian function of the system. It is given by

$$L = T - V = \frac{1}{2} \sum_i a \left[\frac{m}{a} \dot{q}_i^2 - ka \left(\frac{q_{i+1} - q_i}{a} \right)^2 \right] \; . \tag{3.4}$$

The equation of motion of a single particle is obtained by applying the Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \; , \tag{3.5}$$

leading to

$$\frac{m}{a} \ddot{q}_i - ka \frac{q_{i+1} - q_i}{a^2} + ka \frac{q_i - q_{i-1}}{a^2} = 0 \; . \tag{3.6}$$

We take the limit $a \to 0$. This implies the following:

1. The quotient m/a becomes the mass density μ.
2. The normalised distance $\xi = (q_{i+1} - q_i)/a$ is proportional to the force $k(q_{i+1} - q_i)$. The proportionality constant is given by the material constant y, the so-called Young modul. We hence have

$$\frac{q_{i+1} - q_i}{a} y = k(q_{i+1} - q_i) \overset{a \to 0}{\to} y = k \cdot a \; . \tag{3.7}$$

3. We pass over from the discret index i to a continuum index x. Insted of the index i, we take the position in rest, x. And instead of q_i, we now have $q(x)$ as function of the position. We hence have

$$q_i \rightarrow q(x) \tag{3.8}$$

$$\frac{q_{i+1} - q_i}{a} \rightarrow \frac{q(x+a) - q(x)}{a} \stackrel{a\to 0}{\rightarrow} \frac{\partial q(x)}{\partial x} = q'(x) \; . \tag{3.9}$$

Furthermore,

$$a \sum_i \rightarrow \int dx \; . \tag{3.10}$$

We thereby get the following Lagrangian function of the continuum system

$$L = \int dx \left(\frac{1}{2} \mu \, \dot{q}(x)^2 - \frac{y}{2} \left(\frac{\partial q(x)}{\partial x} \right)^2 \right) \; . \tag{3.11}$$

The integrand is called Lagrangian density $\mathcal{L}$. The equation of motion is obtained from Eq. (3.6):

$$\mu \ddot{q} - y \lim_{a \to 0} \left(\frac{q'(x+a) - q'(x)}{a} \right) = 0 \quad \Rightarrow \quad \mu \ddot{q} - y q'' = 0 \; . \tag{3.12}$$

Note that x is <u>not</u> a generalised coordinate, but an index. The canonical variable is given by $q(x) = q(t, \vec{x})$. We call $q = q(t, \vec{x})$ a <u>field</u>. The equations of motion are partial differential equations.

For three-dimensional systems we have

$$L = \int dx \, dy \, dz \, \mathcal{L} \; . \tag{3.13}$$

The Lagrangian density $\mathcal{L}$ is a function of $\vec{\nabla} q$. The canonical momentum is given by

$$\pi = \frac{\partial L}{\partial \dot{q}} \; . \tag{3.14}$$

3.3 The Euler-Lagrange Equation for Fields

We apply the Hamiltonian principle to the Lagrangian density $\mathcal{L}(q, \dot{q}, \vec{\nabla} q)$. It states that the action S

$$S = \int_{t_1}^{t_2} dt \int d^3x \, \mathcal{L} \tag{3.15}$$

has to be minimised, keeping the endpoints $q(t_1)$, $q(t_2)$ fixed. We look at the variation

$$0 \stackrel{!}{=} \delta S = \int_{t_1}^{t_2} dt \int d^3x \, \frac{\partial \mathcal{L}}{\partial q} \delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta \dot{q} + \frac{\partial \mathcal{L}}{\partial (\vec{\nabla} q)} \delta (\vec{\nabla} q) \, ,$$
$$\text{with } \delta \dot{q} = \frac{d}{dt} \delta q \, , \; \delta(\vec{\nabla} q) = \vec{\nabla} \delta q \; . \tag{3.16}$$

We perform a partial integration keeping the terms at the endpoints fixed so that their variation vanishes, i.e. $\delta q(t_1) = \delta q(t_2) = 0$. We further demand that $q(t, \vec{x}) = 0$ for $|\vec{x}| \to \infty$. Thereby, we obtain

$$0 \stackrel{!}{=} \int_{t_1}^{t_2} dt \int d^3x \left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \vec{\nabla} \frac{\partial \mathcal{L}}{\partial \vec{\nabla} q} \right) \delta q \tag{3.17}$$

This has to hold for all variations δq. Thereby we get the Euler-Lagrange equation for fields

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \vec{\nabla} \frac{\partial \mathcal{L}}{\partial \vec{\nabla} q} = 0 \; . \tag{3.18}$$

The Hamiltonian density $\mathcal{H}$ is given by

$$\mathcal{H} = \pi \dot{q} - \mathcal{L} \; , \quad \text{with } \pi = \frac{\partial \mathcal{L}}{\partial \dot{q}} \; . \tag{3.19}$$

We look as example at the following Lagrangian density

$$\mathcal{L} = \frac{\mu}{2}\dot{q}^2 - \frac{y}{2}q'^2 \; . \tag{3.20}$$

We have with

$$\frac{\partial \mathcal{L}}{\partial q} = 0 \; , \quad \frac{\partial \mathcal{L}}{\partial \dot{q}} = \mu \dot{q} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial q'} = -yq' \tag{3.21}$$

the equation of motion

$$\mu \ddot{q} - yq'' = 0 \; . \tag{3.22}$$

3.3.1 Relativistic Notation

We define

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \quad \text{and} \quad \partial^\mu \equiv \frac{\partial}{\partial x_\mu} \quad \text{as well as} \tag{3.23}$$

$$\int dx \equiv \int dt \int d^3x \; . \tag{3.24}$$

The $\int dx$ is Lorentz-invariant (the Lorentz contraction is compensated by the time dilatation). With this notation we get for the field $\phi(t, \vec{x})$ then $\phi(x)$ and for the Lagrangian density

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi) \; . \tag{3.25}$$

The Euler-Lagrange equations can be written as

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \qquad \text{with} \quad \pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \; . \tag{3.26}$$

In case the Lagrangian density $\mathcal{L}$ is Lorentz invariant, the field equations are covariant.

We look at the following examples:

1. Real scalar field without interaction. The Lagrangian density reads

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2 \; . \tag{3.27}$$

We thereby have

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi \; , \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi \tag{3.28}$$

and hence get the equation of motion

$$-m^2\phi - \partial_\mu \partial^\mu \phi = 0 \quad \Rightarrow \quad (\Box + m^2)\phi = 0 \text{ with } \partial_\mu \partial^\mu = \partial_0^2 - \vec{\nabla}^2 = \partial_0^2 - \Delta \tag{3.29}$$

This is the relativistic Klein-Gordon equation known from quantum mechanics.

2. Complex scalar field without interaction. The Lagrangian density reads

$$\mathcal{L}(\phi, \phi^*, \partial_\mu \phi, \partial_\mu \phi^*) = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \; . \tag{3.30}$$

The fields ϕ und ϕ^* can be varied independently of each other. This means, we have

$$\frac{\partial \mathcal{L}}{\partial \phi^*} = -m^2 \phi \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = \partial^\mu \phi \tag{3.31}$$

$$\Rightarrow \partial_\mu \partial^\mu \phi + m^2 \phi = 0 \qquad \text{and analoguously} \quad \partial_\mu \partial^\mu \phi^* + m^2 \phi^* = 0 \; . \tag{3.32}$$

3. Spin-1/2 field (Dirac field) without interaction. The Lagrangian density reads

$$\mathcal{L}(\psi, \bar{\psi}) = \bar{\psi}(i\partial\!\!\!/ - m)\psi \; , \tag{3.33}$$

where

$$a\!\!\!/ \; := \; a^\mu \gamma_\mu = a_\mu \gamma^\mu \tag{3.34}$$

$$\partial\!\!\!/ \; := \; \gamma^\mu \frac{\partial}{\partial x^\mu} = \gamma^\mu \partial_\mu \; . \tag{3.35}$$

We have

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = (i\partial\!\!\!/ - m)\psi \; , \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = 0 \; . \tag{3.36}$$

We thereby get the equation of motion

$$(i\partial\!\!\!/ - m)\psi = 0 \; . \tag{3.37}$$

3.4 The Noether Theorem for Fields

For each symmetry of the action integral under a continuous transformation there exists a conservation law, which can be derived from the Lagrangian density.

Proof: We look at $\mathcal{L} = \mathcal{L}(\varphi, \partial_\mu \varphi)$. Here φ is a field (a scalar field φ or $\varphi = A^\mu$ or a multiplet of fields $\varphi = (\varphi_1, ..., \varphi_n)$). We look at an infinitesimal transformation w.r.t. a Lie group

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu \tag{3.38}$$

with

$$\delta x^\mu = A_k^\mu \delta\omega^k \; . \tag{3.39}$$

Here $\delta\omega^k$ is the parameter of the transformation (e.g. the Euler rotation angle). For a rotation by $\delta\vec{\omega}$ $(\exp(i\delta\omega^k J_k))$ we have for example

$$\vec{x}\,' = \vec{x} + i\delta\omega^k J_k \vec{x} \; , \tag{3.40}$$

so that also

$$A_k^0 = 0 \; , \quad A_k^j = i(J_k \vec{x})^j \; , \quad j = 1,2,3 \; . \tag{3.41}$$

We furthermore have the transformation of the field

$$\varphi(x) \to \varphi'(x') = \varphi(x) + \delta\varphi(x) = \varphi(x) + \Phi_k(x)\delta\omega^k \; . \tag{3.42}$$

(For example we have for a scalar $\varphi'(x') = \varphi(x)$ and hence $\delta\varphi(x) = 0$ and for a vector $\varphi'^\mu(x') = \Lambda^\mu_\nu \varphi^\nu(x)$ etc.) We find

$$\begin{aligned} \varphi'(x') &= \varphi'(x + \delta x) \\ &= \varphi'(x) + \delta x^\nu \partial_\nu \varphi \\ &= \varphi(x) + \delta_0\varphi(x) + \delta x^\nu \partial_\nu \varphi \; , \end{aligned} \tag{3.43}$$

with the variation

$$\delta_0\varphi(x) = \varphi'(x) - \varphi(x) \; . \tag{3.44}$$

Thereby we have

$$\begin{aligned} \delta_0\varphi(x) &\overset{(3.43)}{=} \varphi'(x') - \varphi(x) - \delta x^\nu \partial_\nu \varphi \\ &\overset{(3.42)}{=} \Phi_k(x)\delta\omega^k - \delta x^\nu \partial_\nu \varphi \\ &\overset{(3.39)}{=} [\Phi_k(x) - (\partial_\nu\varphi)A_k^\nu]\delta\omega^k \; . \end{aligned} \tag{3.45}$$

For the variation of the Lagrangian density we have

$$\begin{aligned} \delta\mathcal{L} &= \mathcal{L}'[\varphi'(x'), \partial_\nu\varphi'(x')] - \mathcal{L}[\varphi(x), \partial_\mu\varphi(x)] \\ &= \frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu + \frac{\partial\mathcal{L}}{\partial\varphi}\delta_0\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} \underbrace{\partial_\mu\delta_0\varphi}_{=\delta_0\partial_\mu\varphi} \; . \end{aligned} \tag{3.46}$$

Application of the Euler Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial\varphi} = \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} \tag{3.47}$$

and using Eq. (3.45) and Eq. (3.46) leads to

$$\begin{aligned}\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta_0\varphi\right)\\ &= \frac{\partial\mathcal{L}}{\partial x^\mu}A^\mu_k(x)\delta\omega^k + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}[\Phi_k(x) - (\partial_\nu\varphi)A^\nu_k(x)]\delta\omega^k\right] .\end{aligned} \tag{3.48}$$

And for the transformed action we have

$$S' = \int d^4x'\,\mathcal{L}'(\varphi', \partial_\mu\varphi') \tag{3.49}$$

and

$$\delta S = S' - S = \int \delta(d^4x)\mathcal{L} + \int d^4x\,\delta\mathcal{L} . \tag{3.50}$$

Furthermore we have

$$\begin{aligned}d^4x' &= \left|\det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right)\right| d^4x\\ &= |\det(\delta^\mu_\nu + \partial_\nu A^\mu_k\delta\omega^k)|d^4x\\ &= (1 + \partial_\mu A^\mu_k\delta\omega^k)d^4x ,\end{aligned} \tag{3.51}$$

where we have used

$$\det(\mathbf{1} + \epsilon) = 1 + \mathrm{tr}\epsilon + O(\epsilon^2) . \tag{3.52}$$

Thereby we have

$$\delta(d^4x) = d^4x\,\partial_\mu A^\mu_k\delta\omega^k . \tag{3.53}$$

And hence

$$\begin{aligned}\delta S &= \int d^4x\,(\mathcal{L}\partial_\mu A^\mu_k\delta\omega^k + \delta\mathcal{L})\\ &\overset{(3.48)}{=} \int d^4x\,\partial_\mu\left[\mathcal{L}A^\mu_k + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}[\Phi_k - (\partial_\nu\varphi)A^\nu{}_k]\right]\delta\omega^k .\end{aligned} \tag{3.54}$$

We call the transformations global if $\delta\omega^k$ is independent of x. The integration volume in $S = \int_V d^4x\,\mathcal{L}$ is chosen arbitrarily. If S is invariant, i.e. $\delta S/\delta\omega_k = 0$, then it follows from Eq. (3.54) that

$$j^\mu_k = -\mathcal{L}A^\mu_k + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}(A^\nu_k\partial_\nu\varphi - \Phi_k(x)) \tag{3.55}$$

is a conserved current (Noether theorem):

$$\partial_\mu j^\mu_k = 0 . \tag{3.56}$$

We call j^μ_k Noether current. And the charge

$$Q_k(x) = \int d^3x\,j^0_k(x) \tag{3.57}$$

is constant, because

$$\dot{Q}_k(x) = \int d^3x \, \partial_0 j_k^0 \stackrel{(3.56)}{=} - \int d^3x \, \vec{\nabla} \vec{j}_k = - \oint d\vec{S} \, \vec{j}_k = 0 \tag{3.58}$$

for $\vec{j}_k = 0$ at infinity. Note that the conserved Noether current is not unique. We can e.g. add a current $j_k^{'\mu}$ whose divergence vanishes, hence

$$\partial_\mu j_k^{'\mu} = 0 \; . \tag{3.59}$$

The Noether current Eq. (3.55) can be generalised to the case of several fields φ^a in $\mathcal{L}$. We then have

$$j_k^\mu = -\mathcal{L} A_k^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^a)} (A_k^\nu \partial_\nu \varphi^a - \Phi_k^a(x)) \; . \tag{3.60}$$

3.4.1 Examples

Real Klein-Gordon field: The Lagrangian density reads

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{m^2}{2}\varphi^2 - V(\varphi) \; , \tag{3.61}$$

where $V(\varphi)$ is an arbitrary potential e.g. $V(\varphi) = \lambda/4! \, \varphi^4$. The Lagrangian density is invariant under transformations,

$$x^\mu \to x^\mu + \underbrace{\epsilon^\mu}_{A_k^\mu \delta\omega^k} \quad , \text{ so that thereby } A_\nu^\mu(x) = \delta_\nu^\mu \; . \tag{3.62}$$

Since φ is a scalar field it is invariant, hence $\delta\varphi(x) = 0$ and thereby $\Phi_\nu(x) = 0$. From the Lagrangian density Eq. (3.61) we get

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = \partial^\mu \varphi \; . \tag{3.63}$$

The conserved current, which in case of the translations Eq. (3.62) is denoted by T_ν^μ, is the energy-momentum tensor and given by

$$T_\nu^\mu = -\mathcal{L}\delta_\nu^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi \tag{3.64}$$

$$= -\mathcal{L}\delta_\nu^\mu + \underbrace{\partial^\mu \varphi}_{(3.63)} \partial_\nu \varphi \; . \tag{3.65}$$

Hence

$$T_\nu^\mu = \partial^\mu \varphi \, \partial_\nu \varphi - \delta_\nu^\mu \left(\frac{1}{2}(\partial_\alpha \varphi)^2 - \frac{m^2}{2}\varphi^2 - V(\varphi) \right) \; . \tag{3.66}$$

The conserved charge, the four-moment, is given by

$$P_\nu = \int d^3x \, T_\nu^0 \; . \tag{3.67}$$

We have

$$T^0_{\ 0} = -\mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_0\varphi)}\,\partial_0\varphi\ . \tag{3.68}$$

In classical mechanics we had the Hamilton function

$$H = \sum_i p_i \dot{q}_i - L = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\ . \tag{3.69}$$

The comparison shows: The quantity $T^0_{\ 0}$ is the Hamilton density. The canonical field-momentum density is given by

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi}(x) \quad , \text{ where } \dot{\varphi} = \partial_0\varphi\ . \tag{3.70}$$

Thereby the Hamilton density is

$$T^0_{\ 0} = \pi(x)\,\dot{\varphi}(x) - \mathcal{L}(x)\ . \tag{3.71}$$

The Hamilton operator, which corresponds to the total energy, is given by

$$H = P_0 = \int d^3x\, T^0_{\ 0} = \int d^3x\,(\pi\dot{\varphi} - \mathcal{L}) = \int d^3x\,\left(\frac{1}{2}(\dot{\varphi}^2 + (\vec{\nabla}\varphi)^2 + m^2\varphi^2) + V(\varphi)\right) \tag{3.72}$$

And the 3-momentum reads

$$P_k = \int d^3x\, T^0_{\ k} = \int d^3x\,\dot{\varphi}\,\partial_k\varphi \tag{3.73}$$

$$P^k = -\int d^3x\,\dot{\varphi}\,\partial_k\varphi \tag{3.74}$$

$$\vec{P} = -\int d^3x\,\dot{\varphi}\,\vec{\nabla}\varphi = -\int d^3x \underbrace{\pi(x)\,\vec{\nabla}\varphi}_{\text{-Momentum density}}\ . \tag{3.75}$$

The next examples are infinitesimal rotations. This means

$$\delta\vec{x} = i\delta\omega^k J_k\,\vec{x}\ , \quad \delta x^0 = 0 \tag{3.76}$$

$$A^j_k = i(J_k\vec{x})^j\ , \quad A^0_k = 0\ , \qquad j,k = 1,2,3 \tag{3.77}$$

$$\delta\varphi = 0 = \Phi_k\ . \tag{3.78}$$

Thereby we obtain from Eq. (3.55)

$$\begin{aligned} j^0_k &= \frac{\partial \mathcal{L}}{\partial(\partial_0\varphi)} A^j_k(\partial_j\varphi) = \frac{\partial \mathcal{L}}{\partial\dot{\varphi}}(\vec{\nabla}\varphi)^T iJ_k\vec{x} \\ &\overset{(3.63)}{=} \dot{\varphi}(\vec{\nabla}\varphi)^T iJ_k\vec{x}\ . \end{aligned} \tag{3.79}$$

The momentum density, cf. Eq. (3.75), is defined through

$$\vec{p} = -\dot{\varphi}\vec{\nabla}\varphi\ , \tag{3.80}$$

and thereby

$$\vec{P} = \int d^3x\,\vec{p}(x)\ . \tag{3.81}$$

We hence have

$$\begin{aligned} j_k^0 &= -\vec{p}^T iJ_k\vec{x} = -p_m \underbrace{(iJ_k)_{mn}}_{=\epsilon_{kmn}} x_n \\ &= -\epsilon_{kmn}p_m x_n = -(\vec{p}\times\vec{x})_k \ , \end{aligned} \tag{3.82}$$

where we have used that the generators of the rotation are given by

$$[J_k]_{lm} = -i\epsilon_{klm} \quad \text{and} \quad [J_k]_{0\mu} = [J_k]_{\mu 0} = 0 \ . \tag{3.83}$$

We hence find that $-j_k^0$ is an angular momentum density. The conserved charge hence is the angular momentun, as with Eq. (3.57) we have

$$L_k = \int d^3x\, j_k^0(x) \tag{3.84}$$

$$\vec{L} = \int d^3x\, (\vec{x}\times\vec{p}) \ . \tag{3.85}$$

We now consider the Noether theorem for an inner symmetry. We apply this to a complex field with self-interaction. The Lagrangian density reads

$$\mathcal{L} = \partial_\mu\varphi\partial^\mu\varphi^* - m|\varphi|^2 - V(|\varphi|) \ . \tag{3.86}$$

The Lagrangian density is invariant under a $U(1)$ symmetry, i.e. under the transformation

$$\varphi \rightarrow \varphi\exp(i\delta\vartheta) = \varphi + i\delta\vartheta\,\varphi \tag{3.87}$$

$$\delta x^\mu = 0 \quad \Rightarrow \quad A_k^\mu = 0 \tag{3.88}$$

$$\left.\begin{aligned} \delta\varphi &= i\delta\vartheta\,\varphi \\ \delta\varphi^* &= -i\delta\vartheta\,\varphi^* \end{aligned}\right\} \quad \overset{(3.42)}{\Rightarrow} \begin{pmatrix} \delta\varphi \\ \delta\varphi^* \end{pmatrix} = \Phi\,\delta\vartheta \ , \qquad \text{with} \tag{3.89}$$

$$\Phi = \begin{pmatrix} i\varphi \\ -i\varphi^* \end{pmatrix} = \begin{pmatrix} \Phi^1 \\ \Phi^2 \end{pmatrix} \ . \tag{3.90}$$

The Noether current (3.55) reads

$$\begin{aligned} j^\mu &= -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\,\Phi^1 - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^*)}\,\Phi^2 \\ &= -i(\partial^\mu\varphi^*)\varphi + i(\partial^\mu\varphi)\varphi^* \ . \end{aligned} \tag{3.91}$$

The corresponding charged current is

$$Q = \int d^3x\, j^0 = i\int d^3x\, (\varphi^*\dot{\varphi} - \varphi\dot{\varphi}^*) \ . \tag{3.92}$$

We consider the $U(1)$ symmetry of the Dirac theory. The Lagrangian density is invariant under the transformations

$$\psi \rightarrow \exp(i\theta)\,\psi \ , \qquad \bar{\psi} \rightarrow \exp(-i\theta)\,\bar{\psi} \ . \tag{3.93}$$

The Lagrangian density is given by

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + e\,\bar{\psi}A\!\!\!/\bar{\psi} \ . \tag{3.94}$$

It contains the coupling to the electromagnetic field A^μ. We have for Φ

$$\Phi = \begin{pmatrix} i\psi_a \\ -i\bar{\psi}_a \end{pmatrix} \qquad \text{with the spinor index } a = 1, 2, 3, 4 \; . \tag{3.95}$$

Thereby we obtain the Noether current

$$j^\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_a)}\, i\psi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi}_a)}\, i\bar{\psi}_a = -i\bar{\psi}\gamma^\mu i\psi = \bar{\psi}\gamma^\mu\psi \; . \tag{3.96}$$

This is the $U(1)$-current density of the Dirac field. With ψ denoting the electron field, $ej^\mu = e\bar{\psi}\gamma^\mu\psi$ is the electromagnetic current density. And

$$Q = e\int d^3x\, j^0 = e\int d^3x\, \bar{\psi}\gamma^0\psi = e\int d^3x\, \psi^\dagger\psi \tag{3.97}$$

is the conserved total electric charge. Hence $e\psi^\dagger\psi$ is the charge density.

Chapter 4

Quantisation of the Fields

In electrodynamics we deal with classical fields, which fulfill the Maxwell equations. These equations, on the other hand, also describe the propagation of photons in the quantised theory. The question is how these two views are related.

Furthermore, we want to describe the creation and annihilation of particles.

The formalism, which we look for, has to account for the fact that both the Klein-Gordon equation and the Dirac equation can have states with negative energy, and that there is a spin-statistics relation.

Quantum field theory provides the framework for the calculation of scattering processes. Its predictions are confirmed by experiment.

4.1 Repetition of Quantum Mechanics

4.1.1 Schödinger Picture

In the Schrödinger picture the states $|\psi(t)\rangle_S$ are time-dependent, whereas the operators O_S are independent of time. The state at time t is given by applying the unitary time-development operator $U(t)$ on the state vector at $t_0 = 0$,

$$|\psi(t)\rangle_S = U(t)\,|\psi(0)\rangle_S\ . \tag{4.1}$$

Inserting the time-dependent wave function in the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle_S = H_S|\psi(t)\rangle_S \tag{4.2}$$

leads to

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}U(t)|\psi(0)\rangle_S &= H_S U(t)|\psi(0)\rangle_S \\ i\hbar\frac{\partial}{\partial t}U(t) &= H_S U(t)\ . \end{aligned} \tag{4.3}$$

We get an operator equation which is equivalent to the Schrödinger equation.

4.1.2 Heisenberg Picture

In the Heisenberg picture the states $|\psi\rangle_H$ are independent of time, the operators $O_H(t)$, however, are time-dependent. The transition from the Schrödinger to the Heisenberg picture is done via

$$\begin{aligned} |\psi\rangle_H &= U^\dagger(t)\,|\psi(t)\rangle_S && (4.4)\\ O_H(t) &= U^\dagger(t)\,O_S U(t)\ . && (4.5)\end{aligned}$$

The expectation value o of the operator is in both pictures the same, as

$$\begin{aligned} o &= {}_S\langle\psi(t)|O_S|\psi(t)\rangle_S = {}_S\langle\psi(t)|\underbrace{U(t)U^\dagger(t)}_{1}O_S\underbrace{U(t)U^\dagger(t)}_{1}|\psi(t)\rangle_S \\ &= {}_H\langle\psi|U^\dagger(t)O_S U(t)|\psi\rangle_H = {}_H\langle\psi|O_H(t)|\psi\rangle_H\ . \end{aligned} \tag{4.6}$$

Performing the derivative of $O_H(t)$ w.r.t. time leads to

$$\frac{d}{dt}O_H(t) = \left(\frac{\partial}{\partial t}U^\dagger(t)\right)O_S U(t) + U^\dagger(t) O_S\left(\frac{\partial}{\partial t}U\right) + U^\dagger\left(\frac{\partial}{\partial t}O_S\right)U(t)\ . \tag{4.7}$$

Usage of

$$\frac{\partial}{\partial t}U(t) = -\frac{i}{\hbar}H_S U(t) \quad \text{and} \quad \frac{\partial}{\partial t}U^\dagger(t) = \frac{i}{\hbar}U^\dagger(t)H_S \tag{4.8}$$

results in

$$\frac{d}{dt}O_H(t) = \frac{i}{\hbar}(U^\dagger H_S O_S U - U^\dagger O_S H_S U) + U^\dagger\left(\frac{\partial O_S}{\partial t}\right)U \tag{4.9}$$

Insertion of $UU^\dagger = 1$ leads to the Heisenberg equation of motion

$$\frac{d}{dt}O_H(t) = \frac{i}{\hbar}[H_H, O_H] + \frac{\partial O_H}{\partial t}\ , \tag{4.10}$$

with

$$\frac{\partial O_H}{\partial t} = U^\dagger\left(\frac{\partial O_S}{\partial t}\right)U\ . \tag{4.11}$$

In the lecture, we will use the Heisenberg picture.

4.2 Excursion: Lagrangian densities for Particles with Spin 0, $\frac{1}{2}$, 1

Starting from the corresponding fields, relativistic theories can be constructed for particles with spin $0, \frac{1}{2}, 1$. We have seen already that in high-energy physics all units can be traced back to the unit 'mass'. We have

$$\begin{aligned} [m] &= [\text{mass}] = 1 \\ [p^\mu] &= [\text{momentum}] = 1 \\ [x^\mu] &= [\text{time, length}] = -1 \\ [H] &= [\text{energy}] = 1 \\ \left[d^3x\right] &= -3 \\ [\mathcal{L}] &= [\mathcal{H}] = 4 \qquad (\text{since } H = \int d^3x\,\mathcal{H}) \\ [S] &= [\text{action}] = 0\ . \end{aligned} \tag{4.12}$$

4.2.1 Construction of Lagrangian Densities

For the construction of Lagrangian densities we apply the following principles:

1) The fields, which shall be included in the theory, are specified. (Fields)

2) The Lagrangian density has the form

$$\mathcal{L}(x) = \sum_i g_i \mathcal{O}_i(x) \ . \tag{4.13}$$

The $\mathcal{O}_i$ are products of fields at the same spacetime point. (Locality) They transform like Lorentz scalars. Thereby the action and hence the dynamics are relativistically invariant/covariant. The g_i are constants, whose mass dimension has been chosen such that

$$[g_i \mathcal{O}_i] = 4 \ . \tag{4.14}$$

If the theory shall contain inner symmetries, then we have to demand that the $\mathcal{O}_i(x)$ are also invariant under these symmetries. (Relativistic invariance and symmetries)

3) The Lagrangian density $\mathcal{L}$ has to contain derivatives ∂_μ of the fields. Otherwise the canonical conjugate momentum of the field would disappear, and the Euler-Lagrange equation would not lead to a development in time. We also note, that sometimes for technical reasons it can be useful to introduce 'help fields', onto which no derivative is acting and which hence have no dynamics. (Dynamics)

4) The mass dimension of the field products $\mathcal{O}_i$ shall not be larger than four. It will become clear later, why we demand this. We also note, that this is not a fundamental requirement. We can give it up in so-called 'effective quantum field theories'. (Renormalisability)

5) Furthermore the Lagrangian density has to contain *all* terms, which are compatible with the requirements 2) and 4). (Completeness)

4.3 Quantisation of the Scalar Field

We want to quantise the scalar field such that the Klein Gordon equation

$$(\Box + m^2)\phi = 0 \tag{4.15}$$

still holds. For this we interpret ϕ as operator and determine its eigenvalues and eigenfunctions. This then leads to the particle interpretation.

The classical field φ fulfills the Klein Gordon equation

$$(\Box + m^2)\varphi = 0 \tag{4.16}$$

Special solutions of this equation are given by plane waves of the form $\exp(ikx)$ or $\exp(-ikx)$ where $k^2 = m^2$. This hence means that

$$k_0 = \pm\sqrt{m^2 + \vec{k}^2} \equiv \pm\omega(\vec{k}) = \omega_k \ . \tag{4.17}$$

We then have

$$\Box \exp(ikx) = -k^2 \exp(ikx) \ . \tag{4.18}$$

The linear combination of both solutions leads to the general solution

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \Big(\alpha(\vec{k}) \exp(-ikx) + \alpha^*(\vec{k}) \exp(ikx)\Big) \ . \tag{4.19}$$

Here the factor

$$\frac{1}{(2\pi)^3 2\omega_k} \tag{4.20}$$

is convention. We furthermore note that the measure

$$d^4k\,\delta(k^2 - m^2)\theta(k_0) = \frac{d^3k}{2\omega_k} \tag{4.21}$$

is Lorentz invariant.

Auxiliary calculation: Derivation of Eq. (4.21). We use the following formula: Be $f(x)$ continuously differentiable with simple roots x_j, $j = 1, ..., n$ and $f'(x_j) \neq 0$, then it holds that

$$\delta(f(x)) = \sum_{j=1}^{n} \frac{1}{\left|\frac{df}{dx}(x_j)\right|}\,\delta(x - x_j) \ . \tag{4.22}$$

Here $x = k_0$ and

$$f(k_0) = k_0^2 - \vec{k}^2 - m^2 \ . \tag{4.23}$$

The roots are given by

$$k_{0_{1,2}} = \pm\sqrt{\vec{k}^2 + m^2} \ . \tag{4.24}$$

With

$$\frac{df}{dk_0} = 2k_0 \tag{4.25}$$

we find

$$\delta(k^2 - m^2) = \frac{1}{2\sqrt{\vec{k}^2 + m^2}} \left\{\delta(k_0 - \sqrt{\vec{k}^2 + m^2}) + \delta(k_0 + \sqrt{\vec{k}^2 + m^2})\right\} \ . \tag{4.26}$$

We define

$$\omega(\vec{k})_k \equiv \sqrt{\vec{k}^2 + m^2} \ , \tag{4.27}$$

this means, we have

$$\theta(k_0)\delta(k^2 - m^2) = \frac{1}{2\omega_k}\delta(k_0 - \omega) \tag{4.28}$$

$$\theta(-k_0)\delta(k^2 - m^2) = \frac{1}{2\omega_k}\delta(k_0 + \omega) \ . \tag{4.29}$$

And thereby we find Eq. (4.21).

4.3.1 Transition to the Quantised Field

Be $\varphi(x)$ the classical measured value, which corresponds to the expectation value of the operator $\phi(x)$,

$$\varphi(x) = \langle \text{state}|\phi(x)|\text{state}\rangle \ . \tag{4.30}$$

For ϕ it has to hold that

1. Be ϕ hermitian, hence $\phi = \phi^\dagger$. From this it follows that the eigenvalue φ is real. Furthermore, the ϕ shall fulfill the Klein Gordon equation, i.e.

$$(\Box + m^2)\phi(x) = 0 \ . \tag{4.31}$$

From this it follows that

$$(\Box + m^2)\varphi(x) = 0 \ . \tag{4.32}$$

2. Be P_μ the generator of a translation. The ϕ shall fulfill the following equation,

$$\partial_\mu \phi(x) = i[P_\mu, \phi(x)] \tag{4.33}$$

We are looking for a description of particles by ϕ. Each solution of the Klein Gordon equation can be given by a superposition of plane waves. Here the Fourier coefficients are operators. Hence

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \Big(a(\vec{k}) \exp(-ikx) + a^\dagger(\vec{k}) \exp(ikx) \Big) \ . \tag{4.34}$$

Application of Eq. (4.33) leads to

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \Big(-a(\vec{k})\, ik_\mu \ \exp(-ikx) + a^\dagger(\vec{k})\, ik_\mu \ \exp(ikx) \Big) \tag{4.35}$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \big(i[P_\mu, a] \exp(-ikx) + i[P_\mu, a^\dagger] \exp(ikx) \big) \ . \tag{4.36}$$

Comparison of the coefficients leads to

$$[P_\mu, a(\vec{k})] \ = \ -k_\mu a \tag{4.37}$$

$$\Big[P_\mu, a^\dagger(\vec{k})\Big] \ = \ k_\mu a^\dagger \ . \tag{4.38}$$

We here remind the harmonic oscillator, where the ladder operators are characterised by the commutation relations

$$[H, a^\dagger] = \omega a^\dagger \quad \text{und} \quad [H, a] = -\omega a \ . \tag{4.39}$$

4.3.2 Construction of the States

We now want to construct the states in an algebraic way. Be $|0\rangle$ the vacuum state, with $|0\rangle \neq 0$, and normalised to one, hence

$$\langle 0|0\rangle = 1 \; . \tag{4.40}$$

Since there are no particles in the vacuum, we have $E = 0$ and $\vec{p} = 0$. Hence

$$P_\mu|0\rangle = 0 \; . \tag{4.41}$$

We apply (4.38) on $|0\rangle$ and find

$$\begin{aligned} (P_\mu a^\dagger(\vec{k}) - a^\dagger(\vec{k})P_\mu)|0\rangle &= k_\mu a^\dagger(\vec{k})|0\rangle \\ \Rightarrow \qquad P_\mu a^\dagger(\vec{k})|0\rangle &= k_\mu a^\dagger|0\rangle \; . \end{aligned} \tag{4.42}$$

Hence $a^\dagger|0\rangle$ is an eigenstate of the energy- and momentum-operator with the eigenvalues k_0 and $\vec{k}$ if $a^\dagger|0\rangle \neq 0$. We apply Eq. (4.37) on $|0\rangle$ and find

$$P_0 a|0\rangle = -k_0 a|0\rangle \; . \tag{4.43}$$

Thereby $a|0\rangle$ would be an eigenstate with negative energy eigenvalue. Since we require that always $E \geq 0$, it follows that

$$a(\vec{k})|0\rangle = 0 \qquad \forall \vec{k} \; . \tag{4.44}$$

If $|p\rangle$ is eigenstate of P_μ, then we have

$$P_\mu|p\rangle = p_\mu|p\rangle \; . \tag{4.45}$$

Using Eqs. (4.38) and (4.37) leads to

$$P_\mu a^\dagger(\vec{k})|p\rangle = (p_\mu + k_\mu)a^\dagger(\vec{k})|p\rangle \tag{4.46}$$

$$P_\mu a(\vec{k})|p\rangle = (p_\mu - k_\mu)a(\vec{k})|p\rangle \; . \tag{4.47}$$

And we have for

$$P^\mu a^\dagger(\vec{k}_1)a^\dagger(\vec{k}_2)|0\rangle = (k_1^\mu + k_2^\mu)a^\dagger(\vec{k}_1)a^\dagger(\vec{k}_2)|0\rangle \; . \tag{4.48}$$

We can hence interpret $a^\dagger$ as creation operator and a annihilation operator. Applying $a^\dagger$ on the vacuum we have

$$|\vec{k}\rangle = a^\dagger(\vec{k})|0\rangle \; . \tag{4.49}$$

It is a 1-particle state with momentum $\vec{k}$. A particle is created in momentum space, with the energy $k_0 = \sqrt{\vec{k}^2 + m^2}$. Thereby the entire Hilbert space ($\equiv$ Fock space) can be constructed in the following way:

$$|\vec{k}_1, \vec{k}_2\rangle = a^\dagger(\vec{k}_1)\, a^\dagger(\vec{k}_2)|0\rangle \tag{4.50}$$

$$|\vec{k}_1, ..., \vec{k}_n\rangle = a^\dagger(\vec{k}_1)...a^\dagger(\vec{k}_n)|0\rangle \; . \tag{4.51}$$

The latter is a state in the Fock space, which consists of n particles with the momenta $\vec{k}_i$.

Microcausality: If we have two measurements, one at x and one at y, then they must not influence each other, if x and y have a space-like distance. We hence have to demand

$$[\phi(x), \phi(y)] = 0 \qquad \text{for } (x-y)^2 < 0 \; . \tag{4.52}$$

Thereby we have more specifically also for $\vec{x} \neq \vec{y}$

$$\begin{aligned} [\phi(\vec{x},t), \phi(\vec{y},t)] &= 0 & (4.53)\\ \left[\phi(\vec{x},t), \dot{\phi}(\vec{y},t)\right] &= 0 \; . & (4.54)\end{aligned}$$

And by triviality,

$$[\phi(\vec{x},t), \phi(\vec{x},t)] = 0 \; . \tag{4.55}$$

We will use this to show that from this follows the Bose symmetry for particles, which means

$$\begin{aligned} [a^\dagger(\vec{k}_1), a^\dagger(\vec{k}_2)] &= 0 & (4.56)\\ \left[a(\vec{k}_1), a(\vec{k}_2)\right] &= 0 \; . & (4.57)\end{aligned}$$

We can write the field and its derivative w.r.t. the time as

$$\begin{aligned} \phi(\vec{x},t) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \exp(-i\vec{k}\vec{x}) \left(\exp(i\omega t) a^\dagger(\vec{k}) + \exp(-i\omega t) a(-\vec{k})\right) & (4.58)\\ \dot{\phi}(\vec{x},t) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \exp(-i\vec{k}\vec{x}) \left(i\omega \exp(i\omega t) a^\dagger(\vec{k}) - i\omega \exp(-i\omega t) a(-\vec{k})\right) \; . & (4.59)\end{aligned}$$

The corresponding Fourier transformation is given by

$$\begin{aligned} \exp(i\omega t) a^\dagger(\vec{k}) + \exp(-i\omega t) a(-\vec{k}) &= 2\omega \int d^3x \; \exp(i\vec{k}\vec{x}) \, \phi(\vec{x},t) & (4.60)\\ \exp(i\omega t) a^\dagger(\vec{k}) - \exp(-i\omega t) a(-\vec{k}) &= -2i \int d^3x \; \exp(i\vec{k}\vec{x}) \, \dot{\phi}(\vec{x},t) \; . & (4.61)\end{aligned}$$

Since $[\phi(\vec{x},t), \phi(\vec{y},t)] = 0 \; \forall \vec{x}, \vec{y}$, we get

$$\begin{aligned} 0 &= [(4.60)(\vec{k}_1), (4.60)(\vec{k}_2)]\\ &= \exp(+i(\omega_1+\omega_2)t)[a^\dagger(\vec{k}_1), a^\dagger(\vec{k}_2)]\\ &+ \exp(-i(\omega_1+\omega_2)t)[a(-\vec{k}_1), a(-\vec{k}_2)]\\ &+ \exp(+i(\omega_1-\omega_2)t)[a^\dagger(\vec{k}_1), a(-\vec{k}_2)]\\ &+ \exp(-i(\omega_1-\omega_2)t)[a(-\vec{k}_1), a^\dagger(\vec{k}_2)] \; . & (4.62)\end{aligned}$$

The time dependence of the first two terms is not compensated by any other term. The last two terms, however, have the same time dependence for $\omega_1 = \omega_2$ and cancel each other (see below). Thereby we have

$$[a^\dagger(\vec{k}_1), a^\dagger(\vec{k}_2)] = 0 \qquad \text{and} \qquad [a(\vec{k}_1), a(\vec{k}_2)] = 0 \; . \tag{4.63}$$

These commutation relations contain the Bose symmetry of the particles, since

$$|\vec{k}_1, \vec{k}_2\rangle \equiv a^\dagger(\vec{k}_1) a^\dagger(\vec{k}_2)|0\rangle = a^\dagger(\vec{k}_2) a^\dagger(\vec{k}_1)|0\rangle = |\vec{k}_2, \vec{k}_1\rangle \; . \tag{4.64}$$

This means that two-particle-states are symmetric w.r.t. an exchange of $\vec{k}_1$ and $\vec{k}_2$. We here have the first spin-statistics theorem, namely that particles with unit spin follow the Bose statistics (and particles with half-unit spin, as we will see below, follow the Fermi statistics).

We look at the commutator $[a, a^\dagger]$. Using Eqs. (4.60) and (4.61) we can calculate $a^\dagger(\vec{k})$ and $a(\vec{k})$:

$$a^\dagger(\vec{k}) = \exp(-i\omega t)\int d^3x \exp(+i\vec{k}\cdot\vec{x})\left(\omega\phi(\vec{x},t) - i\dot{\phi}(\vec{x},t)\right) \quad (4.65)$$

$$a(\vec{k}) = \exp(+i\omega t)\int d^3x \exp(-i\vec{k}\cdot\vec{x})\left(\omega\phi(\vec{x},t) + i\dot{\phi}(\vec{x},t)\right) . \quad (4.66)$$

Using (4.53), (4.54), (4.55) and

$$[\phi(\vec{x}), \phi(\vec{x})] = [\dot{\phi}(\vec{x}), \dot{\phi}(\vec{x})] = 0 \quad (4.67)$$

leads to

$$\begin{aligned}[a(\vec{k}_1), a^\dagger(\vec{k}_2)] &= \exp(i(\omega_1-\omega_2)t)\int d^3x\, d^3y\, \exp(-i\vec{k}_1\cdot\vec{x} + i\vec{k}_2\cdot\vec{y}) \\ &\quad \left\{i\omega_2[\dot{\phi}(\vec{x},t), \phi(\vec{y},t)] - i\omega_1[\phi(\vec{x},t), \dot{\phi}(\vec{y},t)]\right\} . \end{aligned} \quad (4.68)$$

Since a and ϕ shall not be simply complex numbers, this expression must not be identically zero. The integrand can be non-zero only for $\vec{x} = \vec{y}$, see Eqs. (4.53)-(4.55). We therefore require as ansatz for the canonical commutation relation

$$[\phi(\vec{x},t), \dot{\phi}(\vec{y},t)] = i\delta(\vec{x}-\vec{y}) . \quad (4.69)$$

Thereby follows for the commutation relation of a and $a^\dagger$

$$[a(\vec{k}_1), a^\dagger(\vec{k}_2)] = (2\pi)^3 2\omega_1 \delta(\vec{k}_1 - \vec{k}_2) . \quad (4.70)$$

We can justify the ansatz, by understanding ϕ as canonical coordinate and $\dot{\phi}$ as canonical momentum and x as index.

We can summarize that ϕ and $\dot{\phi}$ fulfill the following commutation relations:

$$[\phi(\vec{x},t), \phi(\vec{y},t)] = 0 \quad (4.71)$$

$$\left[\phi(\vec{x},t), \dot{\phi}(\vec{y},t)\right] = i\delta(\vec{x}-\vec{y}) . \quad (4.72)$$

And thereby the commutators of a and $a^\dagger$ are

$$[a^\dagger, a^\dagger] = 0 \quad (4.73)$$

$$[a, a] = 0 \quad (4.74)$$

$$\left[a(\vec{k}_1), a^\dagger(\vec{k}_2)\right)] = (2\pi)^3 2\omega_1\delta(\vec{k}_1 - \vec{k}_2) . \quad (4.75)$$

Note that there are also definitions of a in the literature which differ by a factor of $\sqrt{(2\pi)^3 2\omega}$. The one-particle states constructed through $a^\dagger$,

$$a^\dagger(\vec{k})|0\rangle \equiv |\vec{k}\rangle \quad (4.76)$$

are normalised as

$$\langle \vec{k}|\vec{k}'\rangle = (2\pi)^3 2\omega\delta(\vec{k}-\vec{k}') \ . \tag{4.77}$$

They are no proper states. It would be mathematically better to introduce states using wave packages, hence

$$|a^\dagger[f]\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega}} f(\vec{k})\, a^\dagger(\vec{k})|0\rangle \ . \tag{4.78}$$

n-particle states are introduced through

$$Na^\dagger(\vec{k}_1)...a^\dagger(\vec{k}_n)|0\rangle \tag{4.79}$$

respectively,

$$Na^\dagger[f_1]...a^\dagger[f_n]|0\rangle \ . \tag{4.80}$$

Here the f_i are orthonormalised by requiring

$$\int \frac{d^3k}{(2\pi)^3} f_i^*(\vec{k})\, f_j(\vec{k}) = \delta_{ij} \ . \tag{4.81}$$

Here N is a normalisation constant. It has the value $N=1$ if all f_i are different, and $N=(n!)^{-\frac{1}{2}}$ if all f_i are equal. If r_1 of the f_i and r_2 of the f_i are equal, we have

$$N = (r_1!\, r_2!...)^{-\frac{1}{2}} \ . \tag{4.82}$$

Interpretation: The Heisenberg states with one particle and the momentum wave function $f(\vec{k})$ are

$$|f\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega}} f(\vec{k})a^\dagger(\vec{k})|0\rangle \ . \tag{4.83}$$

And the Schrödinger states are

$$|f,t\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega}} \exp(-i\omega t) f(\vec{k})a^\dagger(\vec{k})|0\rangle \tag{4.84}$$

From

$$\langle f,t|f,t\rangle = 1 \tag{4.85}$$

follows

$$\int |f(\vec{k})|^2 \frac{d^3k}{(2\pi)^3} = 1 \tag{4.86}$$

and vice versa.
Remark: We cannot interpret

$$\int \frac{d^3k}{(2\pi)^3} f(\vec{k}) \exp(i\vec{k}\cdot\vec{x} - i\omega t) \tag{4.87}$$

as space wave function $\phi(x)$ of a particle, because $\langle 0|\phi(x)\phi(y)|0\rangle \neq 0$ also for $(x-y)^2 < 0$, which would have to be interpreted as expansion with a velocity larger than the speed of light and which would hence contradict causality.

Construction of operators: Operators like energy, momentum, particle number operators or other operators are constructed from the quantum fields in the following way: The Noether currents are used, the 0-component is integrated, and the *normal order* is performed. This means

$$\text{Normal order:} \quad : a^\dagger a : \equiv : a a^\dagger : \equiv \; a^\dagger a \; . \tag{4.88}$$

It will become clear later, why we perform this normal ordering.

Particle number operator: The particle number operator is given by

$$N \equiv \int d\tilde{k}\, a^\dagger a \; , \tag{4.89}$$

where

$$d\tilde{k} = \frac{d^4k}{(2\pi)^4} 2\pi\, \delta(k^2 - m^2)\theta(k_0) = \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k} \tag{4.90}$$

is a Lorentz-invariant integration measure. Thereby we obtain by using the commutator relations and the definition of the state $|\vec{k}_1, ..., \vec{k}_n\rangle$,

$$N|\vec{k}_1, ..., \vec{k}_n\rangle = n|\vec{k}_1, ..., \vec{k}_n\rangle \; . \tag{4.91}$$

Energy and momentum operators: Starting from the energy-momentum tensor T^μ_ν we obtain the conserved charge, the four-momentum, as (see Eq. (3.67))

$$P_\mu = \int d^3x\, T^0_{\ \mu} = \int d^3x\, \partial^0\phi\partial_\mu\phi - g^0_\mu \mathcal{L} \; . \tag{4.92}$$

The energy operator is obtained from

$$H = \int d^3x\, T^0_{\ 0} = \int d^3x\, \partial^0\phi\partial_0\phi - \mathcal{L} = \frac{1}{2}\int d^3x \left(: \partial^0\phi\partial_0\phi : + : \vec{\nabla}\phi\vec{\nabla}\phi : + m^2 : \phi^2 : \right) \tag{4.93}$$

where the double-dots symbolise the normal order. Usage of

$$\begin{aligned}
\phi &= \int d\tilde{k} \left(\exp(i\omega t)\exp(-i\vec{k}\cdot\vec{x})a^\dagger(\vec{k}) + \exp(-i\omega t)\exp(i\vec{k}\cdot\vec{x})a(\vec{k}) \right) && (4.94)\\
\partial_0\phi &= \int d\tilde{k} \left(i\omega\exp(i\omega t)\exp(-i\vec{k}\cdot\vec{x})a^\dagger(\vec{k}) - i\omega\exp(-i\omega t)\exp(i\vec{k}\cdot\vec{x})a(\vec{k}) \right) && (4.95)\\
\vec{\nabla}\phi &= \int d\tilde{k} \left(-i\vec{k}\exp(i\omega t)\exp(-i\vec{k}\cdot\vec{x})a^\dagger(\vec{k}) + i\vec{k}\exp(-i\omega t)\exp(i\vec{k}\cdot\vec{x})a(\vec{k}) \right) . && (4.96)
\end{aligned}$$

leads to

$$\begin{aligned}
m^2 \int d^3x |\phi(\vec{x},t)|^2|_{t=0} &= (2\pi)^3 m^2 \int d\tilde{k} \int d\tilde{k}' \{\delta(\vec{k} - \vec{k}')[a^\dagger(\vec{k})a(\vec{k}') + a(\vec{k})a^\dagger(\vec{k}')] \\
&\quad + \delta(\vec{k} + \vec{k}')[a^\dagger(\vec{k})a^\dagger(\vec{k}') + a(\vec{k})a(\vec{k}')]\} && (4.97)
\end{aligned}$$

and

$$\int d^3x|\partial_0\phi(\vec{x},t)|^2|_{t=0} = (2\pi)^3\omega^2\int d\tilde{k}\int d\tilde{k}'\{\delta(\vec{k}-\vec{k}')[a^\dagger(\vec{k})a(\vec{k}')+a(\vec{k})a^\dagger(\vec{k}')] \\ -\delta(\vec{k}+\vec{k}')[a^\dagger(\vec{k})a^\dagger(\vec{k}')+a(\vec{k})a(\vec{k}')] \tag{4.98}$$

as well as

$$\int d^3x|\vec{\nabla}\phi(\vec{x},t)|^2|_{t=0} = (2\pi)^3\int d\tilde{k}d\tilde{k}'\{\delta(\vec{k}-\vec{k}')[\vec{k}\cdot\vec{k}'\,a^\dagger(\vec{k})a(\vec{k}')+\vec{k}'\cdot\vec{k}\,a(\vec{k})a^\dagger(\vec{k}')] \\ -\delta(\vec{k}+\vec{k}')[\vec{k}\cdot\vec{k}'\,a^\dagger(\vec{k})a^\dagger(\vec{k}')+\vec{k}'\cdot\vec{k}\,a(\vec{k})a(\vec{k}')]\ . \tag{4.99}$$

And thereby we have

$$\begin{aligned} H &= \frac{1}{2}\int d\tilde{k}\int d\tilde{k}'(2\pi)^3\{\overbrace{[m^2+\omega^2+\vec{k}^2]}^{=2\omega^2}(a^\dagger a+aa^\dagger)\delta(\vec{k}-\vec{k}')+ \\ &\quad +\underbrace{[m^2-\omega^2+\vec{k}^2]}_{=0}(a^\dagger a^\dagger+aa)\delta(\vec{k}+\vec{k}')\} \\ &= \int d\tilde{k}\int d\tilde{k}'(2\pi)^3\delta(\vec{k}-\vec{k}')\omega^2\left(a^\dagger(\vec{k})a(\vec{k}')+a(\vec{k})a^\dagger(\vec{k}')\right) \\ &= \frac{1}{2}\int d\tilde{k}\,\omega\left(a^\dagger(\vec{k})a(\vec{k})+a(\vec{k})a^\dagger(\vec{k})\right) \\ &= \int d\tilde{k}\,\omega\,a^\dagger(\vec{k})a(\vec{k})+\ \text{const}\ . \end{aligned} \tag{4.100}$$

Here the constant corresponds to the vacuum energy, which is irrelevant for physical processes. This infinitely large constant is subtracted, so that the vacuum has the energy zero. The trick that we use here, is the normal order, which means (see above) that all creation operators have to be left of the annihilation operators. Thereby we have

$$:H:=\int d\tilde{k}\frac{\omega}{2}\ :(a^\dagger a+aa^\dagger):=\int d\tilde{k}\,\omega a^\dagger a\ . \tag{4.101}$$

Thereby we have in the vacuum: $\langle 0|H|0\rangle = 0$. Analoguously it is found starting from

$$\vec{P}=\int d^3x\,T^0_j=\int d^3x\,\partial_0\phi\vec{\nabla}\phi \tag{4.102}$$

for

$$\vec{P}=\int d\tilde{k}\,\vec{k}a^\dagger(\vec{k})a(\vec{k})\ . \tag{4.103}$$

4.3.3 The commutator $[\phi(x),\phi(y)]$

We compute the commutator $[\phi(x),\phi(y)]$, by inserting the Fourier decomposition and using the commutator relations of the creation and the annihilation operators:

$$[\phi(x),\phi(y)] = \int d\tilde{k}d\tilde{k}'\left\{[a(\vec{k}),a^\dagger(\vec{k}')]e^{-ikx+ik'y}+[a^\dagger(\vec{k}),a(\vec{k}')]e^{ikx-ik'y}\right\}$$

$$
\begin{aligned}
&= \int \underbrace{d\tilde{k}}_{\frac{d^4k}{(2\pi)^3}\delta(k^2-m^2)\theta(k^0)} \left\{e^{-ik(x-y)} - e^{ik(x-y)}\right\} \\
&= \frac{1}{(2\pi)^3}\int d^4k\,\epsilon(k^0)\delta(k^2-m^2)e^{-ik(x-y)} \\
&\equiv i\Delta(x-y)\ , \qquad (4.104)
\end{aligned}
$$

where

$$\epsilon(k^0) \equiv \theta(k^0) - \theta(-k^0)\ . \qquad (4.105)$$

The Pauli-Jordan distribution has the following properties

1) $(\Box_x + m^2)\Delta(x-y) = (\Box_y + m^2)\Delta(x-y) = 0$ (4.106)
(mass shell condition $k^2 = m^2$)

2) $\Delta(x-y) = -\Delta(y-x)$ (4.107)

3) $\Delta(x-y)|_{x^0=y^0} = 0$ (4.108)

4) $\Delta(x-y) = 0\ ,\ \text{ if } (x-y)^2 = (x_0-y_0)^2 - (\vec{x}-\vec{y})^2 < 0$ (space-like) (4.109)

5) $\dfrac{\partial}{\partial x_0}\Delta(x-y)|_{x_0=y_0} = -\delta(\vec{x}-\vec{y})\ .$ (4.110)

From 4) follows the micro-causality, i.e. $[\phi(x),\phi(y)] = 0$ for $(x-y)^2 < 0$.

4.4 Charged Scalar Field

The field $\phi = \phi^\dagger$ describes self-conjugate particles, i.e. particles which are equal to their antiparticles. Examples for such particles are the uncharged pion π^0 or the Higgs boson. However, there are also spin-0 particles, which are not equal to their antiparticle, like e.g. the charged pions π^+, π^- or the Kaon $K^0, \bar{K}^0$. These particles cannot be described by a hermitian field. We therefore look at a doublet of two hermitian fields ϕ_1, ϕ_2 with $\phi_i^\dagger = \phi_i$ $(i = 1,2)$. The field

$$\phi(x) = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \qquad (4.111)$$

is then not hermitian. The Lagrangian density for a free field ϕ reads

$$
\begin{aligned}
\mathcal{L}(\phi) &= \mathcal{L}(\phi_1) + \mathcal{L}(\phi_2) \\
&= \sum_{i=1}^{2}\frac{1}{2}(\partial_\mu\phi_i\partial^\mu\phi_i - m^2\phi_i\phi_i) \\
&= (\partial_\mu\phi^\dagger)(\partial^\mu\phi) - m^2\phi^\dagger\phi \qquad \text{(check!)}\ . \qquad (4.112)
\end{aligned}
$$

The equations of motion are obtained from the Euler-Lagrange equation

$$
\begin{aligned}
\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} - \frac{\partial\mathcal{L}}{\partial\phi} &= 0 \quad\Rightarrow\quad (\Box + m^2)\phi^\dagger = 0 \qquad (4.113)\\
\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)} - \frac{\partial\mathcal{L}}{\partial\phi^\dagger} &= 0 \quad\Rightarrow\quad (\Box + m^2)\phi = 0\ . \qquad (4.114)
\end{aligned}
$$

The canonical conjugate momentum of ϕ is

$$\Pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi^\dagger = \dot{\phi}^\dagger \ , \tag{4.115}$$

and the canonical conjugate momentum of $\phi^\dagger$ is

$$\Pi^\dagger = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi^\dagger)} = \partial_0 \phi = \dot{\phi} \ . \tag{4.116}$$

The Hamiltonian density operator thereby is

$$\mathcal{H} = \Pi\dot{\phi} + \Pi^\dagger \dot{\phi}^\dagger - \mathcal{L} = \partial_0\phi\partial_0\phi^\dagger + (\vec{\nabla}\phi^\dagger)(\vec{\nabla}\phi) + m^2\phi^\dagger\phi \ . \tag{4.117}$$

The Hamiltonian density operator, respectively the Hamiltonian operator has to be normal ordered, so that $\langle 0|H|0\rangle = 0$. Hence

$$H = \int d^3x \ :\left\{\partial_0\phi\partial_0\phi^\dagger + (\vec{\nabla}\phi^\dagger)(\vec{\nabla}\phi) + m^2\phi^\dagger\phi\right\}: \ . \tag{4.118}$$

The hermitian fields fulfill

$$[\Pi_i(t,\vec{x}), \Pi_j(t,\vec{y})] = [\phi_i(t,\vec{x}), \phi_j(t,\vec{y})] = 0 \tag{4.119}$$

$$[\phi_i(t,\vec{x}), \Pi_j(t,\vec{y})] = i\delta_{ij}\delta(\vec{x}-\vec{y}) \qquad i,j = 1,2 \ . \tag{4.120}$$

Or, in general (having the same mass parameter)

$$[\phi_i(x), \phi_j(y)] = i\delta_{ij}\Delta(x-y) \ . \tag{4.121}$$

And for the non-hermitian field ϕ we obtain

$$[\phi(x), \phi^\dagger(y)] = i\Delta(x-y) \qquad \text{(check!)} \ . \tag{4.122}$$

Differentiation with respect to x_0 leads to

$$(\partial_{x_0}\phi(x))\,\phi^\dagger(y) - \phi^\dagger(y)\,(\partial_{x_0}\phi(x)) = i\partial_{x_0}\Delta(x-y) \ . \tag{4.123}$$

We set $x_0 = y_0$ and obtain with Eq. (4.110)

$$[\Pi^\dagger(x_0,\vec{x}), \phi^\dagger(x_0,\vec{y})] = -i\delta(\vec{x}-\vec{y}) \ . \tag{4.124}$$

And by hermitian adjugation,

$$[\phi(x_0,\vec{x}), \Pi(x_0,\vec{y})] = i\delta(\vec{x}-\vec{y}) \quad \Rightarrow \quad [\Pi(x_0,\vec{x}), \phi(x_0,\vec{y})] = -i\delta(\vec{x}-\vec{y}) \ . \tag{4.125}$$

The Fourier decomposition of the field ϕ, which fulfills the Klein Gordon equation,

$$(\Box + m^2)\phi(x) = 0 \ , \tag{4.126}$$

is given by

$$\phi(x) = \int d\tilde{k}\,[a(\vec{k})e^{-ikx} + \underbrace{b^\dagger(\vec{k})}_{\neq a^\dagger,\,\text{since}\ \phi\neq\phi^\dagger} e^{ikx}] \ . \tag{4.127}$$

And that of $\phi^\dagger$ by

$$\phi^\dagger(x) = \int d\tilde{k}\,[b(\vec{k})e^{-ikx} + a^\dagger(\vec{k})e^{ikx}]\ . \tag{4.128}$$

Insertion of

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \tag{4.129}$$

and expressing ϕ_1, ϕ_2 through their respective Fourier decomposition then leads to

$$a(\vec{k}) \;=\; \frac{1}{\sqrt{2}}(a_1(\vec{k}) + ia_2(\vec{k})) \tag{4.130}$$

$$b^\dagger(\vec{k}) \;=\; \frac{1}{\sqrt{2}}(a_1^\dagger(\vec{k}) + ia_2^\dagger(\vec{k})) \tag{4.131}$$

and with

$$[a_i(\vec{k}), a_j^\dagger(\vec{k}')] = (2\pi)^3 2\omega\delta_{ij}\delta(\vec{k} - \vec{k}')\ ,\ [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 \tag{4.132}$$

finally to

$$[a(\vec{k}), a^\dagger(\vec{k}')] \;=\; [b(\vec{k}), b^\dagger(\vec{k}')] = (2\pi)^3 2\omega\delta(\vec{k} - \vec{k}') \tag{4.133}$$

$$\text{all other commutators} \;=\; 0\ . \tag{4.134}$$

The normal ordered 4-momentum operator P^μ is obtained by (check!)

$$P_\mu = \int d^3x\ :T^0_{\ \mu}: \;=\; \int d^3x\ :\left\{\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}\partial_\mu\phi + \frac{\partial\mathcal{L}}{\partial(\partial_0\phi^\dagger)}\partial_\mu\phi^\dagger - g^0_\mu\mathcal{L}\right\}:$$

$$\;=\; \int d\tilde{k}\, k_\mu[a^\dagger(\vec{k})a(\vec{k}) + b^\dagger(\vec{k})b(\vec{k})]\ . \tag{4.135}$$

And we have the commutator relations (check!)

$$[P_\mu, a^\dagger(\vec{k})] = k_\mu a^\dagger(\vec{k})\ , \quad [P_\mu, a(\vec{k})] = -k_\mu a(\vec{k})\ , \tag{4.136}$$

$$\left[P_\mu, b^\dagger(\vec{k})\right] = k_\mu b^\dagger(\vec{k})\ , \quad [P_\mu, b(\vec{k})] = -k_\mu b(\vec{k})\ . \tag{4.137}$$

The Lagrangian density

$$\mathcal{L} = \partial_\mu\phi^\dagger(x)\partial^\mu\phi(x) - m^2\phi^\dagger(x)\phi(x) \tag{4.138}$$

is invariant under a $U(1)$ symmetry, i.e. under the phase transformation

$$\phi(x) \;\rightarrow\; \phi'(x) = e^{i\lambda}\phi(x) \tag{4.139}$$

$$\phi^\dagger(x) \;\rightarrow\; \phi^{\dagger'}(x) = \phi^\dagger e^{-i\lambda}\ . \tag{4.140}$$

The Noether current is obtained using

$$x^\mu \;\rightarrow\; x'^{\mu} = x^\mu + \delta x^\mu \qquad \delta x^\mu = A^\mu\lambda = 0 \tag{4.141}$$

$$\phi'(x) \;=\; \phi(x) + i\lambda\phi(x) \qquad \delta\phi = i\phi \tag{4.142}$$

$$\phi^{\dagger'}(x) \;=\; \phi^\dagger(x) - i\lambda\phi^\dagger(x) \qquad \delta\phi^\dagger = -i\phi^\dagger \tag{4.143}$$

$$\tag{4.144}$$

(compare with Eqs. (3.87)-(3.90), where the index k is omitted, as it is a one-parametric transformation) as

$$\begin{aligned} j^\mu &= -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} i\phi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)}(-i\phi^\dagger) \\ &= -i(\partial^\mu \phi^\dagger)\phi + i(\partial^\mu \phi)\phi^\dagger \ . \end{aligned} \tag{4.145}$$

The current is normal ordered, so that for the charge density j^0 it is guaranteed that

$$j^0(x)|0\rangle = 0 \ , \tag{4.146}$$

hence

$$j^\mu(x)_{\text{normal ordered}} = \ : -i(\partial^\mu \phi^\dagger)\phi + i(\partial^\mu \phi)\phi^\dagger : \tag{4.147}$$

$$\equiv i \ : \phi^\dagger \overset{\leftrightarrow\mu}{\partial} \phi(x) : \ . \tag{4.148}$$

The charge is given by (check!)

$$Q = \int d^3x j^0(x)_{\text{normal ordered}} = i\int d^3x \ : \phi^\dagger(x) \overset{\leftrightarrow}{\partial_0} \phi(x) : = \int d\tilde{k}[a^\dagger(\vec{k})a(\vec{k}) - b^\dagger(\vec{k})b(\vec{k})] \ . \tag{4.149}$$

It is (check!)

$$\dot{Q} = i[H, Q] = 0 \tag{4.150}$$

and thereby Q is a conserved quantity. The interpretation of the operators $a, a^\dagger, b, b^\dagger$ is (analoguously to the hermitian case)

$a^\dagger$	generates a particle of type a with spin 0 and mass m	(4.151)
$b^\dagger$	generates a particle of type b with spin 0 and mass m	(4.152)
a	destroys a particle of type a with spin 0 and mass m	(4.153)
b	destroys a particle of type b with spin 0 and mass m .	(4.154)

This means that the field

ϕ	destroys a quantum of type a, generates a quantum of type b	(4.155)
$\phi^\dagger$	destroys a quantum of type b, generates a quantum of type a .	(4.156)

We look at the state space (Fock space). The ground state is given by $|0\rangle$. We demand that

$$a(\vec{k})|0\rangle = b(\vec{k})|0\rangle = 0 \ , \tag{4.157}$$

so that

$$P_\mu|0\rangle = 0 \ , \qquad Q|0\rangle = 0 \ . \tag{4.158}$$

The 1-particle states with sharp 4-momentum k_μ are given by

$$|a(\vec{k})\rangle := a^\dagger(\vec{k})|0\rangle \tag{4.159}$$

$$|b(\vec{k})\rangle := b^\dagger(\vec{k})|0\rangle \ . \tag{4.160}$$

The charge of these states is (check!)

$$Q|a(\vec{k})\rangle = Qa^\dagger(\vec{k})|0\rangle = +|a(\vec{k})\rangle \tag{4.161}$$
$$Q|b(\vec{k})\rangle = Qb^\dagger(\vec{k})|0\rangle = -|b(\vec{k})\rangle \; . \tag{4.162}$$

This means that

$$|a(\vec{k})\rangle \qquad \text{is a 1-particle state of mass } m, \text{ spin 0 and charge } + \tag{4.163}$$
$$|b(\vec{k})\rangle \qquad \text{is a 1-particle state of mass } m, \text{ spin 0 and charge } - \; . \tag{4.164}$$

This is a particle and its antiparticle.

4.5 Quantisation of Spinor Fields (Dirac Fields)

The free Lagrangian density without interaction is given by

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \; , \qquad \text{where } \bar{\psi} = \psi^\dagger\gamma^0 \; . \tag{4.165}$$

Reminder:

$$(\gamma^0)^2 = 1 \; . \tag{4.166}$$

The canonical conjugated momentum in component form is given by ($\alpha = 1, ..., 4$)

$$\pi^\alpha(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi_\alpha)} = i\psi^\dagger_\alpha = i\psi^*_\alpha \; . \tag{4.167}$$

The solution of the Dirac equation before quantisation is given by an expansion in plane waves,

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega} \sum_{s=\pm\frac{1}{2}} \left[\exp(ikx)\beta^*_s(\vec{k})\, v_s(\vec{k}) + \exp(-ikx)\, \alpha_s(\vec{k})\, u_s(\vec{k})\right] \tag{4.168}$$
$$\bar{\psi}(x) = \int \frac{d^3k}{(2\pi)^3 2\omega} \sum_{s=\pm\frac{1}{2}} \left[\exp(-ikx)\beta_s(\vec{k})\, \bar{v}_s(\vec{k}) + \exp(ikx)\, \alpha^*_s(\vec{k})\, \bar{u}_s(\vec{k})\right] \; . \tag{4.169}$$

The fields fulfil the Dirac equation

$$(i\partial\!\!\!/ - m)\psi = 0 \tag{4.170}$$
$$\bar{\psi}(i\overleftarrow{\partial\!\!\!/} + m) = 0 \; . \tag{4.171}$$

From (4.170) follows

$$(k\!\!\!/ + m)v_s = 0 \qquad \text{Lösung zu negativer Frequenz} \tag{4.172}$$
$$(k\!\!\!/ - m)u_s = 0 \qquad \text{Lösung zu positiver Frequenz} \; . \tag{4.173}$$

4.5.1 Quantisation

For the quantisation the α_s, β_s^* are replaced by operators,

$$\alpha_s \to a_s \qquad \text{und } \beta_s^* \to b_s^\dagger \ . \tag{4.174}$$

The solution of the Dirac equation in quantised form is then given by

$$\psi(x) = \int d\tilde{k} \sum_{s=\pm\frac{1}{2}} \left[\exp(ikx) b_s^\dagger(\vec{k})\, v_s(\vec{k}) + \exp(-ikx)\, a_s(\vec{k})\, u_s(\vec{k})\right] \tag{4.175}$$

$$\bar{\psi}(x) = \int d\tilde{k} \sum_{s=\pm\frac{1}{2}} \left[\exp(-ikx) b_s(\vec{k})\, \bar{v}_s(\vec{k}) + \exp(ikx)\, a_s^\dagger(\vec{k})\, \bar{u}_s(\vec{k})\right] \ . \tag{4.176}$$

Again the Heisenberg equation Es soll wiederum die Heisenberg Gleichung

$$\partial_\mu \psi = i[P_\mu, \psi] \tag{4.177}$$

shall be fulfilled. From this follows

$$[P_\mu, a_s^\dagger(\vec{k})] = k_\mu a_s^\dagger(\vec{k}) \tag{4.178}$$

$$\left[P_\mu, b_s^\dagger(\vec{k})\right] = k_\mu b_s^\dagger(\vec{k}) \tag{4.179}$$

$$\left[P_\mu, a_s(\vec{k})\right] = -k_\mu a_s(\vec{k}) \tag{4.180}$$

$$\left[P_\mu, b_s(\vec{k})\right] = -k_\mu b_s(\vec{k}) \tag{4.181}$$

Thereby follows, like for the scalar field,

$$a|0\rangle = b|0\rangle = 0 \ , \tag{4.182}$$

as for all states the energy has to be positive. Thereby ψ generates an antiparticle (e.g. the positron e^+) and destroys a particle (e.g. the electron e^-).

4.5.2 Operator Algebra

The translation invariance of the action of the Dirac Lagrangian density leads to the energy-momentum tensor which is given by

$$T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi - g^{\mu\nu}[\bar{\psi} \underbrace{(i\partial\!\!\!/ - m)\psi}_{=0 \text{ due to Dirac equation}}] \ . \tag{4.183}$$

This means that

$$T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi \ . \tag{4.184}$$

The momenum operator is given by

$$P^\nu = \int d^3x\, T^{0\nu} = i\int d^3x\, \psi^\dagger\partial^\nu\psi \ . \tag{4.185}$$

It still has to be normal ordered. Inserting the Fourier expansion leads before normal ordering for the Hamilton operator to

$$\begin{aligned} H &= \int d^3x \psi^\dagger(x)\, i\frac{\partial}{\partial t}\psi(x) = \int d^3x \sum_{\alpha=1}^{4} \psi^\dagger_\alpha\, i\frac{\partial}{\partial t}\psi_\alpha \\ &= \int d\tilde{k}\, k_0 \sum_{s=\pm\frac{1}{2}} [a^\dagger_s(\vec{k})\, a_s(\vec{k}) - b_s(\vec{k})\, b^\dagger_s(\vec{k})] \;. \end{aligned} \tag{4.186}$$

Here, we have used that

$$u^\dagger(k,s)\, u(k,s') = 2k_0\delta_{ss'} \tag{4.187}$$
$$v^\dagger(k,s)\, v(k,s') = 2k_0\delta_{ss'} \tag{4.188}$$
$$u^\dagger(k,s)\, v(\bar{k},s') = 0 \tag{4.189}$$
$$u(k,s)\, v^\dagger(\bar{k},s') = 0 \;, \qquad \text{with } \bar{k}^\mu = (k_0, -\vec{k})^T \;. \tag{4.190}$$

For the creation and annihilation operator we have to deman anti-commutation relations, so that particles and antiparticles have opposite energies. We hence demand that

$$\{a_r(\vec{k}), a^\dagger_s(\vec{k}')\} = \delta_{rs}(2\pi)^3\, 2\omega_k\, \delta(\vec{k}-\vec{k}') \tag{4.191}$$
$$\{b_r(\vec{k}), b^\dagger_s(\vec{k}')\} = \delta_{rs}(2\pi)^3\, 2\omega_k\, \delta(\vec{k}-\vec{k}') \tag{4.192}$$
$$\{a,b\} = \{a,a\} = \{b,b\} = ... = 0 \;. \tag{4.193}$$

Thereby we obtain after normal ordering for the Hamilton operator

$$H = \int d\tilde{k}\, k_0 \sum_{s=\pm\frac{1}{2}} [a^\dagger_s(\vec{k})\, a_s(\vec{k}) + b^\dagger_s(\vec{k})\, b_s(\vec{k})] \;. \tag{4.194}$$

And for the momentum

$$\vec{P} = \int d\tilde{k}\, \vec{k} \sum_{s=\pm\frac{1}{2}} [a^\dagger_s(\vec{k})\, a_s(\vec{k}) + b^\dagger_s(\vec{k})\, b_s(\vec{k})] \tag{4.195}$$

For the charge operator (see Eq. (3.97)) one finds

$$Q = \int d^3x \;: j_0(x) : = \int d^3x \;: \psi^\dagger(x)\psi(x) : = \int d\tilde{k} \sum_{s=\pm\frac{1}{2}} [a^\dagger_s(\vec{k})\, a_s(\vec{k}) - b^\dagger_s(\vec{k})\, b_s(\vec{k})] \tag{4.196}$$

Remarks:

1. The (infinitley large) negative constant is dropped because of the normal ordering.

2. From the anti-commutation relation of the creation and annihilation operators follow by using

$$\sum_{s=\pm\frac{1}{2}} u_\alpha(\vec{k},s)\bar{u}_\beta(\vec{k},s) = (\not{k}+m)_{\alpha\beta} \tag{4.197}$$
$$\sum_{s=\pm\frac{1}{2}} v_\alpha(\vec{k},s)\bar{v}_\beta(\vec{k},s) = (\not{k}-m)_{\alpha\beta} \tag{4.198}$$

the anti-commutation relations

$$\{\psi_\alpha(\vec{x},t),\psi_\beta(\vec{y},t)\} = \{\bar{\psi}_\alpha(\vec{x},t),\bar{\psi}_\beta(\vec{y},t)\} = 0 \tag{4.199}$$
$$\{\psi_\alpha(\vec{x},t),\bar{\psi}_\beta(\vec{y},t)\} = \gamma^0_{\alpha\beta}\delta(\vec{x}-\vec{y}) \; . \tag{4.200}$$

And thereby

$$\{\psi_\alpha(\vec{x},t),\psi^\dagger_\beta(\vec{y},t)\} = \delta_{\alpha\beta}\delta(\vec{x}-\vec{y}) \; . \tag{4.201}$$

Hence $\psi^\dagger$ is the canonical conjugated momentum of ψ.

3. Construction of the states: The one-particle state

$$a^\dagger_s|0\rangle = |\vec{k},s\rangle \tag{4.202}$$

is interpretated as electron with momentum $\vec{k}$ and spin s. The vacuum state is the state with $\vec{k}=\vec{0}$ and $s=0$. Two-particle states are constructed by

$$|\vec{k}_1,s_1;\vec{k}_2,s_2\rangle = a^\dagger_{s_1}(\vec{k}_1)a^\dagger_{s_2}(\vec{k}_2)|0\rangle = -a^\dagger_{s_2}(\vec{k}_2)a^\dagger_{s_1}(\vec{k}_1)|0\rangle = -|\vec{k}_2,s_2;\vec{k}_1,s_1\rangle \; . \tag{4.203}$$

The Pauli principle is hence fulfilled. The Pauli principle results from the second quantisation of the spinor field. The spin-1/2 particles hence obeye the Fermi statistics.

With the identification

$$|1\rangle \equiv a^\dagger_1(\vec{k})|0\rangle \; , \qquad |2\rangle \equiv a^\dagger_2(\vec{k})|0\rangle$$
$$|3\rangle \equiv b^\dagger_1(\vec{k})|0\rangle \; , \qquad |4\rangle \equiv b^\dagger_2(\vec{k})|0\rangle \tag{4.204}$$

hence

$$H|c\rangle = +k_0|c\rangle \; , \qquad c = 1,2,3,4 \tag{4.205}$$

is positive definite. And for the charge operator Q we have

$$Q|c\rangle = \left\{ \begin{array}{ll} +|c\rangle & \text{for } c=1,2 \\ -|c\rangle & \text{für } c=3,4 \end{array} \right. \; . \tag{4.206}$$

4.6 Feynman Propgagator for a Scalar Field

When we apply the field operator $\phi^\dagger(x)$ on an arbitrary Fock state, then

$$\left\{ \begin{array}{l} \text{it generates a particle with charge } +1 \\ \text{or it destroys a particle with charge } -1 \end{array} \right\} \rightsquigarrow \text{ i.e. it “adds” the charge } +1 \; . \tag{4.207}$$

Analogously, the field operator $\phi(y)$ takes of the charge $+1$. For $t' > t$ we look at

$$\theta(t'-t)\,\langle 0|\underbrace{\phi(t',\vec{x}')}_{\substack{\text{Destruction of charge}\\ +1 \text{ at later}\\ \text{time } t' \text{ and at } \vec{x}'}}\underbrace{\phi^\dagger(t,\vec{x})}_{\substack{\text{Generation of charge}\\ +1 \text{ at}\\ \text{time } t \text{ and at } \vec{x}}}|0\rangle\ . \tag{4.208}$$

Wenn $t > t'$ betrachte

$$\theta(t-t')\,\langle 0|\underbrace{\phi^\dagger(t,\vec{x})}_{\substack{\text{Vernichtung von Ladung}\\ -1 \text{ at the later}\\ \text{time } t \text{ and at } \vec{x}}}\underbrace{\phi(t',\vec{x}')}_{\substack{\text{Generation of charge}\\ -1 \text{ zur}\\ \text{the time } t' \text{ and at } \vec{x}'}}|0\rangle\ . \tag{4.209}$$

In both cases the charge is increase at $(t, \vec{x})$ and lowered at $(t', \vec{x}')$. The so-callded Feynman-Propagator $i\Delta_F(x-x')$ is the sum of the amplitudes (4.208) and (4.209). Hence

$$i\Delta_F(x-x') = \theta(t'-t)\langle 0|\phi(x')\phi^\dagger(x)|0\rangle + \theta(t-t')\langle 0|\ \phi^\dagger(x)\phi(x')|0\rangle\ . \tag{4.210}$$

By using the time-ordered product, which for Bose fields is defined by

$$T[A(x)B(y)] = A(x)B(y)\theta(x_0-y_0) + B(y)A(x)\theta(y_0-x_0)\ , \tag{4.211}$$

where A, B are Bose fields, the Feynman propagator can be written as

$$i\Delta_F(x-y) = \langle 0|T[\phi(x)\phi^\dagger(y)]|0\rangle\ . \tag{4.212}$$

We now want to determine the representation of Δ_F. For this we calculate

$$\langle 0|\phi(x)\phi^\dagger(y)|0\rangle = \int d\tilde{k}\, e^{-ik(x-y)} \tag{4.213}$$

$$\langle 0|\ \phi^\dagger(y)\phi(x)|0\rangle = \int d\tilde{k} e^{+ik(x-y)}\ . \tag{4.214}$$

This is obtained by inserting the Fourier decomposition of $\phi, \phi^\dagger$ and using the commutation relations (4.133), (4.134). Insertion of (4.213), (4.214) in (4.210) leads to

$$\begin{aligned}\Delta_F(x-y) &= \frac{1}{i}\int \frac{d^3k}{(2\pi)^3 2\omega}\left\{\theta(x_0-y_0)e^{-ik(x-y)} + \theta(y_0-x_0)e^{+ik(x-y)}\right\}\\ &= \lim_{\epsilon\to 0^+}\int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2-m^2+i\epsilon}\, e^{-ik(x-y)}\ .\end{aligned} \tag{4.215}$$

In order to demonstrate the last line, the integration $\int_{-\infty}^{+\infty} dk_0$ is performed. The denominator exhibits two roots,

$$k_0 = \pm\sqrt{\vec{k}^2+m^2-i\epsilon} \approx \pm\underbrace{\sqrt{\vec{k}^2+m^2}}_{\omega}\mp i\epsilon'\ , \tag{4.216}$$

where $\epsilon' = \epsilon/(2(\vec{k}^2+m^2))$. The $i\epsilon$-prescription corresponds to a deformation of the integration path. It is for

$$x_0 > y_0 : \quad e^{-ik_0(x_0-y_0)} \to 0 \quad \text{if } \mathrm{Im}k_0 \to -\infty\ . \tag{4.217}$$

This means that the integration path can be complemented by a large half circle in the lower half-plain. Thereby on finds that

$$\oint dk_0\, f(k_0) = \underbrace{\int_{\text{lower half-circle}} dk_0 f(k_0)}_{=0} + \int_{-\infty}^{+\infty} dk_0 f(k_0)\ . \tag{4.218}$$

Because of the residuum theorum on the other hand

$$\oint dk_0\, f(k_0) = (-1)\, 2\pi i\, f(k_0)\, (k_0-\omega)|_{k_0=\omega}\ . \tag{4.219}$$

The minus sign results from the fact that the curve is run through in the mathematically negative direction. Thereby we have

$$\begin{aligned} \int_{-\infty}^{+\infty} dk_0\, f(k_0) &= -2\pi i \int \frac{d^3k}{(2\pi)^4} \frac{1}{k^2-m^2+i\epsilon}\, (k_0-\omega)\, e^{-ik(x-y)}|_{k_0=\omega} \\ &= (-i) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega}\, e^{-i\omega(x_0-y_0)} e^{i\vec{k}(\vec{x}-\vec{y})}\ . \end{aligned} \tag{4.220}$$

It is for

$$y_0 > x_0 : \quad e^{-ik_0(x_0-y_0)} \to 0 \quad \text{if } \mathrm{Im}k_0 \to +\infty\ . \tag{4.221}$$

Thereby the integration path can be closed in the upper plain. It encloses the pole at $k_0 = -\sqrt{\vec{k}^2+m^2-i\epsilon}$, i.e.

$$\oint dk_0\, f(k_0) = \underbrace{\int_{\text{upper half-circle}} dk_0 f(k_0)}_{=0} + \int_{-\infty}^{+\infty} dk_0 f(k_0)\ . \tag{4.222}$$

And with the residuum theorem

$$\oint dk_0\, f(k_0) = 2\pi i\, f(k_0)\, (k_0+\omega)|_{k_0=-\omega}\ . \tag{4.223}$$

we obtain

$$\begin{aligned} \int_{-\infty}^{+\infty} dk_0\, f(k_0) &= 2\pi i \int \frac{d^3k}{(2\pi)^4} \frac{1}{k^2-m^2+i\epsilon}\, (k_0+\omega)\, e^{-ik(x-y)}|_{k_0=-\omega} \\ &= (-i) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega}\, e^{+i\omega(x_0-y_0)} e^{i\vec{k}(\vec{x}-\vec{y})} = -i \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega}\, e^{+ik(x-y)} \end{aligned} \tag{4.224}$$

where in the last step a variable transformation $\vec{k} \to -\vec{k}$ was performed. The sum of (4.220) and (4.224) results in (4.215).

Properties of $\Delta_F(x-y)$

1. It is

$$\Delta_F(x-y) = \Delta_F(y-x)\ . \tag{4.225}$$

   It is hence an even distribution.

2. Δ_F is the Greens function of the Klein-Gordon equation, as it is

$$(\Box_x + m^2)\Delta_F(x-y) = -\delta^{(4)}(x-y) \; . \tag{4.226}$$

The $i\epsilon$ prescription corresponds to a boundary condition, namely: Positive frequencies $+\omega$ move forward in time, negative frequencies$-\omega$ move backward in time. This is why Δ_F is also called causal Greensfunktion.

3. The hermitian spin-0 field $\phi = \phi^\dagger$ has the same Feynman propagator:

$$\langle 0|T[\phi(x), \phi(y)]|0\rangle = i\Delta_F(x-y) \; . \tag{4.227}$$

4. The distribution $\Delta_F(x-y)$ is Poincaré-invariant:

$$x' = \Lambda x + b \; , \qquad y' = \Lambda y + b \quad \Rightarrow \quad \Delta_F(x'-y') = \Delta_F(x-y) \; . \tag{4.228}$$

4.7 The Fermion Propagator

We have in analogy to the Klein-Gordon field (cf. Eq.. (4.104), where here the anti-commutator is used)

$$\{\psi_\alpha(x), \bar{\psi}_\beta(y)\} = (i\partial\!\!\!/ + m)_{\alpha\beta} i\Delta(x-y) \; , \tag{4.229}$$

because

$$\begin{aligned}
\{\psi_\alpha(x), \bar{\psi}_\beta(y)\} &= \int d\tilde{k} \int d\tilde{k}' \sum_{s,s'} \{\exp(ikx) v_{s_\alpha}(\vec{k}) b_s^\dagger(\vec{k}) + \exp(-ikx) u_{s_\alpha}(\vec{k}) a_s(\vec{k}), \\
&\quad \exp(-ik'y)\bar{v}_{s'_\beta}(\vec{k}') b_{s'}(\vec{k}') + \exp(ik'y)\bar{u}_{s'_\beta}(\vec{k}') a^\dagger_{s'}(\vec{k}')\} \\
&= \int d\tilde{k} \sum_s \Big(u_{s_\alpha}(\vec{k})\bar{u}_{s_\beta}(\vec{k}) \exp(-ik(x-y)) + v_{s_\alpha}(\vec{k})\bar{v}_{s_\beta}(\vec{k}) \exp(ik(x-y)) \Big) \\
&= \int d\tilde{k} \, ((k\!\!\!/ + m)_{\alpha\beta} \exp(-ik(x-y)) + (k\!\!\!/ - m)_{\alpha\beta} \exp(ik(x-y))) \\
&= (i\partial\!\!\!/_x + m)_{\alpha\beta} \int d\tilde{k} (\exp(-ik(x-y)) - \exp(+ik(x-y))) \\
&= (i\partial\!\!\!/_x + m)_{\alpha\beta} (i\Delta(x-y)) \; .
\end{aligned} \tag{4.230}$$

The Feynman propagator is given by the time-ordered product (for fermions!)

$$S_{F_{\alpha\beta}}(x-y) \equiv \langle 0|T\,\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle = \langle 0|\theta(x^0-y^0)\psi_\alpha(x)\bar{\psi}_\beta(y) - \theta(y^0-x^0)\bar{\psi}_\beta(y)\psi_\alpha(x)|0\rangle \; . \tag{4.231}$$

Moreover, we have

$$\begin{aligned}
&(i\partial\!\!\!/_x + m) i\Delta_F(x-y) \\
&= S_F(x-y) \quad = \quad (i\partial\!\!\!/_x + m) \lim_{\epsilon\to 0^+} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik(x-y)} \\
&\qquad = \quad i \int \frac{d^4k}{(2\pi)^4} \frac{k\!\!\!/ + m}{k^2 - m^2 + i\epsilon} e^{-ik(x-y)} \\
&\qquad \overset{k\!\!\!/^2 = k^2}{=} \quad i \int \frac{d^4k}{(2\pi)^4} \frac{1}{k\!\!\!/ - m + i\epsilon} e^{-ik(x-y)} \; .
\end{aligned} \tag{4.232}$$

Thereby

$$(i\partial\!\!\!/_x - m)S_F(x-y) = i\delta^{(4)}(x-y) \; . \tag{4.233}$$

The fermion propagator hence is the Greens function of the free Dirac equation. And causality holds for $\{\psi, \bar{\psi}\}$.

4.8 Quantisation of Spin-1-Fields (Vector Fields)

We now want to quantise vector fields. From electrodynamics, the classical limit is known. It is given by the Maxwell equations. However, the quantisation of fields, which are defined through the Maxwell equations, is difficult. The problem results from the degrees of freedom. The four-potential

$$A^\mu = \begin{pmatrix} \phi \\ \vec{A} \end{pmatrix} \tag{4.234}$$

has four degrees of freedom. The photon has only two degrees, however. We have to achieve that the non-dynamical degrees of freedom, which do not contribute to the photon, are not quantised.

We will start by quantising the massive vector field. It has three degrees of freedom. Examples for massive vector fields are the $W^\pm$ and Z bosons of the weak interaction. Further examples are the spin-1 mesons ρ, ω, ϕ, which are built up by quarks. The reason for the discrepance in the number of degrees of freedom is an inner symmetry (gauge invariance).

4.8.1 Massive Vector Field

The field A^μ has four degrees of freedom, the massive vector field has only three degrees, however. We therefore need an additional condition to reduce the number of degrees of freedom.

We start by writing up the field equations by departing from the Maxwell equations, which can be written covariantly. We hence have

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0 \; , \tag{4.235}$$

with the field strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \; . \tag{4.236}$$

This equation is called *Proca equation*. Usage of the four-divergence on the equation leads to

$$\partial_\nu(\partial_\mu F^{\mu\nu} + m^2 A^\nu) = \partial_\nu \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) + m^2 \partial_\nu A^\nu = 0 \; . \tag{4.237}$$

The first equation is zero, as here a symmetric tensor is combined with an antisymmetric tensor. This can be checked through explicit calculation:

$$\begin{aligned} \partial_\nu \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) &= \partial_\nu \partial_\mu \partial^\mu A^\nu - \partial_\nu \partial_\mu \partial^\nu A^\mu = \partial_\mu \partial_\nu \partial^\nu A^\mu - \partial_\nu \partial_\mu \partial^\nu A^\mu \\ &= \partial_\nu \partial_\mu \partial^\nu A^\mu - \partial_\nu \partial_\mu \partial^\nu A^\mu = 0 \; . \end{aligned} \tag{4.238}$$

Here, in the first summand, the summation indices μ and ν were interchanged, which is allowed, as we sum over all indices. The partial derivatives ∂_μaund ∂_ν can then be exchanged again because of the Schwarz theorem. Thereby we have $m^2\partial_\nu A^\nu = 0$. By assuming that the mass is non-zero, we have the condition

$$\partial_\nu A^\nu = 0 \; . \tag{4.239}$$

Insertion in Eq. (4.236) leads to

$$\partial_\mu F^{\mu\nu} = \Box A^\nu \tag{4.240}$$

and thereby to

$$(\Box + m^2)A^\nu(x) = 0 \qquad \text{with} \quad \partial_\nu A^\nu = 0 \; . \tag{4.241}$$

We hence have found the Klein Gordon equation for each component of A^ν. Through the condition $\partial_\nu A^\nu$ the number of degreees of freedom is reduced to three. The Lagrangian corresponding to Eq. (4.241) reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu \; , \tag{4.242}$$

where A_μ represents a real vector field. As ansatz for a solution of Eq. (4.241) we choose plane waves of the form

$$A_\nu(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\lambda=0}^{3} \exp(-ikx)\, \epsilon_\nu^{(\lambda)}(\vec{k})\, \alpha(\vec{k}) + \text{ h.c.} \; , \text{ with } \omega_k = k_0 = \sqrt{\vec{k}^2 + m^2} \tag{4.243}$$

Here $\epsilon_\nu^{(\lambda)}$ is a polarisation vector. Applying the additional condition leads to

$$k^\nu \epsilon_\nu^{(\lambda)}(\vec{k}) = 0 \; . \tag{4.244}$$

Thereby we get three linearly independent $\epsilon_\nu^{(\lambda)}$. In the rest system of the particle we have $k^\nu = (m, \vec{0})^T$. This leads to

$$\epsilon_0^{(\lambda)} = 0 \; , \quad \text{and thereby } k^\nu \epsilon_\nu(\vec{k}) = 0 \; . \tag{4.245}$$

This is fulfilled by the choice

$$\epsilon_\nu^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} , \quad \epsilon_\nu^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} , \quad \text{und } \epsilon_\nu^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \; . \tag{4.246}$$

This is the cartesian standard basis. Alternatively, one can choose the following complex basis:

$$\epsilon_\nu^{(\pm)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} , \quad \text{und } \epsilon_\nu^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \; . \tag{4.247}$$

The first two vectors describe the circular polarisation. We have for arbitrary reference systems

$$\epsilon_\mu^{(\lambda)}(\epsilon^*)^{(\lambda')\mu} = \epsilon^{(\lambda)} \cdot \epsilon^{(\lambda')*} = -\delta^{\lambda\lambda'} \quad \text{for } \lambda, \lambda' = 1, 2, 3 \,. \tag{4.248}$$

And the completeness relation reads

$$\sum_\lambda \epsilon_\mu^{(\lambda)} \epsilon_\nu^{*(\lambda)} = -\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2} \right) \,. \tag{4.249}$$

For the general solution in quantised form we have

$$A_\mu(x) = \int d\tilde{k} \sum_{\lambda=1,2,3} \left(\exp(-ikx)\, \epsilon_\mu^{(\lambda)}(\vec{k})\, a^{(\lambda)}(\vec{k}) + \exp(ikx)\, \epsilon_\mu^{*(\lambda)}(\vec{k})\, a^{\dagger(\lambda)}(\vec{k}) \right) \,. \tag{4.250}$$

The operator $a^{\dagger(\lambda)}(\vec{k})$ generates a particle with momenum $\vec{k}$ and polarisation λ. The following commutation relations hold,

$$[A_\nu(x), A_\nu(y)] = 0 \quad \text{für } (x-y)^2 < 0 \,. \tag{4.251}$$

And for the operators

$$[a^{(\lambda)}(\vec{k}), a^{\dagger(\lambda')}(\vec{k}')] = \delta_{\lambda\lambda'}\, (2\pi)^3\, 2\omega_k\, \delta(\vec{k} - \vec{k}') \,. \tag{4.252}$$

4.8.2 Massless Vector Field (Photon Field)

Gauge freedom

The Maxwell equations read

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= j^\nu \qquad \text{inhomogeneous Maxwell equation} & (4.253)\\ \partial_\mu \tilde{F}^{\mu\nu} &= 0 \qquad \text{homogeneous Maxwell equation ,} & (4.254) \end{aligned}$$

with the dual field strength tensor

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \,. \tag{4.255}$$

The field strength tensor expressed through the potentials reads

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{4.256}$$

Thereby the equation (4.254) is automatically fullfilled (check!). The vector potential hereby, however, is not yet uniquely fixed. We hence have the gauge freedom. Thus $F_{\mu\nu}$ is unchanged upon replacement

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x) \,, \tag{4.257}$$

where $\Lambda(x)$ is an arbitrary scalar field. We can choose the

$$\underline{\text{Lorenz gauge:}} \quad \partial_\mu A^\mu = 0 \tag{4.258}$$

Because, if

$$\partial_\mu A^\mu = G(x) \neq 0 \,, \tag{4.259}$$

then choose $\Lambda(x)$ such that

$$\Box\Lambda(x) = -G(x) \; . \tag{4.260}$$

For the new field $A'_\mu(x)$ then (4.258) is fulfilled. In the Lorenz gauge the Maxwell equation (4.253) is equivalent to

$$\Box A^\nu = j^\nu \tag{4.261}$$

or in the free case ($j = 0$)

$$\Box A^\nu = 0 \; . \tag{4.262}$$

As additional freedom in the choice of the gauge such Λ's can be chosen, for which we have

$$\Box\Lambda(x) = 0 \; . \tag{4.263}$$

Apart from the Lorenz gauge there is also the Coulomb gauge. Here, one demands

$$\underline{\text{Coulomb gauge:}} \quad \vec{\nabla}\cdot\vec{A} = 0 \; . \tag{4.264}$$

This gauge, however, is not covariant. Another example for a gauge is the axial gauge, for which we have

$$\underline{\text{Axial gauge:}} \quad A_z = 0 \; . \tag{4.265}$$

In the following, the Lorenz gauge will be used.

Lagrangian Formalism

As simplest ansatz for the Lagrangian density we choose

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \; . \tag{4.266}$$

We furthermore demand the commutation relations

$$[A_\mu(x), \Pi_\nu(y)]_{x_0=y_0} \stackrel{!}{=} -ig_{\mu\nu}\delta(\vec{x}-\vec{y}) \; , \quad [A_\mu(x), A_\nu(y)]_{x_0=y_0} = [\Pi_\mu(x), \Pi_\nu(y)]_{x_0=y_0} = 0 \; . \tag{4.267}$$

The minus sign will be explained later. The canonical conjugate momentum is given by

$$\Pi^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_\mu)} = -\partial^0 A^\mu + \partial^\mu A^0 \; . \tag{4.268}$$

As we see, the zero component Π^0 is vanishing. We hence cannot work with this Lagrangian density. We hence look that the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F^2 - \frac{1}{2}\lambda(\partial_\mu A^\mu)^2 \; . \tag{4.269}$$

The corresponding field equations read

$$\partial_\mu F^{\mu\nu} + \lambda\partial^\nu\partial_\mu A^\mu = 0 \quad \leftrightarrow \quad \Box A^\nu - (1-\lambda)\partial^\nu\partial_\mu A^\mu = 0 \; . \tag{4.270}$$

The equations ressemble at the Klein Gordon equation. The canical conjugate momentum is obtained as

$$\Pi^\mu = -\partial^0 A^\mu + \partial^\mu A^0 - \lambda g^{0\mu}(\partial_\nu A^\nu) \;\Rightarrow\; \Pi^0 = -\lambda\,\partial_\nu A^\nu \neq 0 \text{ if } \lambda \neq 0 \text{ and } \partial_\nu A^\nu \neq 0\,. \tag{4.271}$$

In the following, we set $\lambda = 1$ (Feynman gauge).

We now do not demand the Lorenz gauge $\partial_\mu A^\mu = 0$, but $\partial_\mu A^\mu|\psi\rangle = 0$, where $|\psi\rangle$ is a physical state. The expectation value of the required commutator relation (4.267) is

$$\langle\psi|[A_\mu(x), \Pi_\nu(y)]_{x_0=y_0}|\psi\rangle = -ig_{\mu\nu}\delta(\vec{x}-\vec{y})\langle\psi|\psi\rangle\,. \tag{4.272}$$

This is, however, inconsistent with the requirement $\partial_\nu A^\nu = 0$. The reason is, that because of $\Pi^0 = -\partial_\nu A^\nu$ the left side of (4.272) is identical to 0 (for $\nu = 0$).

We now demand

$$\underline{\text{Gupta Bleuler condition:}}\quad \partial_\mu A^{(+)\mu}|\text{phys}\rangle = \partial_\mu A^{(+)\mu}|\psi\rangle = 0\,, \tag{4.273}$$

where

$$A_\mu = A_\mu^{(+)} + A_\mu^{(-)} \tag{4.274}$$

with

$$A_\mu^{(+)}(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\lambda=0}^{3} \epsilon_\mu^{(\lambda)}(\vec{k})\, a^{(\lambda)}(\vec{k})\, \exp(-ikx)\,. \tag{4.275}$$

This is hence the part with positive frequencies. We here use the notation, that a plane wave travelling to the right is described by $\exp(-ikx)$. The part

$$A_\mu^{(-)}(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\lambda=0}^{3} \epsilon_\mu^{*(\lambda)}(\vec{k})\, a^{\dagger(\lambda)}(\vec{k})\, \exp(ikx) \tag{4.276}$$

contains the negative frequencies. The sum is performed over the four linearly independent polarisations. For these we have

$$\text{Orthogonality:}\qquad \epsilon_\mu(\lambda)\epsilon^{*\mu}(\lambda') = -\zeta^{(\lambda)}\delta^{\lambda\lambda'} \;\text{ with } \zeta^{(0)} = -1,\, \zeta^{(1)} = \zeta^{(2)} = \zeta^{(3)} = 1 \tag{4.277}$$

$$\text{Completeness:}\qquad \sum_{\lambda=0}^{3} \zeta^{(\lambda)} \epsilon_\mu^{(\lambda)} \epsilon_\nu^{*(\lambda)} = -g_{\mu\nu}\,. \tag{4.278}$$

We now use a specific reference system with

$$\epsilon^{(0)\mu} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{und} \quad \epsilon^{(\lambda)\mu} = \begin{pmatrix} 0 \\ \vec{\epsilon}_\lambda \end{pmatrix} \quad \text{für } \lambda = 1,2,3\,. \tag{4.279}$$

In case only the zero component $\epsilon^{(0)\mu}$ is present, we call it scalar polarisation. Furthermore, we have

$$\vec{\epsilon}_3 = \frac{\vec{k}}{|\vec{k}|} \quad \text{and} \quad \vec{k}\cdot\vec{\epsilon}_{1,2} = 0\,. \tag{4.280}$$

The $\vec{\epsilon}_3$ describes the longitudinal polarisation and $\vec{\epsilon}_{1,2}$ the transversal polarisations.

From the requirement (4.273), which is equivalent to

$$\langle\psi|\partial_\mu A^{(-)\mu} = 0 \tag{4.281}$$

it follows that

$$\langle\psi|\partial_\mu A^\mu|\psi\rangle = 0 \, . \tag{4.282}$$

This is the Lorenz condition for the expectation value.

Insertion of the expansion in plane waves in

$$\begin{aligned} [A_\mu(x), \Pi_\nu(y)]_{x_0=y_0} &= -ig_{\mu\nu}\delta(\vec{x}-\vec{y}) \, , \\ [\partial_{x_i} A_\mu(x), A_\nu(y)]_{x_0=y_0} &= [A_\mu(x), \partial_{y_j} A_\nu(y)]_{x_0=y_0} = [\partial_{x_i} A_\mu(x), \partial_{y_j} A_\nu(y)]_{x_0=y_0} = 0 \end{aligned} \tag{4.283}$$

leads to

$$[a^{(\lambda)}(\vec{k}), a^{\dagger(\lambda')}(\vec{k}')] = \zeta^{(\lambda)}\delta^{\lambda\lambda'}(2\pi)^3\, 2\omega\, \delta(\vec{k}-\vec{k}') \, . \tag{4.284}$$

For $\lambda = 0$ the annihilation and creation operators interchange their roles, as $\zeta^{(0)} = -1$. We have

$$[a, a] = [a^\dagger, a^\dagger] = 0 \, . \tag{4.285}$$

We interpret $a^{(\lambda)}(\vec{k})$ as annihilation operator and $a^{(\lambda)\dagger}(\vec{k})$ as creation operator. The scalar polarisation is described by $\lambda = 0$, the longitudinal polarisation by $\lambda = 3$ and the transversal polarisation by $\lambda = 1, 2$. We have

$$\text{for the vacuum state:} \qquad a^{(\lambda)}(\vec{k})|0\rangle = 0 \tag{4.286}$$

$$\text{for the one-photon state:} \qquad a^{(\lambda)\dagger}(\vec{k})|0\rangle = |\vec{k}, \lambda\rangle \, . \tag{4.287}$$

(Here for simplicity we omitted the smearing with the function $f(\vec{k})$, which, however, is necessary for the correct normalisation. Hence, $\int d\tilde{k} f(\vec{k}) a^\dagger(\vec{k})|0\rangle = |\vec{k}, \lambda\rangle$.) In the following, we want to justify this interpretation. For this, we apply the Hamiltonian operator on a state $|\vec{k}', \lambda'\rangle$,

$$\begin{aligned} :H:\; |\vec{k}', \lambda'\rangle &= \int d\tilde{k}\omega_k \sum_{\lambda=0}^{3} \zeta^{(\lambda)}\, a^{\dagger(\lambda)}(\vec{k}) a^{(\lambda)}(\vec{k}) a^{(\lambda')\dagger}(\vec{k}')|0\rangle \\ &= \int d\tilde{k}\omega_k \sum_{\lambda=0}^{3} \zeta^{(\lambda)} a^{\dagger(\lambda)}(\vec{k})(2\pi)^3 2\omega_k \zeta^{(\lambda)}\delta^{\lambda\lambda'}\delta(\vec{k}-\vec{k}')|0\rangle = \omega_{k'}|\vec{k}', \lambda'\rangle \end{aligned} \tag{4.288}$$

We have for the massless photon

$$\omega_k = |\vec{k}| \, . \tag{4.289}$$

Thereby $\omega_k > 0$ and hence the Hamilton operator is positive definite. However, there are states with negative norm. Because

$$\langle\vec{k}, \lambda|\vec{k}, \lambda\rangle = \langle 0|a^{(\lambda)}(\vec{k}) a^{\dagger(\lambda)}(\vec{k})|0\rangle = \zeta^{(\lambda)} < 0 \;\text{ for }\; \lambda = 0 \, . \tag{4.290}$$

Scalar photons hence have the norm -1. However, it follows from

$$k_\mu \sum_{\lambda=0}^{3} \epsilon^{\mu(\lambda)} a^{(\lambda)} = \sum_{\lambda=0,3} k_\mu \epsilon^{\mu(\lambda)} a^{(\lambda)} \tag{4.291}$$

with the requirement

$$\partial_\mu A^{(+)\mu} |\psi\rangle = 0 \; , \tag{4.292}$$

that

$$\left(a^{(3)}(\vec{k}) - a^{(0)}(\vec{k}) \right) |\text{phys}\rangle = \left(a^{(3)}(\vec{k}) - a^{(0)}(\vec{k}) \right) |\psi\rangle = 0 \; . \tag{4.293}$$

This is a condition on the scalar and the longitudinal photons. The Gupta Bleuler condition is not condition on the transversal photons. In particular, we have

$$\langle \psi | a^{\dagger(3)} a^{(3)} - a^{\dagger(0)} a^0 | \psi \rangle = \langle \psi | a^{\dagger(3)} (a^{(3)} - a^{(0)}) | \psi \rangle = 0 \; . \tag{4.294}$$

This follows from $\left(a^{(3)}(\vec{k}) - a^{(0)}(\vec{k}) \right) |\psi\rangle = 0$. And for the expectation value of the energy we get

$$\langle \psi | H | \psi \rangle = \left\langle \psi \left| \int d\tilde{k} \sum_{\lambda=0}^{3} \omega_k \, \zeta^{(\lambda)} a^{(\lambda)\dagger} a^{(\lambda)} \right| \psi \right\rangle = \left\langle \psi \left| \int d\tilde{k} \sum_{\lambda=1}^{2} \omega_k \, a^{(\lambda)\dagger} a^{(\lambda)} \right| \psi \right\rangle \; . \tag{4.295}$$

Hence, only transversal parts contribute. Through the Gupta Bleuler condition we achieve that in physical quantities only the physical degrees of freedom contribute. This can be shown analoguously also for other physical quantities. From $\partial_\mu A^{(+)\mu}|\psi\rangle = 0$ follows that only transversal photons contribute to observable quantities.

Summary

- Apart from the two physical there are two additional degrees of freedom (longitudinal and scalar).

- One is forbidden by the Gupta Bleuler condition.

- The other one corresponds to an additional gauge freedom, which still exists despite the Lorenz condition, namely $A^\mu \to A^\mu + \partial^\mu \Lambda$ with $\Box \Lambda = 0$.

- In scattering processes the scalar and the longitudinal photons play an important role. For gauge theories it holds in general that the requirement of covariance leads to unphysical states (states with negative norm) with so-called ghost particles. In quantum electrodynamics (QED) the solution is trivial as here the ghosts decouple. In quantum chromodynamics (QCD) and in the electroweak interaction these states appear in the calculation. However, for incoming waves without ghosts, there are also only outgoing waves without ghosts, so that the probability interpretation is guaranteed.

4.9 The Feynman Propagator for the Photon Field

For the Feynman propagator of the photon field we look at the commutator

$$[A^\mu(x), A^\nu(y)] \equiv iD^{\mu\nu}(x-y) \; . \tag{4.296}$$

The Feynman propagator of the photon field is then given by

$$D_F^{\mu\nu}(x-y) = -g^{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+i\epsilon} \exp(-ik(x-y)) = \langle 0|TA^\mu(x)A^\nu(y)|0\rangle \; . \tag{4.297}$$

This is found by inserting the photon field Eq. (4.243) and using the commutator relations Eqs. (4.284), (4.285) as well as the completeness relation (4.278). We want to interpret the propagator in analogy to the scalar field. However, here four types of photons are exchanged, two transversal ones, a longitudinal one and a scalar photon.

We look at the Feynman propagator in the momentum space in the reference system with the polarisation vectors

$$n_\mu = \epsilon_\mu^{(0)}(\vec{k}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \epsilon_\mu^{(3)} = \frac{k_\mu - (n\cdot k)n_\mu}{\sqrt{(n\cdot k)^2 - k^2}} \; . \tag{4.298}$$

The Feynman propagator in momentum space is given by

$$\begin{aligned} D_F^{\mu\nu} &= -\frac{g^{\mu\nu}}{k^2+i\epsilon} \overset{(4.278)}{=} \frac{1}{k^2+i\epsilon} \sum_{\lambda=0}^{3} \zeta^{(\lambda)} \epsilon^{(\lambda)\mu} \epsilon^{(\lambda)\nu} \\ &= \frac{1}{k^2+i\epsilon} \left[\sum_{\lambda=1}^{2} \zeta^{(\lambda)} \epsilon^{(\lambda)\mu} \epsilon^{(\lambda)\nu} + \frac{(k^\mu - (n\cdot k)n^\mu)(k^\nu - (n\cdot k)n^\nu)}{(n\cdot k)^2 - k^2} - n^\mu n^\nu \right] \\ &= \frac{1}{k^2+i\epsilon} \left[\sum_{\lambda=1}^{2} \zeta^{(\lambda)} \epsilon^{(\lambda)\mu} \epsilon^{(\lambda)\nu} + \underbrace{\frac{k^2 n^\mu n^\nu}{(n\cdot k)^2 - k^2}}_{D_{F,C}^{\mu\nu}} + \underbrace{\frac{k^\mu k^\nu - k\cdot n(k^\mu n^\nu + k^\nu n^\mu)}{(n\cdot k)^2 - k^2}}_{D_{F,R}^{\mu\nu}} \right] \end{aligned} \tag{4.299}$$

The first term describes the exchange of transversal photons. Thereby we hence have

$$D_F^{\mu\nu} = D_{F,T}^{\mu\nu} + D_{F,C}^{\mu\nu} + D_{F,R}^{\mu\nu} \; . \tag{4.300}$$

The second term in local space is

$$D_{F,C}^{\mu\nu} = g^{\mu 0} g^{\nu 0} \int \frac{d^4k}{(2\pi)^4} \frac{1}{|\vec{k}|^2} \exp(-ikx) = g^{\mu 0} g^{\nu 0} \frac{1}{4\pi|\vec{x}|} \delta(x_0) \; . \tag{4.301}$$

This corresponds to an instantaneous Coulomb potential. The third term $D_{F,R}^{\mu\nu}$ vanishes as the photon couples to a conserved current. This means that $\partial_\mu j^\mu = 0$ and thereby in momentum space $k_\mu j^\mu = 0$.

Chapter 5

Interaction, Perturbation Theory

5.1 Free Theory

So far we only looked at the free theory. It contains in the Lagrangian density only terms quadratic in the fields. By using the Euler Lagrange Equation we obtain from the Lagrangian density homogeneous linear field equations. These can be solved exactly through Fourier expansion. This then leads to the computation of the propagator. E.g. a scalar field is described by the Klein-Gordon equation. The Lagrangian density reads

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2 \ . \tag{5.1}$$

Application of the Euler-Lagrange equation leads to the field equation

$$(\Box + m^2)\phi = 0 \ . \tag{5.2}$$

The Feynman propagator is given by

$$D_F(x-y) = \langle 0|T\phi(x)\phi(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} \exp(-ik(x-y)) \ . \tag{5.3}$$

This is so to say the propability amplitude for the propagation of a particle from location $\vec{x}$ to location $\vec{y}$. Fields without interaction cannot be detected, however. Therefore, in the following, we will look at the interaction between fields. However, for this case so far there is no exact solution possible, only in lattice gauge theory by applying certain assumptions and for certain Lagrangian densities. The interaction phenomena are therefore led back, by the help of perturbation theory, to the description through free fields.

5.2 Interaction Terms

The Lagrangian density is split up into a free Lagrangian density $\mathcal{L}_0$ and a Lagrangian density $\mathcal{L}_I$, which describes the interaction,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I \ . \tag{5.4}$$

The free Lagrangian density is quadratic in the fields, the interaction Lagrangian density contains terms with more than two fields.

We look as example at electrodynamics. In classical physics, the electromagnetic fields interact with the electric current,

$$\partial_\mu F^{\mu\nu} = j^\nu \ . \tag{5.5}$$

The current j^ν is e.g. caused by electrons. Thereby, an interaction term between photons (A_μ) and electrons (ψ) is constructed. However, $\mathcal{L}$ has to be a Lorentz respectively Dirac scalar. Possible Dirac scalars are e.g. $\bar{\psi}\psi$, $\bar{\psi}\gamma^\mu\psi$ etc. We hence introduce the interaction term

$$\mathcal{L}_I = -e\bar{\psi}\gamma^\mu\psi A_\mu \,, \tag{5.6}$$

where e is a coupling constant and determines the strength of the interaction. Thereby the Lagrangian density of quantum electrodynamics (QED) is given by

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu + \text{ gauge fixing} \,. \tag{5.7}$$

The first term is the kinetic term for the photon, the second term is the kinetic term for the electron (Dirac field). The Euler-Lagrange equations of $\mathcal{L}_{QED}$ read

$$(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi(x) \;=\; 0 \tag{5.8}$$

$$\partial_\mu F^{\mu\nu} = e\bar{\psi}\gamma^\nu\psi \;\equiv\; j^\nu \,. \tag{5.9}$$

The second equation has the form which is already known from electrodynamics. Thereby we can interpret

$$j^\nu = e\bar{\psi}\gamma^\nu\psi \tag{5.10}$$

as Noether current. It is here the conserved current for a Dirac theory. The two equations are coupled non-linear field equations! We have to answer the question, how these can be solved.

Let us remark that $\mathcal{L}_{QED}$ is invariant under a local gauge transformation. Thus $\psi(x)$ can be replaced by

$$\psi(x) \;\rightarrow\; \exp(i\alpha(x))\,\psi(x)) \,, \quad \text{with } \alpha(x) \in \mathbb{R} \text{ , and simultaneously} \tag{5.11}$$

$$A_\mu(x) \;\rightarrow\; A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x) \tag{5.12}$$

Through this gauge transformation, QED is characterised.

Further examples for interactions are

- ϕ^4-theory:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{5.13}$$

- Yukawa theory (ineraction between a scalar and fermions):

$$\mathcal{L} = \mathcal{L}_\phi^{free} + \mathcal{L}_\psi^{free} - g\bar{\psi}\psi\phi \tag{5.14}$$

- Scalar electrodynamics

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + [(\partial_\mu + ieA_\mu)\phi]^*(\partial^\mu + ieA^\mu)\phi - m^2\phi^*\phi \tag{5.15}$$

So far we only looked at cubic and quartic terms in the interaction Lagrangian density. The question is if there are also higher powers like e.g. ϕ^6. The answer is that products of fields in the Lagrangian density which have mass dimension > 4, are not renormalisable. They lead to observables which depend on a cut-off. Equivalent to this is that in interaction terms the mass dimension of the coupling constant must always be ≥ 0. Remind the mass dimensions that we derived, were

$$[\phi] = 1\,,\ [A_\mu] = 1\,,\ [\psi] = \frac{3}{2}\,,\ [x] = -1\,,\ [\partial_\mu] = 1\,,\ ... \tag{5.16}$$

5.3 Interaction Picture

The Hamiltonian operator can be written, just like the Lagrangian density, as a sum of a free operator H_0 and an operator H_I, which describes the interaction, i.e.

$$H = H_0 + H_I\ . \tag{5.17}$$

The comparison between Schrödinger, Heisenberg and interaction picture is given in Tab. 5.1.

Picture	States	Operators
Schrödinger picture	$i\partial_t\|\psi\rangle^S = H\|\psi\rangle^S$	$i\partial_t O^S = 0$
Heisenberg picture	$i\partial_t\|\psi\rangle^H = 0$	$i\partial_t O^H = [O^H, H]$
Interaction picture	$i\partial_t\|\psi\rangle^I = H_I\|\psi\rangle^I$	$i\partial_t O^I = [O^I, H_0]$

Table 5.1: Comparison between Schrödinger, Heisenberg, and interaction picture.

This holds in particular for $O = \pi, \phi, A_\mu, ...$ The operators fulfill in the interaction picture hence the equations of motion of the free theory. From this follows that the field operators have a Fourier decomposition like before. Thus for the field operator of the scalar field in the interaction picture we have

$$\begin{aligned}\phi^I(x) &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[\exp(ikx)\,a^\dagger(\vec{k}) + \exp(-ikx)\,a(\vec{k})\right] \\ &\qquad \text{with} \quad \omega_k = k_0 = \sqrt{\vec{k}^2 + m^2}\ .\end{aligned} \tag{5.18}$$

In the following, we will always work in the interaction picture and drop the index I.

5.4 Time Evolution of the States - the S-Matrix

We have to solve the interaction theory approximately in the framework of perturbation theory. This is given by a power series in the coupling constant. The assumption (hope, experience) is, that the series converges so that an approximation through the leading terms in the expansion is possible. Let us look e.g. at the anomalous magnetic moment a of the electron, which was calculated as

$$\begin{aligned}a \equiv \frac{g_e - 2}{2} &= \frac{\alpha}{2\pi} - 0.32...\left(\frac{\alpha}{\pi}\right)^2 + 1.18...\left(\frac{\alpha}{\pi}\right)^3 - 1.51...\left(\frac{\alpha}{\pi}\right)^4 \\ &= 0.0011596521866\end{aligned} \tag{5.19}$$

Here α is the coupling constant $\alpha = e^2/(4\pi)$. Experimentally it is found that $a = 0.0011596521884(43)$.

We want to describe scattering experiments. For this the interaction shall only be active during a certain period of time $[T_1, T_2]$. The asymptotic states $|\phi(t \to -\infty)\rangle$ and $|\phi(t \to +\infty)\rangle$ fulfill the free equation of motion with $H_I = 0$, hence

$$|\phi(-\infty)\rangle \longrightarrow \text{ Interaction area } \longrightarrow |\phi(\infty)\rangle \; . \tag{5.20}$$

The interaction here only takes place between T_1 and T_2. We hence will not describe interaction states. If $|n\rangle$ are the eigenstates of H_0, then we can write

$$|\phi(-\infty)\rangle = \sum_n a_n |n\rangle \quad \text{with} \quad \sum_n |a_n|^2 = 1 \tag{5.21}$$

$$|\phi(+\infty)\rangle = \sum_n b_n |n\rangle \quad \text{with} \quad \sum_n |b_n|^2 = 1 \; . \tag{5.22}$$

This means that the state is changed by the interaction, but that its norm is conserved. This corresponds to a rotation in the state space. There exists hence a unitary transformation S with

$$|\phi(+\infty)\rangle = S|\phi(-\infty)\rangle \; . \tag{5.23}$$

In the following, the so-called S-matrix will be determined.

5.5 Determination of the S-Matrix

Our starting point is the Schrödinger equation

$$i\partial_t |\phi(t)\rangle = H_I |\phi(t)\rangle \quad \text{with the initial condition } |\phi(-\infty)\rangle = |i\rangle \; . \tag{5.24}$$

The differential equation can be re-written as an integral equation, from which an iterative solution will be determined, by repeatedly inserting $|\phi(t)\rangle$ with $t = t_1, t_2, ...$ into the equation. Hence,

$$\begin{aligned} |\phi(t)\rangle &= |i\rangle + (-i)\int_{-\infty}^{t} dt_1 H_I(t_1)|\phi(t_1)\rangle = |i\rangle + (-i)\int_{-\infty}^{t} dt_1 H_I(t_1)|i\rangle \\ &\quad + (-i)^2 \int_{-\infty}^{t} dt_1 H_I(t_1) \int_{-\infty}^{t_1} dt_2 H_I(t_2)|\phi(t_2)\rangle \\ &= |i\rangle + \sum_{n=1}^{\infty} (-i)^n \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 ... \int_{-\infty}^{t_{n-1}} dt_n H_I(t_1)\, H_I(t_2) ... H_I(t_n)|i\rangle \; . \end{aligned} \tag{5.25}$$

For $t \to \infty$ we can extract from this the S-matrix. First, we transform the above expression for $t \to \infty$ into a more compact form by using:

$$\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 H_I(t_1) H_I(t_2) = \frac{1}{2}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \, T(H_I(t_1)H_I(t_2)) \; , \tag{5.26}$$

where the time-ordering operator T , which is defined as

$$T(H_I(t_1)H_I(t_2)) = \begin{cases} H_I(t_1)H_I(t_2) & t_2 \le t_1 \\ H_I(t_2)H_I(t_1) & t_2 \ge t_1 \end{cases} \; , \tag{5.27}$$

was used. Thereby, we obtain

$$|\phi(\infty)\rangle = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 ... \int_{-\infty}^{\infty} dt_n T(H_I(t_1)...H_I(t_n))|i\rangle \ . \tag{5.28}$$

Thereby we obtain for the S-matrix the expression

$$S = T\left[\exp\left(-i\int_{-\infty}^{\infty} dt H_I(t)\right)\right] = T\left[\exp\left(i\int_{-\infty}^{\infty} d^4x\, \mathcal{L}_I(x)\right)\right] \ . \tag{5.29}$$

The transition probabilities are given by

$$\begin{aligned} \langle f|S|i\rangle &= \langle f|\phi(+\infty)\rangle \\ &= \left\langle f\left|\left(1 + i\int d^4x\, T\mathcal{L}_I(x) + \frac{i^2}{2!}T\int d^4x \int d^4y T\mathcal{L}_I(x)\mathcal{L}_I(y) + ...\right)\right| i\right\rangle \end{aligned} \tag{5.30}$$

For example in quantum electrodynamics we have a Lagrangian density of the form $\mathcal{L}_I \sim e\bar{\psi}\gamma^\mu\psi A_\mu$, which describes the interaction. We hence have to compute e.g.

$$\langle f|T(\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x))(\bar{\psi}(y)\gamma^\nu\psi(y)A_\nu(y))...|i\rangle \ . \tag{5.31}$$

Required are the matrix elements of the form $\langle f|T\varphi_1(x_1)...\varphi_n(x_n)|i\rangle$, where the $\varphi_i(x_i)$ in general shall be quantum fields in the interaction picture. Moreover, the φ shall represent both bosonic and fermionic fields, $\varphi_i \in \{\psi, A_\mu, \phi, ...\}$. The time evolution is analogous to the one of the free fields. In the following, we will have a closer look at the time-ordered product.

5.6 The Wick Theorem

We write

$$\varphi_i(x_i) = \varphi_i^{(+)}(x_i) + \varphi_i^{(-)}(x_i) \ , \tag{5.32}$$

with

$$\varphi_i^{(+)}(x_i) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} a_i(\vec{k}) \exp(-ikx_i) \tag{5.33}$$

$$\varphi_i^{(-)}(x_i) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} a_i^\dagger(\vec{k}) \exp(ikx_i) \text{ mit} \tag{5.34}$$

$$\omega_k = k_0 = \sqrt{\vec{k}^2 + m^2} \ . \tag{5.35}$$

The $(+)$ stands for positive, the $(-)$ for negative frequencies. In the following be $\varphi_i \equiv \varphi_i(x_i)$. We have

$$:\varphi_i^{(+)}\varphi_j^{(-)}: \;=\; \begin{cases} -\varphi_j^{(-)}\varphi_i^{(+)} & \text{if } \varphi_i, \varphi_j \text{ fermionic fields} \\ +\varphi_j^{(-)}\varphi_i^{(+)} & \text{otherwise} \end{cases} \tag{5.36}$$

$$:\varphi_i^{(-)}\varphi_j^{(+)}: \;=\; \varphi_i^{(-)}\varphi_j^{(+)} \ . \tag{5.37}$$

It is

$$\varphi_1\varphi_2 = (\varphi_1^{(+)} + \varphi_1^{(-)})(\varphi_2^{(+)} + \varphi_2^{(-)}) = \varphi_1^{(+)}\varphi_2^{(+)} + \varphi_1^{(-)}\varphi_2^{(+)} + \varphi_1^{(-)}\varphi_2^{(-)} + \varphi_1^{(+)}\varphi_2^{(-)} \ . \tag{5.38}$$

All terms apart from the last one are normal-ordered. This one can be rewritten by using the (anti-)commutator ('+' relates to the commutor, '-' relates to the anti-commutator) relations. Because

$$\varphi_1^{(+)}\varphi_2^{(-)} = [\varphi_1^{(+)}, \varphi_2^{(-)}]_\pm \pm \varphi_2^{(-)}\varphi_1^{(+)} \; . \tag{5.39}$$

And thereby

$$\varphi_1\varphi_2 = :\varphi_1\varphi_2: \; + \begin{cases} \{\varphi_1^{(+)}, \varphi_2^{(-)}\} & \text{if } \varphi_1, \varphi_2 \text{ fermion fields} \\ \left[\varphi_1^{(+)}, \varphi_2^{(-)}\right] & \text{otherwise} \end{cases} \tag{5.40}$$

In case φ_1, φ_2 are bosonic fields, we can also write for $x_1^0 > x_2^0$ (note that the commutator is a c-number)

$$[\varphi_1^{(+)}, \varphi_2^{(-)}] = \langle 0|[\varphi_1^{(+)}, \varphi_2^{(-)}]|0\rangle = \langle 0|\varphi_1^{(+)}\varphi_2^{(-)}|0\rangle = \langle 0|\varphi_1\varphi_2|0\rangle \overset{x_1^0>x_2^0}{\equiv} \langle 0|T\varphi_1\varphi_2|0\rangle \; . \tag{5.41}$$

This holds analogously in case φ_1, φ_2 are fermionic (show this!). In case $x_1^0 < x_2^0$, the indices 1 and 2 are interchange, and one obtains the same equation:

$$\varphi_2\varphi_1 = :\varphi_2\varphi_1: \; + \langle 0|T\varphi_2\varphi_1|0\rangle \; . \tag{5.42}$$

And also

1. Be φ_1, φ_2 fermions, then it is($x_2^0 > x_1^0$)

$$T(\varphi_1\varphi_2) = -\varphi_2\varphi_1 = -:\varphi_2\varphi_1: \; -\langle 0|T\varphi_2\varphi_1|0\rangle = :\varphi_1\varphi_2: \; +\langle 0|T\varphi_1\varphi_2|0\rangle \; . \tag{5.43}$$

2. Be φ_1 or φ_2 a boson, then one obtains an analogous result.

It follows hence

$$T(\varphi_1\varphi_2) = :\varphi_1\varphi_2: \; +\langle 0|T\varphi_1\varphi_2|0\rangle \quad \forall\varphi_1, \varphi_2 \text{ und } \forall x_1, x_2 \; . \tag{5.44}$$

The fields φ_i behave like free fields. This means that

$$\langle 0|T\varphi_i(x)\varphi_j(y)|0\rangle = D_F^i(x-y) \tag{5.45}$$

is the Feynman propagators for fields of type i. If φ_1, φ_2 are different fields, then it holds that

$$\langle 0|T\varphi_1\varphi_2|0\rangle = 0 \; . \tag{5.46}$$

We introduce the following abbreviation:

$$\underbrace{\varphi_1\varphi_2} \equiv \langle 0|T\varphi_1\varphi_2|0\rangle \; . \tag{5.47}$$

This is called Wick contraction.
For three bosonic fields it holds

$$\begin{aligned} T\phi(x_1)\phi(x_2)\phi(x_3) &= \; :\phi(x_1)\phi(x_2)\phi(x_3): + :\phi(x_1): \; \langle 0|T\phi(x_2)\phi(x_3)|0\rangle \\ &+ :\phi(x_2): \; \langle 0|T\phi(x_1)\phi(x_3)|0\rangle \\ &+ :\phi(x_3): \; \langle 0|T\phi(x_1)\phi(x_2)|0\rangle \; . \end{aligned} \tag{5.48}$$

In general it is

$$
\begin{aligned}
T\phi(x_1)...\phi(x_n) \;=\; & :\phi(x_1)...\phi(x_n): \\
& + \sum_{k<l} :\phi(x_1)\phi(x_2)...\cancel{\phi(x_k)}...\cancel{\phi(x_l)}...\phi(x_n): \;\langle 0|T\phi(x_k)\phi(x_l)|0\rangle + ... \\
& + \sum_{p\geq 2} \left(\sum_{\substack{k_1<k_2\\ <..<k_{2p}}} :\phi(x_1)...\cancel{\phi(x_{k_1})}...\cancel{\phi(x_{k_{2p}})}...\phi(x_n): \;\star \right. \\
& \left. \star \sum_{\text{Alle Perm}} \langle 0|T\phi(x_{k_1})\phi(x_{k_2}|0\rangle ... \langle 0|T\phi(x_{k_{2p-1}})\phi(x_{k_{2p}})|0\rangle \right) \; . \qquad (5.49)
\end{aligned}
$$

Be aware of the minus sign for fermionic fields. For example it holds in case of three fields:

$$
\underbrace{\varphi_1\varphi_2\varphi_3} \equiv \begin{cases} -\underbrace{\varphi_1\varphi_3}\varphi_2 & \text{falls } \varphi_2,\varphi_3 \text{ Fermionfelder} \\ +\underbrace{\varphi_1\varphi_3}\varphi_2 & \text{sonst} \end{cases} \qquad (5.50)
$$

It general it is

$$
\varphi_1...\varphi_k...\varphi_l...\varphi_n = (-1)^p \underbrace{\varphi_k\varphi_l}\varphi_1...\varphi_n \; , \qquad (5.51)
$$

where p is the number of fermionic interchanges.

We look at an example for $n = 4$:

$$
\begin{aligned}
T(\varphi_1\varphi_2\varphi_3\varphi_4) \;=\; & :\varphi_1\varphi_2\varphi_3\varphi_4: + :\underbrace{\varphi_1\varphi_2}\varphi_3\varphi_4: + :\underbrace{\varphi_1\varphi_2\varphi_3}\varphi_4: + :\underbrace{\varphi_1\varphi_2\varphi_3\varphi_4}: \\
& + :\varphi_1\underbrace{\varphi_2\varphi_3}\varphi_4: + :\varphi_1\underbrace{\varphi_2\varphi_3\varphi_4}: + :\varphi_1\varphi_2\underbrace{\varphi_3\varphi_4}: \qquad (5.52) \\
& + :\underbrace{\varphi_1\varphi_2}\underbrace{\varphi_3\varphi_4}: + :\underbrace{\varphi_1\underbrace{\varphi_2\varphi_3}\varphi_4}: + :\underbrace{\varphi_1\varphi_2\varphi_3}\varphi_4 \;(\varphi_2\text{–}\varphi_4):
\end{aligned}
$$

We look at the vaccum expectation value for $\varphi_i \equiv \varphi(x_i)$, hence two identical bosonic fields:

$$
\begin{aligned}
\langle 0| :\varphi_1...\varphi_n: |0\rangle &= 0 \qquad (5.53) \\
\langle 0|T(\varphi_1\varphi_2\varphi_3\varphi_4)|0\rangle &= D_F(x_1-x_2)D_F(x_3-x_4) + D_F(x_1-x_4)D_F(x_2-x_3) \\
&+ D_F(x_1-x_3)D_F(x_2-x_4) \qquad (5.54)
\end{aligned}
$$

5.7 Computation of S-Matrix Elements

The evaluation of $\langle f|S|i\rangle$ with $S = T\exp\left(i\int d^4x\mathcal{L}_W(x)\right)$ leads to S-matrix elements of the form $\langle f|T\varphi_1(x_1)...\varphi_n(x_n)|i\rangle$. This is reduced via the Wick theorem to normal-ordered products and propagators,

$$
T\varphi_1...\varphi_n \;=\; :\varphi_1...\varphi_n: + \sum :\varphi_1...\underbrace{\varphi_i...\varphi_k}...\varphi_n: + ... \text{ (all contractions)} \; . \qquad (5.55)
$$

We first look at a real scalar field ϕ. Be the initial state

$$|i\rangle = |\vec{p}_1, \vec{p}_2, ...\vec{p}_n\rangle = a^\dagger(\vec{p}_1)a^\dagger(\vec{p}_2)...a^\dagger(\vec{p}_n)|0\rangle \ . \tag{5.56}$$

And analogously for the final state $|f\rangle$. The scalar field is decomposed into components, which only contain creation and annihilation operators, i.e.

$$\begin{aligned}
\phi &= \phi^{(+)} + \phi^{(-)} \quad \text{with} & (5.57)\\
\phi^{(+)}(x) &= \int \frac{d^3p}{(2\pi)^3 2\omega_k} a(\vec{p}) \exp(-ipx) \equiv \int_p a(\vec{p}) \exp(-ipx) \quad \text{and} & (5.58)\\
\phi^{(-)}(x) &= \int_p a^\dagger(\vec{p}) \exp(ipx) \ , & (5.59)
\end{aligned}$$

where $\omega = p_0 = \sqrt{\vec{p}^2 + m^2}$. As $\phi^{(+)}$ only contains annihilation operators, it holds that

$$\phi^{(+)}(x)|0\rangle = 0 \ . \tag{5.60}$$

And furthermore

$$\begin{aligned}
[\phi^{(+)}(x), a^\dagger(\vec{p})] &= \int_{p'} [a(\vec{p}\,') \exp(-ip'x), a^\dagger(\vec{p})] = \int_{p'} \exp(-ip'x)[a(\vec{p}\,'), a^\dagger(\vec{p})] \\
&= \int d^3p' \exp(-ip'x)\delta(\vec{p} - \vec{p}\,') = \exp(-ipx) \ . & (5.61)
\end{aligned}$$

Thereby we have

$$\phi^{(+)}|\vec{p}\rangle = \phi^{(+)}a^\dagger(\vec{p})|0\rangle = \exp(-ipx)|0\rangle \tag{5.62}$$

and

$$\begin{aligned}
\phi^{(+)}|\vec{p}_1...\vec{p}_k\rangle &= \phi^{(+)}a^\dagger(\vec{p}_1)...a^\dagger(\vec{p}_k)|0\rangle \\
&= a^\dagger(\vec{p}_1)\phi^{(+)}a^\dagger(\vec{p}_2)...a^\dagger(\vec{p}_k)|0\rangle + \exp(-ip_1x)a^\dagger(\vec{p}_2)...a^\dagger(\vec{p}_k)|0\rangle \\
&= \sum_{n=1}^{k} \underbrace{\phi^{(+)}a^\dagger(\vec{p}_n)}_{\text{I-----I}}\ a^\dagger(\vec{p}_1)...\not{a^\dagger(\vec{p}_n)}...a^\dagger(\vec{p}_k)|0\rangle \quad \text{with} & (5.63)\\
\underbrace{\phi^{(+)}(x)a^\dagger(\vec{p})}_{\text{I-----I}} &\equiv \exp(-ipx) \ . & (5.64)
\end{aligned}$$

From this follows that

$$\phi^{(+)}(x_1)...\phi^{(+)}(x_N)|\vec{p}_1...\vec{p}_n\rangle = 0 \qquad \text{for } N > n \ . \tag{5.65}$$

For example we can have for N scalar fields

$$\begin{aligned}
\langle f| \ :\phi...\phi: \ |i\rangle &= \langle f|\phi^{(+)}...\phi^{(+)}|i\rangle + \langle f|\phi^{(-)}\phi^{(+)}...\phi^{(+)}|i\rangle + ... + ... \\
&= \langle f|\underbrace{\phi^{-}...\phi^{(-)}}_{N_-}\underbrace{\phi^{(+)}...\phi^{(+)}}_{N_+}|i\rangle + ... & (5.66)
\end{aligned}$$

This expression is only then non-zero, if $N_+ \leq n_i$ and $N_- \leq n_f$. Here n_i (n_f) is the number of particles in the initial (final) state and N_+ (N_-) is the number of fields with positive and negative frequencies, respectively, in the individual expressions above ($N_+ + N_- = N$). If $N_+ < n_i$ and $N_- < n_f$, then

$$\langle f|\phi^{(-)}...\phi^{(-)}\phi^{(+)}...\phi^{(+)}|i\rangle \sim \langle f'|i'\rangle \text{ only } \neq 0 \ , \text{ if } |i'\rangle = |f'\rangle \ . \tag{5.67}$$

5.7.1 Modifications for Dirac and Vector fields

We had the Fourier decompositions

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 2\omega_p} \left(a(\vec{p}) \exp(-ipx) + a^\dagger \exp(ipx)\right) \equiv \phi^{(+)} + \phi^{(-)} \tag{5.68}$$

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3 2\omega_p} \sum_{s=\pm\frac{1}{2}} \left(a_s(\vec{p}) u_s(\vec{p}) \exp(-ipx) + b_s^\dagger(\vec{p}) v_s(\vec{p}) \exp(ipx)\right) \equiv \psi^{(+)} + \psi^{(-)} \tag{5.69}$$

$$A_\mu(x) = \int d\tilde{p} \sum_\lambda (c_\lambda(\vec{p}) \epsilon_\mu^{(\lambda)}(\vec{p}) \exp(-ipx) + c_\lambda^\dagger(\vec{p}) \epsilon_\mu^{(\lambda)*}(\vec{p}) \exp(ipx)) \equiv A_\mu^{(+)} + A_\mu^{(-)} \ . \tag{5.70}$$

Analogously as before, we obtain

$$\underbrace{\psi^{(+)}(x) a_s^\dagger(\vec{p})} = u_s(\vec{p}) \exp(-ipx) \tag{5.71}$$

$$\underbrace{\bar{\psi}^{(-)}(x) b_s^\dagger(\vec{p})} = \bar{v}_s(\vec{p}) \exp(-ipx) \tag{5.72}$$

$$\underbrace{a_s(\vec{p}) \bar{\psi}^{(+)}(x)} = \bar{u}_s(\vec{p}) \exp(ipx) \tag{5.73}$$

$$\underbrace{b_s(\vec{p}) \psi^{(-)}(x)} = v_s(\vec{p}) \exp(ipx) \tag{5.74}$$

$$\underbrace{A_\mu^{(+)} c_\lambda^\dagger(\vec{p})} = \epsilon_\mu^{(\lambda)}(\vec{p}) \exp(-ipx) \tag{5.75}$$

$$\underbrace{c_\lambda(\vec{p}) A_\mu^{(-)}} = \epsilon_\mu^{(\lambda)*}(\vec{p}) \exp(ipx) \ . \tag{5.76}$$

We furthermore have to take into account that

$$\psi^{(+)}(x) a_s^\dagger(\vec{p}_1) a_t^\dagger(\vec{p}_2)|0\rangle = \underbrace{\psi^{(+)} a_1^\dagger} a_2^\dagger|0\rangle - \underbrace{\psi^{(+)} a_1^\dagger a_2^\dagger}|0\rangle \ . \tag{5.77}$$

5.7.2 Example: Quantum Elektrodynamics

Starting point is the interaction Lagrangian

$$\mathcal{L}_I = -e\bar{\psi}\gamma^\mu \psi A_\mu = -e\bar{\psi}_\alpha \gamma^\mu_{\alpha\beta} \psi_\beta A_\mu = -e\bar{\psi}_\alpha \psi_\beta \gamma^\mu_{\alpha\beta} A_\mu \ . \tag{5.78}$$

In the following the spinor indices (α, β) will be suppressed. We look at Möller scattering, i.e. the process

$$e^-(p_1, r_1) + e^-(p_2, r_2) \to e^-(p_3, r_3) + e^-(p_4, r_4) \ . \tag{5.79}$$

This means that we have in the initial and in the final state

$$|i\rangle = a_{r_1}^\dagger(\vec{p}_1) a_{r_2}^\dagger(\vec{p}_2)|0\rangle \quad \text{und} \quad |f\rangle = a_{r_3}^\dagger(\vec{p}_3) a_{r_4}^\dagger(\vec{p}_4)|0\rangle \ . \tag{5.80}$$

There are four particles, which participate in the process. We hence have to have at least four fields ψ in the T product, so that $N_+ = n_i$ und $N_- = n_f$. We hence have to look at least at $\mathcal{L}_I^2$. We have

$$\begin{aligned}\langle f|S|i\rangle &= \langle f|i\rangle + \left\langle f \left| iT \int d^4x\, \mathcal{L}_W(x) \right| i \right\rangle \\ &+ \frac{i^2e^2}{2!} \int d^4x \int d^4y \langle f|T(\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x))(\bar{\psi}(y)\gamma^\nu\psi(y)A_\nu(y))|i\rangle + \ldots \end{aligned} \quad (5.81)$$

The following remarks are at order:

1. The fields A_μ have to be Wick-contracted, as there are no photons in the external states. Otherwise they are in the normal product which vanishes:

$$\langle 0|\ :A_\mu A_\nu:\ |0\rangle \ = \ 0 \quad \text{and} \quad \langle 0|A_\mu A_\nu|0\rangle \neq 0\ . \quad (5.82)$$

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2. No field ψ must be Wick-contracted, as for the external states four ψ are required.

3. Thereby the following possible terms are obtained: a) First term, cf. Fig. 5.1 (left). b) Second term, cf. Fig. 5.1 (right). Through a fermionic exchange the relative minus sign compared to the first case is obtained.

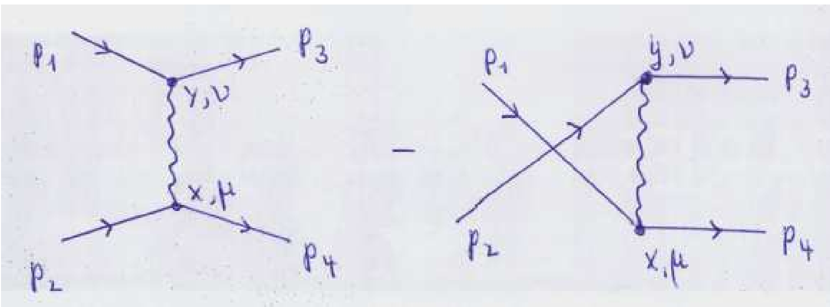

Figure 5.1: Feynman diagrams, which contribute to the process $e^-e^- \to e^-e^-$.

4. Third term: This term corresponds to the first one with the interchange $x \leftrightarrow y$.

5. Fourth term This term corresponds to the second term with the interchange $x \leftrightarrow y$.

Thereby one obtains an additional factor 2, which cancels the prefactor 1/(2!) in the expansion of the S matrix. Expressed through formulae, one obtains for the diagrams 1 and 3:

$$-e^2 \int d^4x \int d^4y \langle 0|a(3)a(4)T[\ \bar{\psi}(x)\gamma^\mu\psi(x)\ A_\mu(x)\ \bar{\psi}(y)\gamma^\nu\psi(y)\ A_\nu(y)]a^\dagger(1)a^\dagger(2)|0\rangle$$

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$$= \ -e^2 \int d^4x \int d^4y\, (A_\mu(x)A_\nu(y)) \cdot \bar{u}(4)\gamma^\mu u(2) \exp(-ip_2x + ip_4x)\bar{u}(3)\gamma^\nu u(1) \exp(-ip_1y + ip_3y) \quad (5.83)$$

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Here $a(1)$ stands for $a_{r_1}(\vec{p}_1)$ and $u(1)$ for $u_{r_1}(\vec{p}_1)$ etc. We use

$$\underbrace{(A_\mu(x)A_\nu(y))} \equiv \langle 0|TA_\mu(x)A_\nu(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{\exp(-ik(x-y))}{k^2+i\epsilon}(-ig_{\mu\nu}) \tag{5.84}$$

Performing the integration leads to

$$\int d^4x \to (2\pi)^4\delta^{(4)}(p_4-p_2-k) \quad \text{und} \quad \int d^4y \to (2\pi)^4\delta^{(4)}(p_3-p_1+k)\ . \tag{5.85}$$

This describes the momentum conservation at the vertices. Performing the integration of k leads to

$$\int \frac{d^4k}{(2\pi)^4}(2\pi)^8\delta^{(4)}(p_4-p_2-k)\delta^{(4)}(p_3-p_1+k) = (2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)\ . \tag{5.86}$$

This corresponds to the conservation of the total momentum. Thereby we have

$$(1)+(3) = -e^2(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)\,\bar{u}(\vec{p}_4)\gamma^\mu u(\vec{p}_2)\bar{u}(\vec{p}_3)\gamma^\nu u(\vec{p}_1)\frac{-ig_{\mu\nu}}{k^2+i\epsilon}\ , \tag{5.87}$$

$$\text{with } k = p_1-p_3 = p_4-p_2\ . \tag{5.88}$$

$(2)+(4)$ is calculated analogously, where $1 \leftrightarrow 2$. We thus have for the total matrix element

$$-e^2(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4) \left[\frac{-ig_{\mu\nu}}{(p_4-p_2)^2+i\epsilon}(\bar{u}_4\gamma^\mu u_2)(\bar{u}_3\gamma^\nu u_1) - \frac{-ig_{\mu\nu}}{(p_2-p_3)^2+i\epsilon}(\bar{u}_3\gamma^\nu u_2)(\bar{u}_4\gamma^\mu u_1)\right]\ . \tag{5.89}$$

Fields that are not contracted, do not contribute beause of the normal ordering, as $\langle 0|\ :\ \phi...\phi\ :\ |0\rangle = 0$. *All* contractions of the fields among each other (propagators) and with external fields ($\exp(\pm ipx), u(p), \epsilon_\mu(p)$) have to be performed. At the order e^2 we had obtained the contributions Fig. 5.1. The order e^3 vanishes. The order e^4 then delivers again a contribution:

$$\langle 0|a_4a_3T[\bar{\psi}_1\gamma_\mu\psi_1\,A_1^\mu\,\bar{\psi}_2\gamma_\nu\psi_2\,A_2^\nu\,\bar{\psi}_3\gamma_\rho\psi_3\,A_3^\rho\,\bar{\psi}_4\gamma_\sigma\psi_4\,A_4^\sigma]a_1^\dagger a_2^\dagger|0\rangle\ . \tag{5.90}$$

Here $\psi_1 \equiv \psi(x_1)$, $A_1^\mu \equiv A^\mu(x_1)$ etc. In the following, the contractions are looked at graphically. The four terms $\bar{\psi}\psi A$ are represented by four vertices, cf. Fig. 5.2. Thereby the Feynman diagram Fig. 5.3 is obtained. The loop corresponds to the expression

$$(\tilde{\bar{\psi}}(x_2)\psi(x_2)A(x_2))(\tilde{\bar{\psi}}(x_3)\psi(x_3)A(x_3)) = AA\tilde{\bar{\psi}}_\alpha\psi_\alpha\tilde{\bar{\psi}}_\beta\psi_\beta = AA(-1)\psi_\beta\tilde{\bar{\psi}}_\alpha\psi_\alpha\tilde{\bar{\psi}}_\beta\ . \tag{5.91}$$

Here α and β are Spinor indices and

$$\tilde{\bar{\psi}}(x_2) \equiv \bar{\psi}(x_2)\gamma^\nu \equiv \tilde{\bar{\psi}}_\alpha \tag{5.92}$$

$$\psi(x_2) \equiv \psi_\alpha \tag{5.93}$$

$$\tilde{\bar{\psi}}(x_3) \equiv \bar{\psi}(x_3)\gamma^\rho \equiv \tilde{\bar{\psi}}_\beta \tag{5.94}$$

$$\psi(x_3) \equiv \psi_\beta \tag{5.95}$$

$$AA \equiv A_\nu(x_2)A_\rho(x_3)\ . \tag{5.96}$$

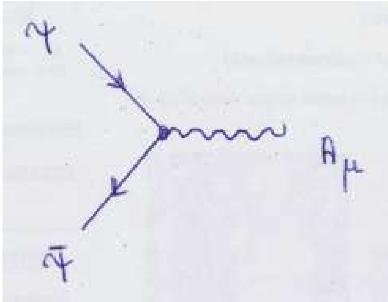


Figure 5.2: Vertex $\bar{\psi}\psi A$.

The expression Eq. (5.91) is a number $\in \mathbb{C}$, so that the trace of this number can be performed,

$$AA(-1) \cdot \mathrm{Tr}(\underbrace{\psi_\beta \tilde{\bar{\psi}}_\alpha} \underbrace{\psi_\alpha \tilde{\bar{\psi}}_\beta}) = A_\nu(x_2) A_\rho(x_3)(-1) \cdot \mathrm{Tr}(S_F(x_3 - x_2)\gamma^\nu \cdot S_F(x_2 - x_3)\gamma^\rho) \, . \tag{5.97}$$

The expression $(-1)\mathrm{Tr}$ corresponds to a closed Fermion loop. The following integrations have to be performed:

$$\int d^4x_1 ... \int d^4x_4 \underbrace{\int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4l_1}{(2\pi)^4} \int \frac{d^4l_2}{(2\pi)^4}}_{\text{Propagators}} \exp(-i(p_1 - p_3)x_1)$$
$$\exp(-i(p_2 - p_4)x_4) \, \exp(-ik_1(x_2 - x_1)) \, \exp(-ik_2(x_3 - x_4)) \, \exp(-il_1(x_2 - x_3))$$
$$\exp(-il_2(x_3 - x_2)) \cdot \text{Functions of } k_1, k_2, l_1, l_2, p_1, p_2, p_3, p_4 \, . \tag{5.98}$$

Performing the integration over x_1, x_2, x_3 and x_4 leads to

$$\begin{aligned} & \int d^4k_1 \, d^4k_2 \, d^4l_1 \, d^4l_2 \, \delta^{(4)}(p_1 - p_3 - k_1) \, \delta^{(4)}(p_2 - p_4 - k_2) \, \delta^{(4)}(k_2 - l_1 + l_2) \, \delta^{(4)}(k_1 + l_1 - l_2) \\ = & \int d^4l_1 \, d^4l_2 \, \delta^{(4)}(p_1 - p_3 + l_1 - l_2) \, \delta^{(4)}(p_2 - p_4 - l_1 + l_2) \\ = & \int d^4l_1 \, \delta^{(4)}(p_2 - p_4 - l_1 + p_1 - p_3 + l_1) = (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \int \frac{d^4l_1}{(2\pi)^4} \, . \end{aligned} \tag{5.99}$$

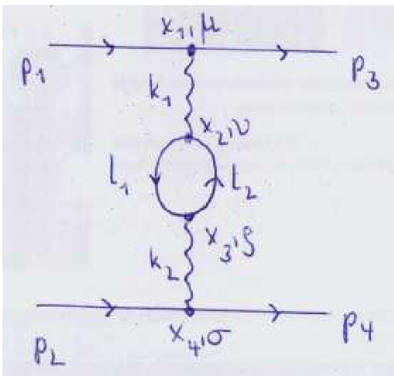


Figure 5.3: Feynman Diagramm, which contrtibutes at order e^2 in $e^-e^- \to e^-e^-$.

Hence for each closed loop, one has to integrate over the respective loop momentum. Here, this is l_1. Further contractions at order e^4 are shown in the diagrams Fig. 5.4

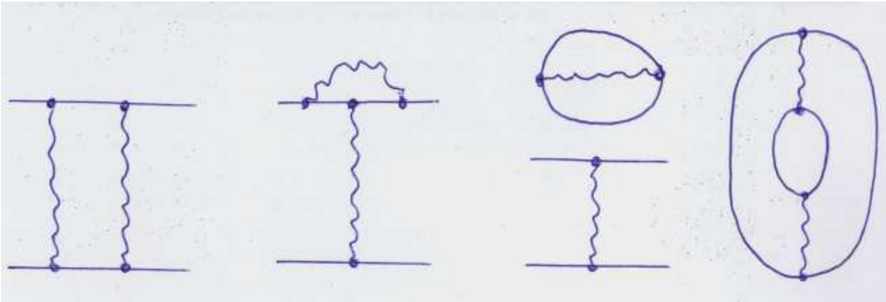
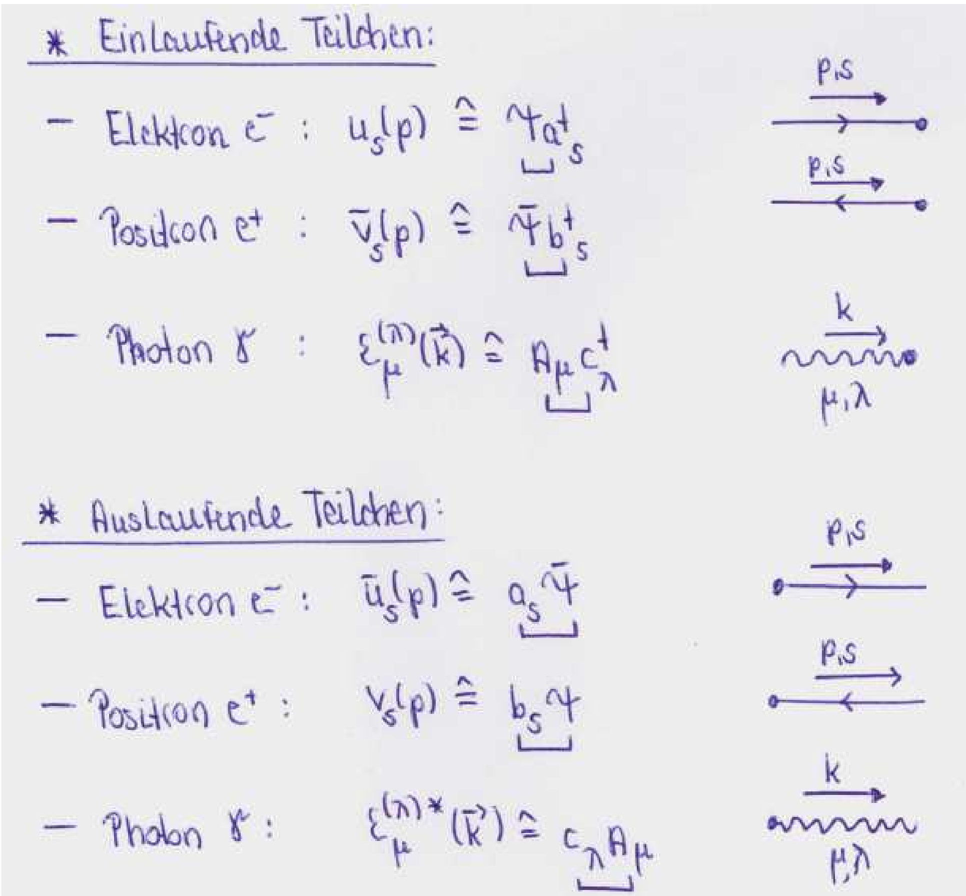
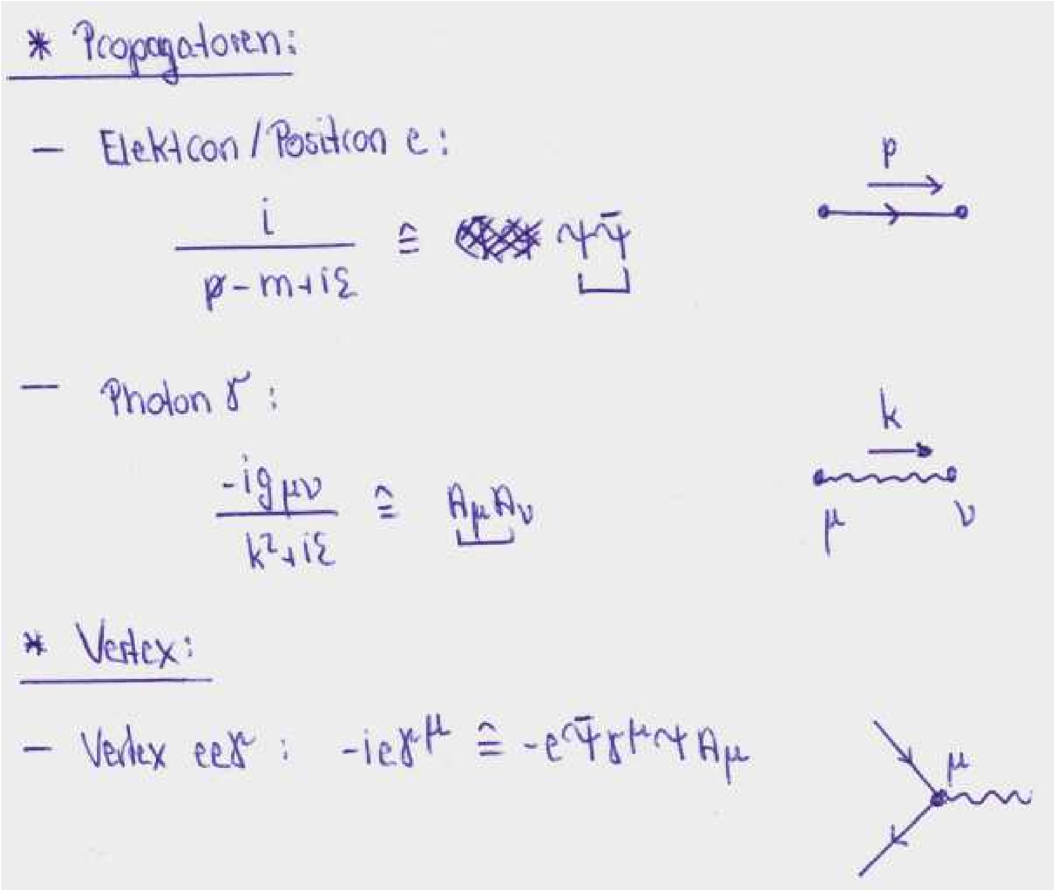


Figure 5.4: Further Feynman diagrams, which contribute at order e^4.

5.7.3 Feynman Rules of the QED

Thereby we can now list the Feynman rules of quantum electrodynamics:





Furthermore, we have the following rules

* At each vertex we have four-momentum conservation.
* Multiply the amplitude (see below) by $(2\pi)^4\delta^{(4)}(P_f - P_i)$, where P_f (P_i) is the sum over the momenta of the incoming (outgoing) particles.
* For each closed loop an integral over the corresponding four-momentum has to be performed, $\frac{d^4 l}{(2\pi)^4}$.
* Each fermion loop receives a factor (-1) and the trace has to be performed.
* When external fermions are interchanged, a factor (-1) has to be added.

We write

$$S = 1 + iT \; . \tag{5.100}$$

And we have the matrix element

$$\langle f|T|i\rangle = (2\pi)^4\delta^{(4)}(P_f - P_i)\mathcal{M}_{fi} \; . \tag{5.101}$$

Here $\mathcal{M}_{fi}$ is the *Feynman amplitude*. It is obtained by connecting the external particles through vertices and propagators and summing up all possibilities. Here only *connected* diagrams are taken into account (without proof). The mathematical expression is written up according to the Feynman rules. Here fermion lines are run through in the **opposite direction of the fermion flow**!

5.7.4 Example: Pair Annihilation

We look at pair annihilation, i.e. the process

$$e^-(p_1, s_1) + e^+(p_2, s_2) \to \gamma^\mu(k_1, \lambda_1) + \gamma^\nu(k_2, \lambda_2) \tag{5.102}$$

The diagrams that contribute to the process are shown in Fig. 5.5.

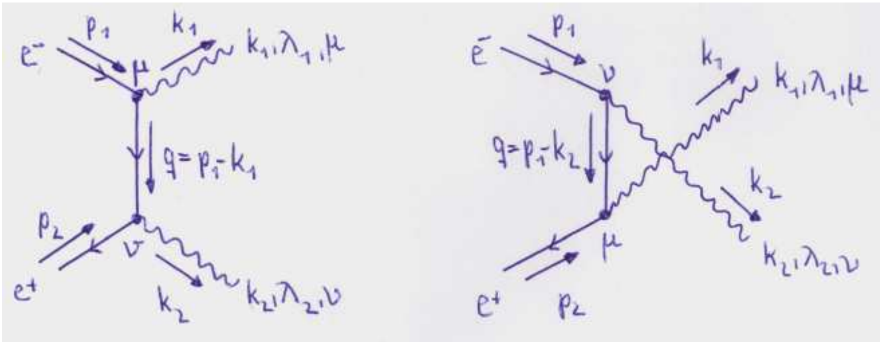


Figure 5.5: Diagrams contributing to pair annihilation. (Remark: In the second diagram the momentum of the fermion propagator between the two photon vertices has to read $q' = p_1 - k_2$.)

The total amplitude $\mathcal{M}$ is given by the sum of the amplitudes corresponding to the two diagrams, i.e.

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 \,. \tag{5.103}$$

For $\mathcal{M}_1$ we get

$$\mathcal{M}_1 = \bar{v}(p_2)(-ie\gamma^\nu)\frac{i}{\not{q} - m + i\epsilon}(-ie\gamma^\mu)u(p_1)\epsilon_\mu^{(\lambda_1)*}(k_1)\,\epsilon_\nu^{(\lambda_2)*}(k_2)\,. \tag{5.104}$$

The calculation of the amplitude $\mathcal{M}_2$ is left as an exercise.

5.7.5 Example: Compton Scattering

We look at Compton scattering, i.e. the process

$$e^-(p_1, s_1) + \gamma^\mu(k_1, \lambda_1) \to e^-(p_2, s_2) + \gamma^\nu(k_2, \lambda_2) \tag{5.105}$$

The Feynman diagrams that contribute are shown in Fig. 5.6.
The total amplitude $\mathcal{M}$ is given by the sum of the two amplitudes corresponding to the two diagrams, i.e.

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 \,. \tag{5.106}$$

For $\mathcal{M}_1$ we get by using the Feynman rules,

$$\mathcal{M}_1 = \bar{u}(p_2)(-ie\gamma^\nu)\frac{i}{\not{q}_1 - m}(-ie\gamma^\mu)u(p_1)\,\epsilon_\mu^{(\lambda_1)}(k_1)\,\epsilon_\nu^{(\lambda_2)*}(k_2) \quad \text{with } q_1 = p_1 + k_1 \,. \tag{5.107}$$

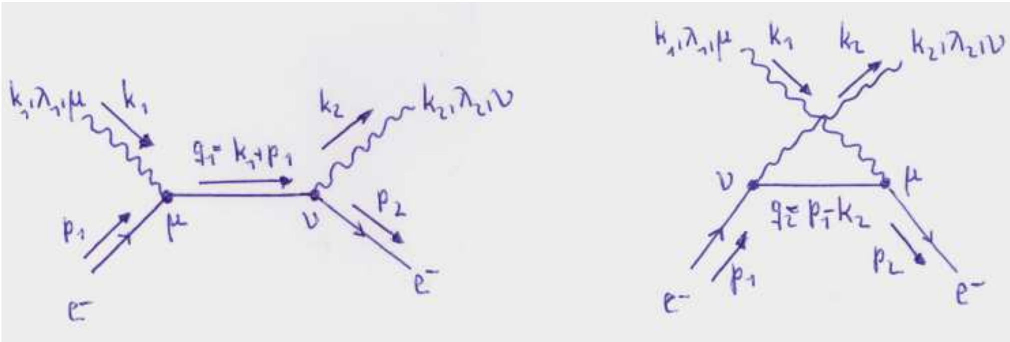


Figure 5.6: Feynman diagrams contributing to Compton scattering.

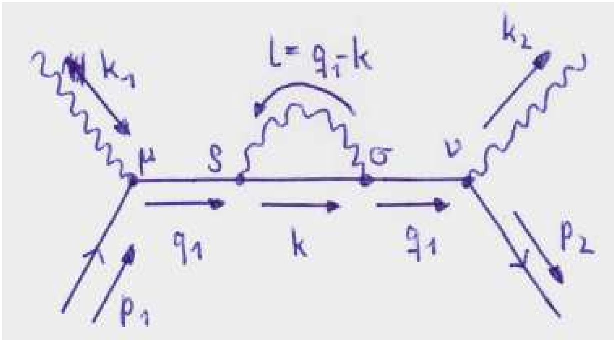


Figure 5.7: Diagram contributing to Compton scattering at order e^4. (Remark: The arrow for the momentum l has to be in the opposite direction.)

And for $\mathcal{M}_2$ we find

$$\mathcal{M}_2 = \bar{u}(p_2)(-ie\gamma^\mu)\frac{i}{\not{q}_2 - m}(-ie\gamma^\nu)u(p_1)\,\epsilon_\mu^{(\lambda_1)}(k_1)\,\epsilon_\nu^{(\lambda_2)*}(k_2) \quad \text{with } q_2 = p_1 - k_2\,. \tag{5.108}$$

A contribution of order e^4 is given by the diagram shown in Fig. 5.7. It contains a loop. The corresponding amplitude reads

$$\begin{aligned}\mathcal{M} &= (-ie)^4 \int \frac{d^4k}{(2\pi)^4}\,\bar{u}(p_2)\gamma^\nu \frac{i}{\not{q}_1 - m}\,\gamma^\sigma \frac{i}{\not{k} - m}\,\gamma^\rho \frac{i}{\not{q}_1 - m}\,\gamma^\mu\, u(p_1)\,\epsilon_\mu^{(\lambda_1)}(k_1)\epsilon_\nu^{(\lambda_2)*}(k_2) \\ &\quad \times \frac{-ig_{\rho\sigma}}{l^2 + i\epsilon}\,.\end{aligned} \tag{5.109}$$

And we have

$$q_1 = p_1 + k_1 = p_2 + k_2\,. \tag{5.110}$$

5.8 The Cross Section

5.8.1 Scattering Cross Section

We look at two bunches of particles, which scatter with each other, cf. Fig. 5.8. For the number N_{event} of events it holds

$$N_{event} \sim \rho_A l_A \rho_B l_B \cdot F\,. \tag{5.111}$$

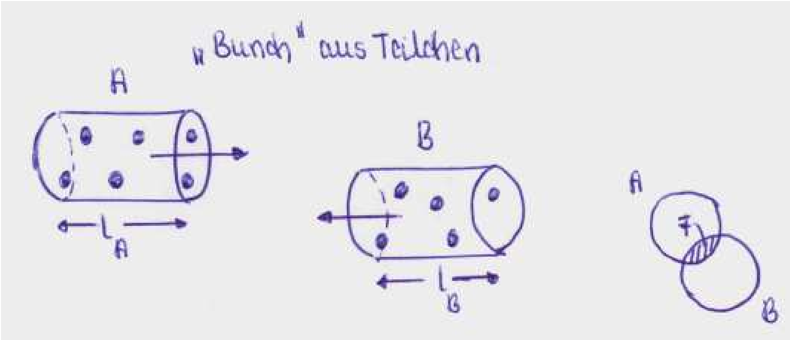


Figure 5.8: Particle Scattering.

Here ρ denotes the particle number density, i.e. the number of particles per volume. And F is the scattering cross section area. The proportionality factor σ is the cross section. We have

$$\sigma = \frac{N_{event} \cdot F}{(\rho_A l_A F)(\rho_B l_B F)} = \frac{N_{event} \cdot F}{N_A \cdot N_B} \,. \tag{5.112}$$

Here $N_{A,B}$ are the particles in the effective area. Be $N_B = 1$, then the scattering experiment looks as in Fig. 5.9.

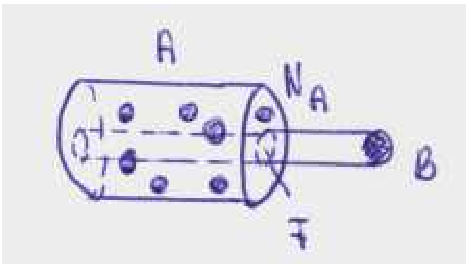


Figure 5.9: Particle scattering with $N_B = 1$.

And we have

$$\sigma = \frac{N_{event}}{N_A} \cdot F \,. \tag{5.113}$$

If B is a steel ball, then $N_A = N_{event}$ and thereby $\sigma = F$. Otherwise this is the effection cross section area the particles which are scattered off the bunch A. We hence have

$$\sigma = \frac{N_{event}}{\frac{N_A}{F}} = \frac{\frac{N_{event}}{T}}{j_A} = \frac{\text{Number of events per time} T}{\text{incoming current density}} \,. \tag{5.114}$$

The cross section has the dimension of a surface in natural units, $1/\text{mass}^2$.

5.8.2 Phase Space Flux Factor

The S mastrix element for the transition from the inital to the final state is given by

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \, \delta \left(\sum_f p_f - \sum_i p_i \right) \langle f|\mathcal{T}|i\rangle \,. \tag{5.115}$$

Through this the Lorentz-invariant $\mathcal{T}$ matrix is defined. The δ function describes the energy-momentum conservation. We look at the following reaction,

$$a_1(p_1) + a_2(p_2) \to b_1(q_1) + ... + b_n(q_n) \ . \tag{5.116}$$

The cross section is given by

$$\sigma = \frac{\text{Transition rate } (\equiv \text{ number of events/time})}{\text{flux of the incoming particles}} \ . \tag{5.117}$$

Nominator and denominator are defined in the given Lorenz-system. For example a_2 can be at rest, while a_1 is incoming. Be $i \neq f$. We work with momentum eigenstates in the initial and final state. Alle 1-particle states $|p_i\rangle, |q_i\rangle$ are normalized to 1 w.r.t. the Lorentz-invariant measure

$$\frac{d^3q_i}{(2\pi)^3 2q_i^0} = \frac{d^4q_i}{(2\pi)^3}\delta(q_i^2 - m_i^2)\theta(q_i^0) \ . \tag{5.118}$$

We first look at distinguishable paricles $b_1, ..., b_n$ in the final state. Then in the calculation of S_{fi} resp.$\mathcal{T}_{fi}$ the used final state

$$|b_1...b_n\rangle = |b_1\rangle \otimes ... \otimes |b_n\rangle \tag{5.119}$$

is normalized to 1 with respect to the measure

$$\Pi_{j=1}^n d^3\tilde{q}_j \ . \tag{5.120}$$

In the initial state we have

$$|a_1, a_2\rangle = |a_1(\vec{p}_1)\rangle \otimes |a_2(\vec{p}_2)\rangle \ . \tag{5.121}$$

We assume that we have one particle a_1 and one particle a_2, respectively, in one normalization volume. Be V the volume (e.g. the laboratory), then we have

$$\langle a_1(\vec{p}_1)|a_1(\vec{p}_1)\rangle = 2p_1^0(2\pi)^3\delta(\vec{p}_1 - \vec{p}_1) = (2\pi)^3 \cdot 2p_1^0 \frac{1}{(2\pi)^3}\int_V d^3x \exp(i(\vec{p}_1 - \vec{p}_1)\vec{x}) = 2p_1^0 V \ . \tag{5.122}$$

Thereby

$$\frac{||a_1(\vec{p}_1)\rangle|^2}{V} = \frac{\text{probability density in momentum space for this state}}{V} \ . \tag{5.123}$$

We now look at the transition probability for the reaction (5.116). With the requirement of one particle a_1 and a_2, respectively, per normalization volume V we have

$$dw_{fi} = \frac{[(2\pi)^4\delta^{(4)}(Q-P)]^2|\mathcal{T}_{fi}|^2\Pi_{j=1}^n d\tilde{q}_j}{\underbrace{2p_1^0 2p_2^0 V^2}_{\text{norm of the initial state}}} \ , \tag{5.124}$$

with $P \equiv p_1 + p_2$ und $Q \equiv \sum_{j=1}^n q_j$. For the square of the δ function we use Fermi's trick: We consider the interaction to be turned on in a volume V and during the time T. Thereby we have

$$[(2\pi)^4\delta^{(4)}(Q-P)]^2 = \int_{VT} d^4x\, e^{i(Q-P)\cdot x}\,(2\pi)^4\delta^{(4)}(Q-P) = (2\pi)^4\delta^{(4)}(Q-P)\, V\cdot T \tag{5.125}$$

And thereby we have the transition rate

$$\frac{dw_{fi}}{T} = \frac{1}{V \cdot 4p_1^0 p_2^0}(2\pi)^4 \, \Pi_{j=1}^n d\tilde{q}_j \, \delta^{(4)}(Q-P) \, |\mathcal{T}_{fi}|^2 \, . \tag{5.126}$$

This formula holds in each interial system. We now those the rest frame of particle a_2. Be $m_{1,2}$ the masses of $a_{1,2}$. In the rest frame of a_2 we have

$$p_{2L}^0 = m_2 \tag{5.127}$$

and for the momentum of particle a_1 it holds in this system ($\equiv$ laboratory system) that

$$|\vec{p}_{1L}| = \left((p_{1L}^0)^2 - m_1^2\right)^{\frac{1}{2}} = \frac{w(s, m_1^2, m_2^2)}{2m_2} \, , \tag{5.128}$$

with the function

$$w(x,y,z) \equiv (x^2 + y^2 + z^2 - 2xy - 2xz - 2yz)^{\frac{1}{2}} \, . \tag{5.129}$$

This can be seen as following: The center-or-mass energy in the c.m. (center-of-mass) frame be s. It is given by

$$s = (p_1 + p_2)^2 \, , \tag{5.130}$$

where p_1, p_2 are the four-momenta in the c.m. frame. The four-momenta in the laboratory frame are given by

$$p_{1L} = \begin{pmatrix} p_{1L}^0 \\ \vec{p}_{1L} \end{pmatrix} \quad \text{and} \quad p_{2L} = \begin{pmatrix} m_2 \\ 0 \end{pmatrix} \tag{5.131}$$

Because of Lorentz invance it holds

$$s = (p_1 + p_2)^2 = p_{1L}^2 + 2p_{1L}^0 m_2 + m_2^2 = m_1^2 + 2p_{1L}^0 m_2 + m_2^2 \Rightarrow \tag{5.132}$$

$$p_{1L}^0 = \frac{s - m_1^2 - m_2^2}{2m_2} \, . \tag{5.133}$$

Since we only assumed 1 particle a_1 per volume V, the flux of a_1 in the laboratory frame L is

$$\Phi_{a_{1L}} = \frac{1}{V}|\vec{v}_{1L}| = \frac{1}{V}\frac{|\vec{p}_{1L}|}{p_{1L}^0} = \frac{1}{V p_{1L}^0}\frac{w(s, m_1^2, m_2^2)}{2m_2} \, . \tag{5.134}$$

Thereby we have

$$V 2p_1^0 2p_2^0 = V \, 2m_2 \, \frac{2w(s, m_1^2, m_2^2)}{\Phi_{a_{1L}} V \, 2m_2} = 2\frac{w(s, m_1^2, m_2^2)}{\Phi_{a_{1L}}} \, . \tag{5.135}$$

And hence we find for the differential cross section

$$\begin{aligned} d\sigma \equiv \frac{dw_{fi}/T}{\Phi_{a_1}} &= \frac{1}{2w(s, m_1^2, m_2^2)} \Pi_{j=1}^n \frac{d^3 q_j}{(2\pi)^3 2q_j^0} \\ &\cdot (2\pi)^4 \delta^{(4)}\left(\sum_{j=1}^n q_j - p_1 - p_2\right) \cdot |\langle b_1(\vec{q}_1)...b_n(\vec{q}_n)| \, \mathcal{T} \, |a_1(\vec{p}_1) a_2(\vec{p}_2)\rangle|^2 \, . \end{aligned} \tag{5.136}$$

The various elements are

$$\frac{1}{2w(s,m_1^2,m_2^2)} \equiv \text{ flux factor } , \tag{5.137}$$

and

$$|\langle b_1(\vec{q}_1)...b_n(\vec{q}_n)|\,\mathcal{T}\,|a_1(\vec{p}_1)a_2(\vec{p}_2)\rangle|^2 = \text{ matrix element } . \tag{5.138}$$

sowie

$$(2\pi)^4\delta^{(4)}\left(\sum_{j=1}^n q_j - p_1 - p_2\right)\Pi_{j=1}^n \frac{d^3q_j}{(2\pi)^3 2q_j^0} = \text{ phase space } \equiv \text{dLIPS}(n) \ , \tag{5.139}$$

The phase space is universal and Lorentz invarian. Also the matrix element is Lorentz invariant, depends on the specific process, however. When we integrate over the phase space, we will write in the following

$$\int \text{dLIPS}(n) \equiv \text{ Lorentz-invariant phase space } . \tag{5.140}$$

We can write the flux factor as

$$2w(s,m_1^2,m_2^2) = 4m_2|\vec{p}_{1L}| = 4(E_1^2m_2^2 - m_1^2m_2^2)^{\frac{1}{2}} = 4[(p_{1L}p_{2L})^2 - m_1^2m_2^2]^{\frac{1}{2}} \ . \tag{5.141}$$

The $d\sigma$ is the differential cross section, which is integrable over all momentum configurations. We hence have for the cross section (we omit the index L in the following) im folgenden weg)

$$\int d\sigma = \frac{1}{4[(p_1p_2)^2 - m_1^2m_2^2]^{\frac{1}{2}}}\int |\langle f|\mathcal{T}|i\rangle|^2\,\text{dLIPS}(n) \ . \tag{5.142}$$

<u>Further remarks:</u>

1. For particles with spin we proceed as follows: For each unpolarised particle in the initial state, one has to average over its spin states. For each non-observed spin polarisation of a particle in the final state one has to sum over the corresponding spin states. This means in this case we have:

$$|\mathcal{T}_{fi}|^2 \to \sum_{\text{Spins}}{}' |\mathcal{T}_{fi}|^2 \equiv \frac{1}{(2s_{a_1}+1)(2s_{a_2}+1)} \sum_{\text{Spins im Endzustand}} |\mathcal{T}_{fi}|^2 \ , \tag{5.143}$$

   where s_{a_1}, s_{a_2} is the spin of a_1 and a_2, respectively. For the spin sums one has to use (see exercise sheet)

$$\sum_{s=1}^{2} u(\vec{p},s)_\alpha \bar{u}(\vec{p},s)_\beta = (\not{p}+m)_{\alpha\beta} \tag{5.144}$$

$$\sum_{s=1}^{2} v(\vec{p},s)_\alpha \bar{v}(\vec{p},s)_\beta = (\not{p}-m)_{\alpha\beta} \tag{5.145}$$

   Note that the weight factor reads $1/(2s+1)$ only for massive particles. If it is a photon or a gluon, then the weight factor is 2, as massless particles only have 2 physical polarisation states.

2. If n_0 of the n final state particles are identical, no matter if it is bosons or fermions, then the state (5.119) is not correctly normalised. The factor $1/\sqrt{n_0!}$ is missing. This means that in this case $d\sigma$ or dw_{fi} has to be multiplied by $1/n_0!$. Hence

$$d\sigma = \frac{1}{2w} \left(\Pi_j d\tilde{q}_j\right) (2\pi)^4 \delta^{(4)}(Q-P) \sum_{\text{Spins}}{}^{'} |\mathcal{T}_{fi}|^2 \cdot \frac{1}{n_0!} \; . \tag{5.146}$$

3. The cross section usually is given in the unit *barn*. It holds that

$$\frac{1}{\text{GeV}^2} = 0.389 \cdot 10^{-3} \text{ barn} \; . \tag{5.147}$$

The cross section is connected with the number of events via the so-called luminosity L:

$$N = L \cdot \sigma \; . \tag{5.148}$$

The luminosity can be through a very exactly known reference cross section, which has been calculated very precisely in theory. For example in electron-positron colliders the cross section of the Bhabha-scatterin $e^+ + e^- \to e^+ + e^-$ is used, at hadron colliders like the LHC e.g. the production of W bosons, $pp \to W^+W^-$, is used. The luminosity hence is roughly spoken a measure for the incoming current. Thus we have for LEP (1993-1998) an integrated luminosity of about 200 pb^{-1}, for the Tevatron Run II (4/02-9/11) - D0 - 11.9 fb^{-1}/delivered (10.7 fb^{-1}/recorded), LHC ATLAS (1/18-6/18) 20.8 fb^{-1} delivered (19.5 fb^{-1} recorded), LHC CMS (July '23) 245.54 fb^{-1} delivered (266.42 fb^{-1} recorded), the high-luminosity phase of the LHC 3000 fb^{-1} shall be gathered per experiment.

Chapter 6

The Process $e^+e^- \to \mu^+\mu^-$

6.1 The Muon

◇ The muon has a mass of

$$m_\mu = 105.6 \text{ MeV} \;\approx 200\, m_e \,. \tag{6.1}$$

◇ All other properties (quantum numbers, interactions etc.) are identical with those of the electron.

◇ The interaction with the photon is given by (Fig. 6.1) $-ie\gamma^\nu$.

Figure 6.1: The photon-muon-muon vertex.

◇ However, the muon is unstable. It decays via the weak interaction as (cf. Fig. 6.2)

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu \,. \tag{6.2}$$

Here $\bar{\nu}_e$ is the anti-electron-neutrino and ν_μ the muon-neutrino. We set $m_{\nu_{e/\mu}} \approx 0$.

◇ The computation of the decay width results in

$$\Gamma_\mu = \frac{G_F^2 \cdot m_\mu^5}{192\,\pi^3} = 3.001 \cdot 10^{-19} \text{ GeV} \;\;, \tag{6.3}$$

where G_F is the Fermi constant,

$$G_F = 1.16637 \cdot 10^{-5} \frac{1}{\text{GeV}^2} \,. \tag{6.4}$$

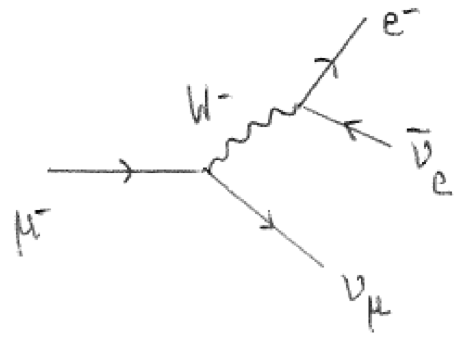


Figure 6.2: The muon decay.

And for the lifetime one gets

$$\frac{\hbar}{\Gamma_\mu} = \frac{6.582 \cdot 10^{-25} \text{ GeV } \cdot \text{ s}}{3.001 \cdot 10^{-19} \text{ GeV}} \approx 2.19 \cdot 10^{-6} \text{ s} . \tag{6.5}$$

6.2 The cross section for $e^+e^- \to \mu^+\mu^-$

There is only one diagram, which contributes to the process

$$e^-(p_1, s_1) + e^+(p_2, s_2) \to \mu^-(p_3, s_3) + \mu^+(p_4, s_4) , \tag{6.6}$$

cf. Fig. 6.3. Because of energy-momentum conservation we have for the photon momentum

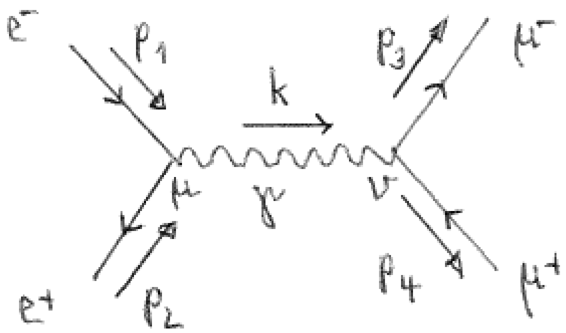


Figure 6.3: The process $e^+e^- \to \mu^+\mu^-$.

k,

$$k = p_1 + p_2 = p_3 + p_4 . \tag{6.7}$$

We introduce the abbreviation $u_1 \equiv u(\vec{p}_1, s_1)$ etc. Thereby we get for the diagram by applying the Feynman rules,

$$\mathcal{M} = \bar{v}_{2\alpha}(-ie)\gamma^\mu_{\alpha\beta} u_{1\beta} \frac{-ig_{\mu\nu}}{k^2 + i\epsilon} \bar{u}_{3\delta}(-ie)\gamma^\nu_{\delta\rho} v_{4\rho} . \tag{6.8}$$

For$\mathcal{M}^\dagger$ we get

$$\mathcal{M}^\dagger = (ie)^2 \bar{v}_{4\sigma}\gamma^{\nu'}_{\sigma\tau}u_{3\tau}\frac{ig_{\mu'\nu'}}{k^2 - i\epsilon}\bar{u}_{1\eta}\gamma^{\mu'}_{\eta\omega}v_{2\omega} \; . \tag{6.9}$$

Here we have used that

$$(\bar{v}_a\gamma^\mu u_b)^\dagger = (v_a^\dagger\gamma_0\gamma^\mu u_b)^\dagger = u_b^\dagger\gamma^{\mu\dagger}\gamma_0^\dagger v_a = u_b^\dagger\gamma_0\gamma^\mu \underbrace{\gamma_0\gamma_0}_{=1} v_a = \bar{u}_b\gamma^\mu v_a \; , \tag{6.10}$$

because

$$\bar{v} \equiv v^\dagger\gamma_0 \qquad \text{and} \qquad \bar{u} \equiv u^\dagger\gamma_0 \tag{6.11}$$

$$\gamma_0^\dagger = \gamma_0 \tag{6.12}$$

$$\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0 \; . \tag{6.13}$$

We calculate the unpolarised cross section. This means, that we average over the polarisations in the initial state and sum over the polarisations in the final state,

$$\frac{1}{4}\sum_{s_1,s_2,s_3,s_4}|\mathcal{M}|^2 \; . \tag{6.14}$$

Application of the relations (5.144), (5.145) leads to

$$\begin{aligned}\frac{1}{4}\sum_{s_1,s_2,s_3,s_4}|\mathcal{M}|^2 &= \frac{e^4}{4k^4}(\not{p}_2 - m_e)_{\omega\alpha}\gamma^\mu_{\alpha\beta}(\not{p}_1 + m_e)_{\beta\eta}\gamma^{\mu'}_{\eta\omega}g_{\mu\nu}g_{\mu'\nu'}(\not{p}_3 + m_\mu)_{\tau\delta}\gamma^\nu_{\delta\rho}(\not{p}_4 - m_\mu)_{\rho\sigma}\gamma^{\nu'}_{\sigma\tau} \\ &= \frac{e^4}{4k^4}\mathrm{Tr}[(\not{p}_2 - m_e)\gamma_\mu(\not{p}_1 + m_e)\gamma_{\nu'}]\cdot\mathrm{Tr}[(\not{p}_3 + m_\mu)\gamma^\mu(\not{p}_4 - m_\mu)\gamma^{\nu'}] \; .\end{aligned} \tag{6.15}$$

Since the trace of an odd number of gamma matrices vanishes, we have

$$\mathrm{Tr}[(\not{p}_3 + m_\mu)\gamma^\mu(\not{p}_4 - m_\mu)\gamma^{\nu'}] = \mathrm{Tr}(\not{p}_3\gamma^\mu\not{p}_4\gamma^{\nu'}) - m_\mu^2\mathrm{Tr}(\gamma^\mu\gamma^{\nu'}) \; . \tag{6.16}$$

For the further evaluation we use

$$\mathrm{Tr}(\gamma^\mu\gamma^{\nu'}) = 4g^{\mu\nu'} \tag{6.17}$$

$$\begin{aligned}p_{3\alpha'}p_{4\beta'}\mathrm{Tr}(\gamma^{\alpha'}\gamma^\mu\gamma^{\beta'}\gamma^{\nu'}) &= p_{3\alpha'}p_{4\beta'}4[g^{\mu\beta'}g^{\nu'\alpha'} + g^{\mu\alpha'}g^{\nu'\beta'} - g^{\mu\nu'}g^{\alpha'\beta'}] \\ &= 4[p_3^{\nu'}p_4^\mu + p_3^\mu p_4^{\nu'} - g^{\mu\nu'}p_3\cdot p_4] \; .\end{aligned} \tag{6.18}$$

Thereby, we have for the trace altogether

$$4[p_3^{\nu'}p_4^\mu + p_3^\mu p_4^{\nu'} - p_3\cdot p_4 g^{\mu\nu'} - m_\mu^2 g^{\mu\nu'}] \; . \tag{6.19}$$

And for the other trace we find analoguously

$$4[p_{1\nu'}p_{2\mu} + p_{1\mu}p_{2\nu'} - p_1\cdot p_2 g_{\mu\nu'} - m_e^2 g_{\mu\nu'}] \; . \tag{6.20}$$

In the following, the electron mass is set to zero, $m_e = 0$. The uncertainty encountered thereby is of the order $m_e^2/m_\mu^2 = 1/200^2$. Multiplication of (6.19) and (6.20) finally leads to (check!)

$$\frac{1}{4}\sum_{s_1,s_2,s_3,s_4}|\mathcal{M}|^2 = \frac{e^4}{4k^4}16[2(p_2\cdot p_3)(p_1\cdot p_4) + 2(p_1\cdot p_3)(p_2\cdot p_4) + 2m_\mu^2(p_1\cdot p_2)] \; . \tag{6.21}$$

In the c.m. system, cf. Fig. 6.4, with the c.m. energy $\sqrt{s}$ we have for the four-vectors,

$$\begin{aligned}
p_1 &= \frac{\sqrt{s}}{2}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \; p_2 = \frac{\sqrt{s}}{2}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}, \\
p_3 &= \frac{\sqrt{s}}{2}\begin{pmatrix}1\\ \beta\sin\theta\cos\phi\\ \beta\sin\theta\sin\phi\\ \beta\cos\theta\end{pmatrix}, \; p_4 = \frac{\sqrt{s}}{2}\begin{pmatrix}1\\ -\beta\sin\theta\cos\phi\\ -\beta\sin\theta\sin\phi\\ -\beta\cos\theta\end{pmatrix},
\end{aligned} \tag{6.22}$$

with

$$\beta = \sqrt{1-\frac{4m_\mu^2}{s}}\,. \tag{6.23}$$

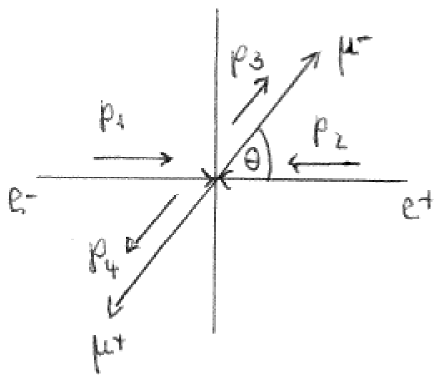


Figure 6.4: Center-of-mass frame.

For the individual needed dot-products we thereby have

$$p_1\cdot p_2 = \frac{s}{2}\,, \quad k^2 = (p_1+p_2)^2 = s \tag{6.24}$$

$$p_1\cdot p_3 = p_2\cdot p_4 = \frac{s}{4}(1-\beta\cos\theta) \quad \text{und} \quad p_1\cdot p_4 = p_2\cdot p_3 = \frac{s}{4}(1+\beta\cos\theta)\,. \tag{6.25}$$

Thereby we find

$$\frac{1}{4}\sum_{s_1,s_2,s_3,s_4}|\mathcal{M}|^2 = e^4[1+\beta^2\cos^2\theta+(1-\beta^2)]\,. \tag{6.26}$$

Here we used that

$$\frac{m_\mu^2}{s} = \frac{1}{4}(1-\beta^2)\,. \tag{6.27}$$

For the differential cross section we still need the phase space. It is for the $2\to 2$ process given by

$$\int \mathrm{dLIPS}(2) = \frac{1}{2w(s,m_e^2,m_e^2)}\int \frac{d^3p_3}{(2\pi)^3 2p_3^0}\frac{d^3p_4}{(2\pi)^3 2p_4^0}(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)$$

$$\begin{aligned}
&= \frac{1}{(2\pi)^2}\frac{1}{2w}\int\frac{d^3p_3}{2p_3^0}\int d^4p_4\,\delta(p_4^2-m_4^2)\,\theta(p_4^0)\,\delta^{(4)}(p_1+p_2-p_3-p_4)\\
&= \frac{1}{4\pi^2}\frac{1}{2s}\int\frac{d^3p_3}{2p_3^0}\underbrace{\delta((p_1+p_2-p_3)^2-m_\mu^2)}_{s-2\sqrt{s}p_3^0+m_\mu^2-m_\mu^2}\\
&= \frac{1}{8\pi^2 s}\int\underbrace{\frac{|\vec{p}_3|^2 d|\vec{p}_3|}{2p_3^0}}_{\frac{|\vec{p}_3|p_3^0 dp_3^0}{2p_3^0}}\underbrace{d\cos\theta d\phi}_{d\Omega}\frac{1}{2\sqrt{s}}\delta\left(p_3^0-\frac{\sqrt{s}}{2}\right)\\
&= \frac{1}{8\pi^2 s}\frac{1}{4\sqrt{s}}\frac{\sqrt{s}\beta}{2}\int d\Omega\\
&= \frac{\beta}{64\pi^2 s}\int d\Omega\ . \qquad (6.28)
\end{aligned}$$

Here we have used that

$$w(s,m_e^2=0,m_e^2=0)=s \qquad \text{and} \qquad |\vec{p}_3|=\frac{\sqrt{s}\beta}{2}\ . \qquad (6.29)$$

The thereby one finds for the unpolarised differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right)^{\text{unpol}} = \frac{\beta}{64\pi^2 s}\left(\frac{1}{4}\sum_{s_1,s_2,s_3,s_4}|\mathcal{M}|^2\right) = \frac{e^4\beta}{64\pi^2 s}[2+\beta^2(\cos^2\theta-1)]\ . \qquad (6.30)$$

The total cross section is obtained through integration over the space angle Ω,

$$\int d\Omega = 2\pi\int_{-1}^{1} d\cos\theta\ . \qquad (6.31)$$

With the finestructure constant

$$\alpha=\frac{e^2}{4\pi}\approx\frac{1}{137} \qquad (6.32)$$

and $\beta\approx 1$ the total cross section is obtained as

$$\sigma=\frac{4\pi\alpha^2}{3s}\ . \qquad (6.33)$$

At a c.m. energy of e.g. $\sqrt{s}=90$ GeV one obtains

$$\sigma=\frac{4\pi}{3}\left(\frac{1}{137}\right)^2\cdot\left(\frac{1}{90}\right)^2\cdot 0.389\cdot 10^{-3}\ \text{barn}\ \approx 11\ \text{pb}\ . \qquad (6.34)$$

6.3 The Cross Section for $e^+e^- \to$ Hadrons

Hadrons are particles which participate in the strong interaction. Hadrons are e.g. p, n, π, η, ρ etc. They are composed of quarks and gluons.

If one calculates the cross section of the process

$$e^+ + e^- \to \ \text{hadrons}\ , \qquad (6.35)$$

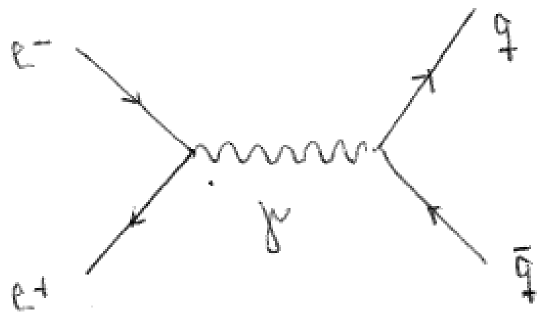


Figure 6.5: Feynman diagram for the process $e^+ + e^- \to q + \bar{q}$.

then in first order of the strong coupling constant it is identical with the process

$$e^+ + e^- \to q + \bar{q} \ . \tag{6.36}$$

The Feynman diagram contributing to the process is depicted in Fig. 6.5. There are 6 quarks, up (u), down (d), charm (c), strange (s), top (t) and bottom (b). The Feynman rule for the $\gamma - q - \bar{q}$ vertex, Fig. 6.6, reads

$$ieQ_q\gamma^\mu \qquad \text{with} \quad Q_U = \frac{2}{3}(U = u, c, t) \text{ and } Q_D = -\frac{1}{3}(D = d, s, b) \ . \tag{6.37}$$

Taking into account the three colors of the quarks one obtains

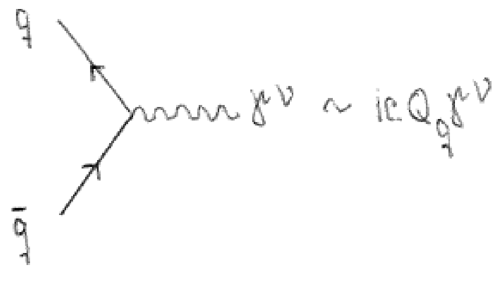


Figure 6.6: The vertex $\gamma - q - \bar{q}$.

$$\sigma = \frac{4\pi\alpha^2}{3s} \sum_q Q_q^2 \cdot 3 \ . \tag{6.38}$$

And thereby

$$R = \frac{\sigma(e^+ + e^- \to \text{ hadrons})}{\sigma(e^+ + e^- \to \mu^+ + \mu^-)} = 3 \cdot \sum_q |Q_q|^2 \ . \tag{6.39}$$

Note that for $2m_s < \sqrt{s} < 2m_c$ we have

$$R = 3 \cdot [Q_u^2 + Q_d^2 + Q_s^2] = 3 \cdot \left[\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right] = 2 \ . \tag{6.40}$$

For $2m_c < \sqrt{s} < 2m_b$ we have

$$R = 2 + 3 \cdot \left(\frac{2}{3}\right)^2 = \frac{10}{3} \tag{6.41}$$

and for $2m_b < \sqrt{s} < 2m_t$

$$R = \frac{11}{3} \; . \tag{6.42}$$

6.4 Higher Order Corrections to $e^+ + e^- \to \mu^+ + \mu^-$

QED corrections of different orders in e to $e^+ + e^- \to \mu^+ + \mu^-$ are depicted in Fig. 6.7.

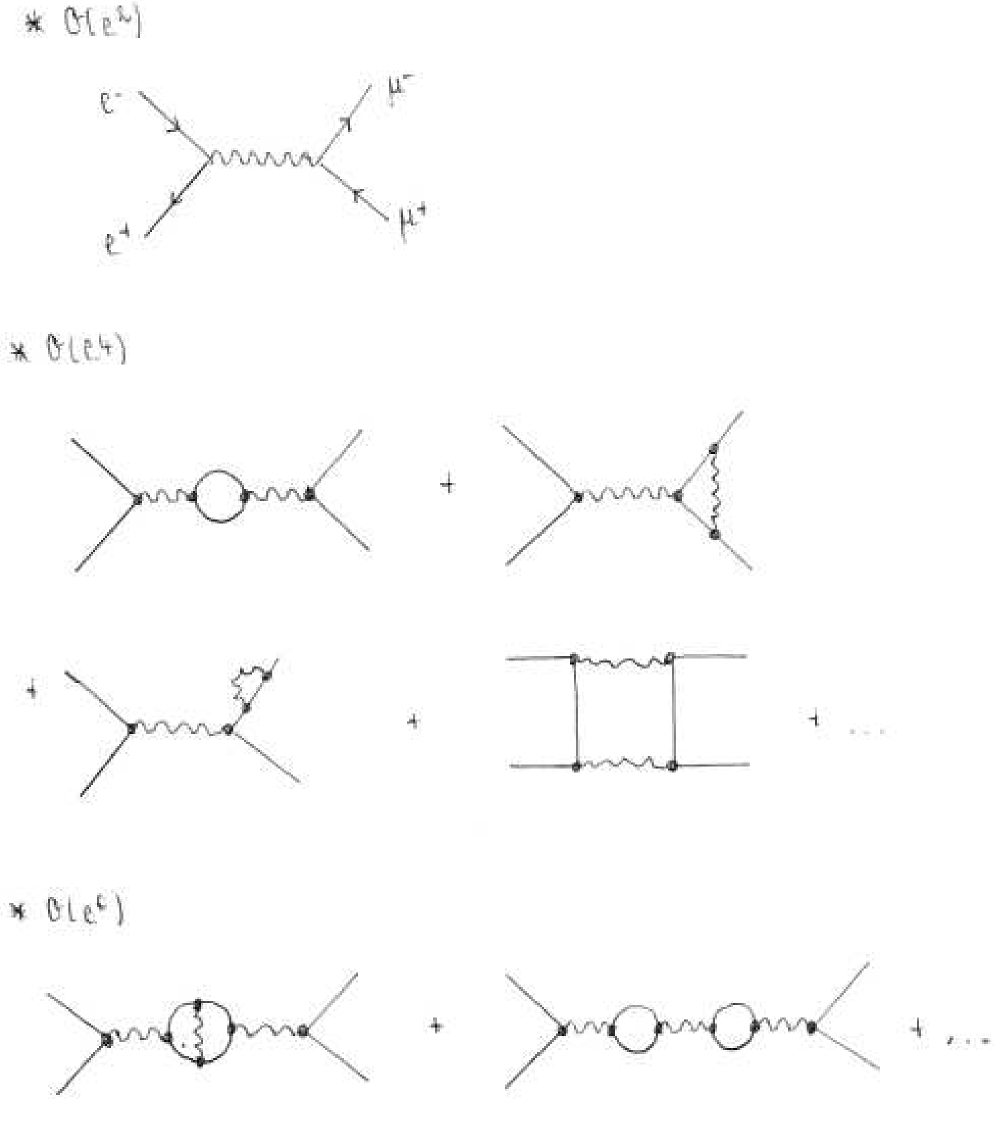


Figure 6.7: Higher-order corrections to $e^+ + e^- \to \mu^+ + \mu^-$.

6.5 Corrections to the Photon Propagator

The corrections to the photon propagator are depicted in Fig. 6.8. The Feynman diagrams

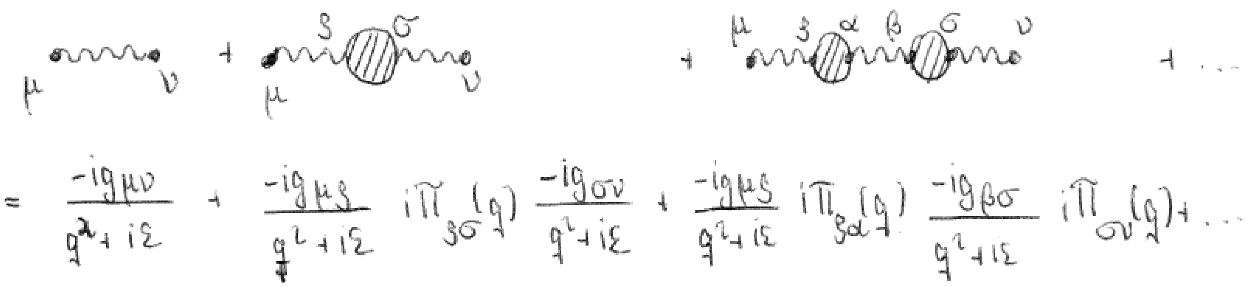


Figure 6.8: Higher-order corrections to the photon propagator. The last summand below the picture is wrong. It has to read: $\frac{-ig_{\mu\rho}}{q^2+i\epsilon} i\Pi_{\rho\alpha}(q) \frac{-ig_{\alpha\beta}}{q^2+i\epsilon} i\Pi_{\beta\sigma}(q) \frac{-ig_{\sigma\nu}}{q^2+i\epsilon}$.

contributing to the vacuum polarisation $i\Pi_{\mu\nu}(q)$ are shown in Fig. 6.9. The vacuum polari-

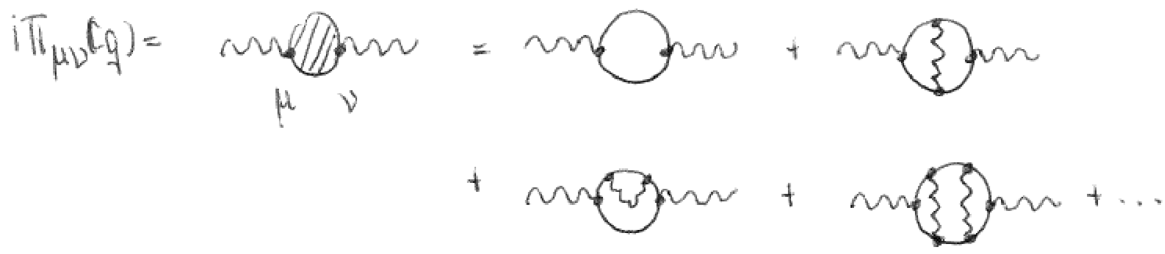


Figure 6.9: Diagrams contributing to the vacuum polarisation $i\Pi_{\mu\nu}(q)$.

sation tensor $\Pi^{\mu\nu}(q)$ can be rewritten as

$$\Pi_{\mu\nu}(q) = Ag_{\mu\nu} + Bq_\mu q_\nu \; . \tag{6.43}$$

This follows from the Ward identity (see below)

$$q_\mu \Pi^{\mu\nu} = 0 = q_\nu \Pi^{\mu\nu} \tag{6.44}$$

so that

$$A = -Bq^2 = q^2\Pi(q^2) \; . \tag{6.45}$$

Thereby $\Pi_{\mu\nu}(q)$ can be expressed through a scalar quantity $\Pi(q^2)$:

$$\Pi_{\mu\nu} = (q^2 g_{\mu\nu} - q_\mu q_\nu)\Pi(q^2) \; . \tag{6.46}$$

Note, however, that $\Pi_{\mu\nu}$ always couples to a conserved current, cf. Fig. 6.10. The equation $q_\nu j^\nu = 0$ follows from $\partial_\nu j^\nu = 0$. The $q_\mu q_\nu$ terms hence do not give any contribution. Thereby Fig. 6.11 follows by means of the geometric series.

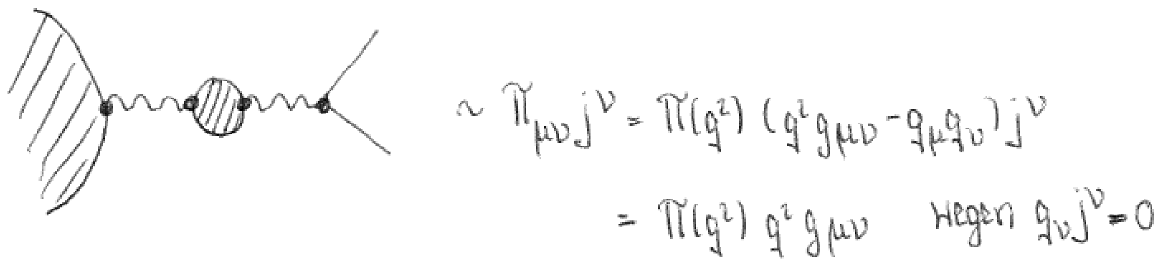


Figure 6.10: Coupling to a conserved current.

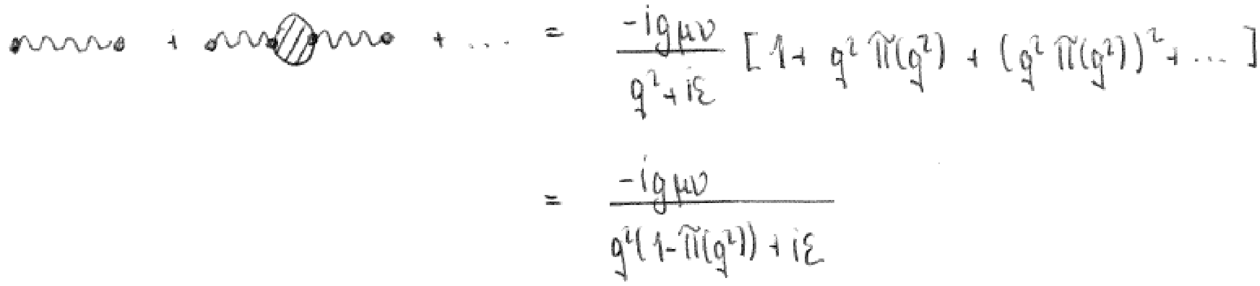


Figure 6.11: Loop-corrected photon propagator. The first line in the formula is wrong. It has to read $\frac{-ig_{\mu\nu}}{q^2+i\epsilon}[1 + \Pi(q^2) + (\Pi(q^2))^2 + ...]$.

Ward identity: We look at a T-matrix element with one incoming photon. It is given by (cf. Fig. 6.12)

$$\langle ...|T|...\gamma_\mu(k)\rangle = \mathcal{M}_\mu(k, ...)\epsilon^\mu(k) \ . \tag{6.47}$$

Because of gauge invariance ($A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu\Lambda(x)$) we can replace the photon state by an equivalent state without changing the physics. We look at the polarisation vector

$$\epsilon'_\mu = \epsilon_\mu + ck_\mu \ , \quad \text{with } c \in \mathbb{C} \ . \tag{6.48}$$

The polarisation vectors ϵ'_μ and ϵ_μ describe equivalent states,

$$\epsilon^\mu a^\dagger_\mu(k)|0\rangle \sim \epsilon'^\mu a^\dagger_\mu(k)|0\rangle \ . \tag{6.49}$$

This means that it has to hold that

$$\mathcal{M}_\mu(k, ...)\epsilon^\mu = \mathcal{M}_\mu(k, ...)\epsilon'^\mu \ , \tag{6.50}$$

hence

$$k^\mu \mathcal{M}_\mu(k, ...) = 0 \ . \tag{6.51}$$

This is an important relation to check your calculation.

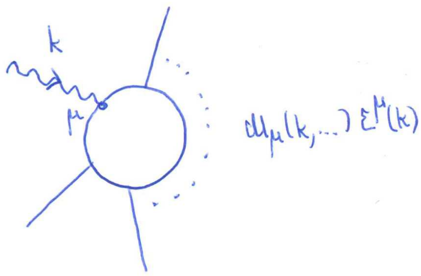


Figure 6.12: Diagram with one photon.

Chapter 7

Loop Diagrams

7.1 Example: ϕ^4-Theory

We consider the scalar ϕ^4 theory with the Lagrangian

$$\mathcal{L} = :\frac{1}{2}(\partial_\mu\phi)^2: \; -\frac{1}{2}m^2 \; :\phi^2: \; -\frac{\lambda}{4!} \; :\phi^4: \quad \text{with} \quad \mathcal{L}_{WW} = -\frac{\lambda}{4!} \; :\phi^4: \; . \tag{7.1}$$

We have for the S-matrix up to order n

$$S = \sum_n \frac{i^n}{n!} T \int dx' \int dx'' ... \int dx^{(n)} \mathcal{L}_{WW}(x') \, \mathcal{L}_{WW}(x'') ... \mathcal{L}_{WW}(x^{(n)}) \; . \tag{7.2}$$

For the $2 \to 2$ scattering we have to determine the matrix element

$$\langle 0|a(3)a(4) \, S \, a^\dagger(1)a^\dagger(2)|0\rangle \; . \tag{7.3}$$

We have in Born approximation (also tree level $\equiv$ contribution of lowest order in the coupling constant/at leading order) Fig. 7.1. It is

$$S_{fi}^{\text{Born}} = \langle 0 | a(3) \, a(4) \, \frac{(-i)\lambda}{4!} \int dx' \, :\phi(x')\,\phi(x')\,\phi(x')\,\phi(x'): \, a^\dagger(1) \, a^\dagger(2) \, |0\rangle$$

Figure 7.1: Contribution at tree level (leading order).

$$\underbrace{\phi(x')a^\dagger(1)} \;=\; \exp(-ip_1x') \tag{7.4}$$

We have to consider 4! permutations. Thereby we obtain

$$S_{fi}^{\text{Born}} = -i\lambda \int dx' \; \exp[i(p_3+p_4-p_1-p_2)x'] = -i\lambda \cdot (2\pi)^4\delta^{(4)}(p_3+p_4-p_1-p_2) \tag{7.5}$$

And thereby

$$\mathcal{T}_{fi} = -\lambda \; . \tag{7.6}$$

At order λ^2 we have to determine

$$\left\langle 0 \left| a(3)a(4)\frac{1}{2}\cdot\frac{(-i\lambda)^2}{(4!)^2}\int dx'dx''T\left[: \phi(x')^4 : : \phi(x'')^4 :\right] a^\dagger(1)a^\dagger(2)\right| 0\right\rangle . \tag{7.7}$$

Via the Wick theorem we obtain typical contributions:

1. The contractions of $a^\dagger(1), a^\dagger(2)$ with $a(3), a(4)$ vanish, as because of $\vec{p}_3 \neq \vec{p}_1$

$$\langle 0|a(3)a^\dagger(1)|0\rangle \sim \delta(\vec{p}_3 - \vec{p}_1) = 0 . \tag{7.8}$$

   Thereby only the contractions of ϕa, $\phi a^\dagger$ or $\phi(x')\phi(x'')$ are left.

2. Since $\mathcal{L}_{WW}$ is normal ordered, there are no contributions, in which $\phi(x')\phi(x')$ are contracted (so-called tadpole diagrams). A non-vanishing contribution is hence only obtained, if 2 of the operators $a, a^\dagger$, respectively, are contracted with 2 fields $\phi(x')$ and the other two operators are contracted with $\phi(x'')$. We hence have

$$S_{fi}^{\text{1-Schleifen}} = \frac{(-i\lambda)^2}{2\cdot(4!)^2}\int dx'dx''T[\text{Term 1 } + \text{Term 2} + \text{Term3 }] . \tag{7.9}$$

A contribution to term 1 is given in Fig. 7.2. There are altogether $(4!)^2$ possibilities of contraction, which lead to the same result. For example the contribution in Fig. 7.3 once

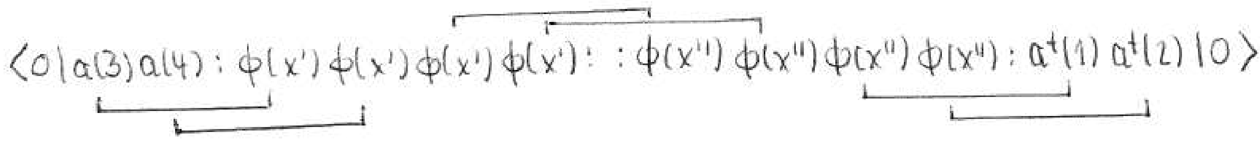


Figure 7.2: Contribution to term 1.

again gives the same result. A contribution to term 2 is given in Fig 7.4. There are again

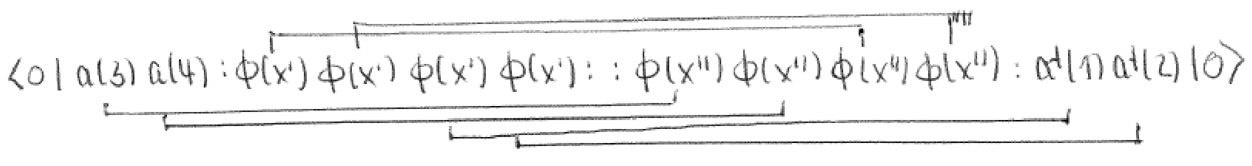


Figure 7.3: Further contribution of type term 1.

$(4!)^2$ possibilities of contraction, which lead to the same result. A contribution to term 3 is given in Fig 7.5. There are again $(4!)^2$ possibilities of contraction, which lead to the same result.

Evaluation of the first term S_1 leads to

$$\begin{aligned} S_1 &= \frac{(-i\lambda)^2}{2}\int dx'dx''\exp[i(p_3x' + p_4x' - p_1x'' - p_2x'')]\cdot\langle 0|T\phi(x')\phi(x'')|0\rangle \\ &\quad \cdot\langle 0|T\phi(x')\phi(x'')|0\rangle . \end{aligned} \tag{7.10}$$

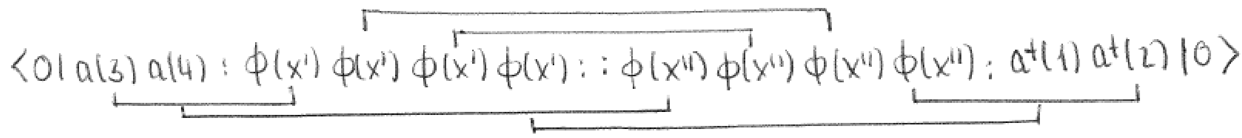


Figure 7.4: Contribution to term 2.

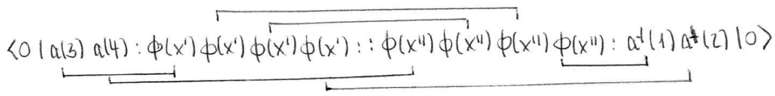


Figure 7.5: Contribution to term 3.

We have

$$\langle 0|T\phi(x')\phi(x'')|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}\,\exp[-ip(x'-x'')]\;. \tag{7.11}$$

And thereby

$$\begin{aligned} S_1 &= \frac{(-i\lambda)^2}{2}\int dx'\int dx''\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\,\frac{i}{p^2-m^2+i\epsilon}\frac{i}{q^2-m^2+i\epsilon} \\ &\quad\cdot\exp[ix'(p_3+p_4-p-q)+ix''(-p_1-p_2+p+q)]\;. \end{aligned} \tag{7.12}$$

Integration over x' und x'' leads to two δ functions, hence

$$\begin{aligned} S_1 &= \frac{(-i\lambda)^2}{2}\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\,\frac{i}{p^2-m^2+i\epsilon}\frac{i}{q^2-m^2+i\epsilon} \\ &\quad\cdot(2\pi)^4\delta^{(4)}(p_3+p_4-p-q)\cdot(2\pi)^4\delta^{(4)}(-p_1-p_2+p+q)\;. \end{aligned} \tag{7.13}$$

Integration over p leads to

$$\begin{aligned} S_1 &= \frac{(-i\lambda)^2}{2}\cdot(2\pi)^4\delta^{(4)}(p_3+p_4-p_1-p_2) \\ &\quad\cdot\int\frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m^2+i\epsilon}\frac{i}{[(q-p_1-p_2)^2-m^2+i\epsilon]}\;. \end{aligned} \tag{7.14}$$

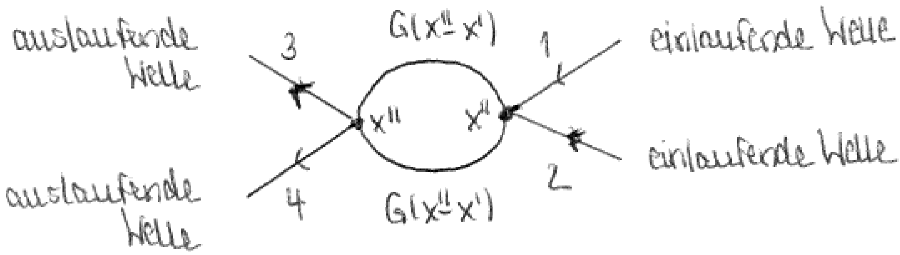


Figure 7.6: Interpretation of S_1 in local space.

The interpretation of S_1 in local space is depicted in Fig. 7.6. The interpretation of S_1 in momentum space is depicted in Fig. 7.7. The contribution of S_2 is depicted in Fig. 7.8. The

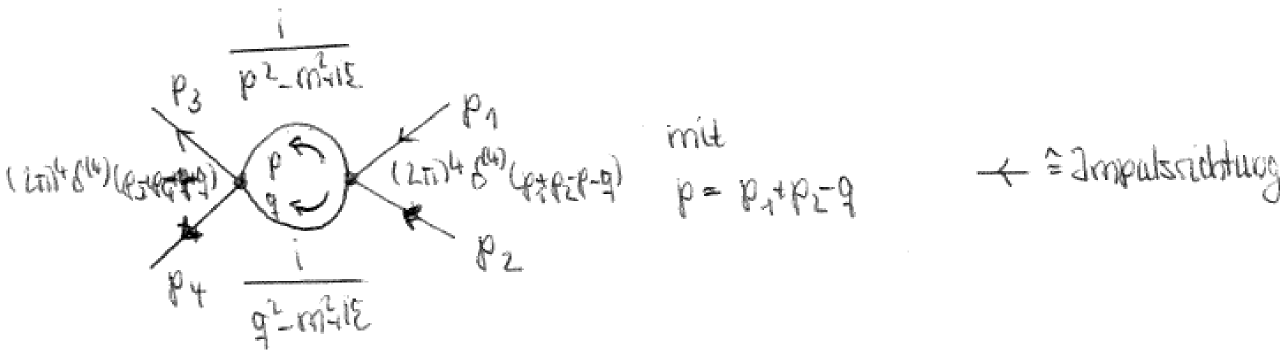


Figure 7.7: Interpretation of S_1 in momentum space.

contribution of term 1 only depends on

$$s = (p_1 + p_2)^2 \ . \tag{7.15}$$

The contribution of term 2 only depends on

$$t = (p_1 - p_3)^2 \ . \tag{7.16}$$

The contribution of term 3 (not shown here) only depends on

$$u = (p_1 - p_4)^2 \ . \tag{7.17}$$

We have the Feynman rules

$$\text{Propagator:} \quad \frac{i}{p^2 - m^2 + i\epsilon} \tag{7.18}$$

and

$$\text{Vertex:} \quad (-i\lambda) \ . \tag{7.19}$$

At each vertex we have energy-momentum conservation. We have to integrate over loop momenta. And we have one δ function for the overall energy-momentum conservation.

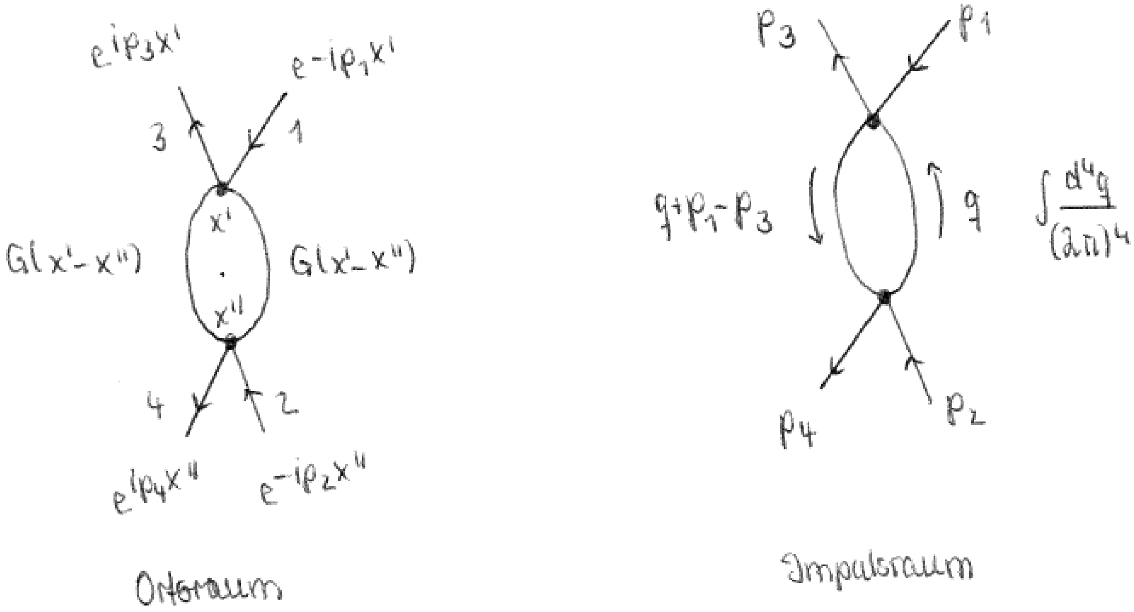


Figure 7.8: Interpretation of S_2 in local space (left) and in momentum space (right).

7.2 Divergence Behaviour

The evaluation of the loop integrals shows that some of them are divergent for four-momenta $\to \infty$. Such divergences are called ultraviolet (UV) divergences. We assume that the divergence behaviour of the integrals can be determined by counting the powers of the momenta (=”power counting”). We look at the following example:

$$\int d^4q \frac{1}{q^2 - m^2 + i\epsilon} \cdot \frac{1}{(q - p_1 - p_2)^2 - m^2 + i\epsilon} \; . \tag{7.20}$$

For fixed p_1, p_2, m and large q^2 the integrand is proportional to $1/(q^2)^2$. Im momentum space we get a logarithmic divergence. In local space this corresponds to the square of a singular function. Next we look at a six-particle reaction, cf. Fig. 7.9, and investigate it w.r.t. possible divergences. The higher-energy behaviour of the corresponding integral is given by

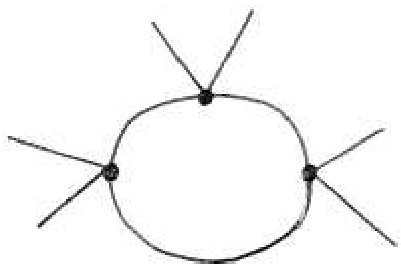


Figure 7.9: Loop diagram with six external legs.

$$\int \frac{d^4q}{(2\pi)^4} \left(\frac{1}{q^2}\right)^3 . \tag{7.21}$$

This integral is UV-convergent. Also the 8-particle reactions are convergent. We now look at two-loop contributions to the propagator, cf. Fig. 7.10. The corresponding loop integral

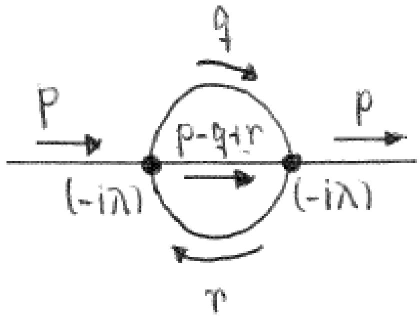


Figure 7.10: Two-loop contribution to the propagator.

is given by

$$(-i\lambda)^2 \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4r}{(2\pi)^4} \frac{1}{(q^2 - m^2 + i\epsilon)} \cdot \frac{1}{(r^2 - m^2 + i\epsilon)} \cdot \frac{1}{(p - q + r)^2 - m^2 + i\epsilon} \tag{7.22}$$

This integral is quadratically divergent. UV-divergences appear only in corrections to the propagator and the four-particle vertex (modulo subdiagrams). The divergence can, in a so-called renormalisable theory, be absorbed through a redefinition of the parameters. The thus defined parameters are the physical parameters, i.e. the parameters, which are measured in experiment. The unrenormalised parameters are called bare parameters. The ϕ^4 theory is renormalisable. The divergences of the S matrix elements can be aborbed through redefinition (renormalisation) of the parameters λ and m. In contrast, in non-renormalisable theories, there are at higher orders always new types of divergent scattering amplitudes.

Chapter 8

Radiative Corrections in Quantum Electrodynamics

The S matrix is given by

$$S = T\left[\exp\left(i\int_{-\infty}^{\infty} d^4x\, \mathcal{L}_W(x)\right)\right] . \tag{8.1}$$

Expansion of the S matrix in the coupling constants contained in the interactions of $\mathcal{L}_W$ leads to Feynman diagrams of the corresponding order in the coupling constant. Thus we have the diagrams contributing to electron-positron scattering at order e^4 depicted in Fig. 8.1. The divergence behaviour is estimated via power counting.

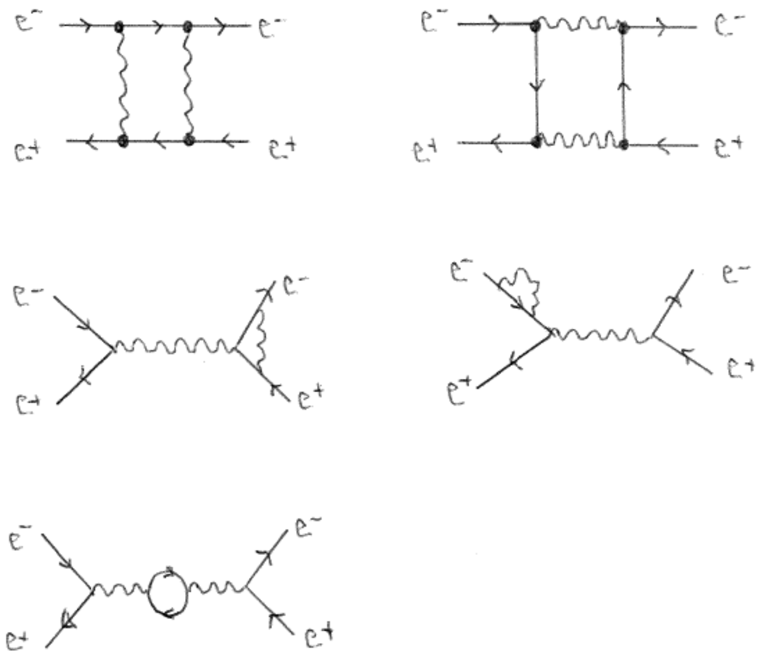


Figure 8.1: Contributions of order e^4 to electron positron scattering (no complete list).

(a),(b)

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{\not k}\frac{1}{k^2}\frac{1}{\not k}\frac{1}{k^2}\ :\ \text{convergent in the UV-Limit.} \tag{8.2}$$

(c)

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{\not k}\frac{1}{k^2}\frac{1}{\not k}\ :\ \text{logarithmic divergence.} \tag{8.3}$$

(d)

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{\not k}\frac{1}{k^2}\ :\ \text{linear divergence.} \tag{8.4}$$

(e)

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{\not k}\frac{1}{\not k}\ :\ \text{quadratic divergence.} \tag{8.5}$$

In the following we will only look at loop parts, and namely those, which exhibit a UV divergence. We hence investigate the diagrams shown in Fig. 8.2. In the tree-level parts the

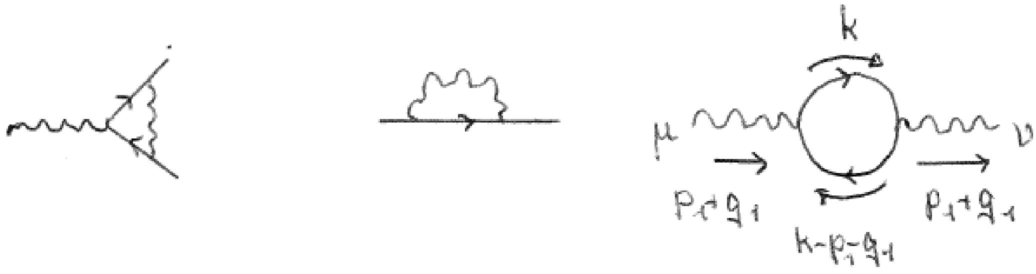


Figure 8.2: Divergent Loop Diagrams.

momenta of the propagators are fixed through the external momenta, cf. Fig. 8.3.

8.1 The Vacuum Polarisation

In Fig. 8.1 (e) the photon propagator

$$\frac{-ig_{\mu\nu}}{(p_1+q_1)^2+i\epsilon} \tag{8.6}$$

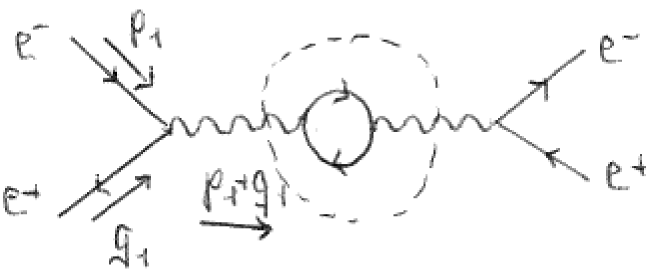


Figure 8.3: Fixing of the momenta.

of the Born appoximation is replaced through

$$(-1)\frac{-ig_{\mu\rho}}{(p_1+q_1)^2+i\epsilon}\underbrace{\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[(-ie\gamma_\rho)\cdot\frac{i}{\not k-\not p_1-\not q_1-m+i\epsilon}\cdot(-ie\gamma_\sigma)\cdot\frac{i}{\not k-m+i\epsilon}\right]}_{\equiv i\Pi_{\rho\sigma}(q)}\frac{-ig_{\sigma\nu}}{(p_1+q_1)^2+i\epsilon}$$

$$=\frac{-i}{q^2+i\epsilon}\underbrace{\int\frac{d^4k}{(2\pi)^4}(-1)\,\mathsf{Sp}\left[(-ie\gamma_\mu)\cdot\frac{i}{\not k-\not p_1-\not q_1-m+i\epsilon}\cdot(-ie\gamma_\nu)\cdot\frac{i}{\not k-m+i\epsilon}\right]}_{\equiv i\Pi_{\mu\nu}(q)}\frac{-i}{q^2+i\epsilon}\ , \tag{8.7}$$

with

$$q=p_1+q_1\ . \tag{8.8}$$

The prefactor (-1) stems from the closed fermion loop. We had already seen in Section 6.5 that the vacuum polarisation can be written as

$$\Pi_{\mu\nu}=(q^2g_{\mu\nu}-q_\mu q_\nu)\Pi(q^2)\ . \tag{8.9}$$

Because of the coupling with an external conserved current the $q_\mu q_\nu$-term does not contribute. And for the photon propagator we found using the geometric series (cf. Fig. 6.11)

$$\frac{-ig_{\mu\nu}}{q^2}\left(\frac{1}{1-\Pi(q^2)}\right)\ . \tag{8.10}$$

As long as $\Pi_{\mu\nu}$ has the form $(q^2g_{\mu\nu}-q_\mu q_\nu)\Pi(q^2)$ and $\Pi(q^2)$ is regular at $q^2=0$, the pole of the propagator remains at $q^2=0$ and thereby $m_\gamma=0$.

a) Scattering at small q^2: We consider the scattering at small q^2, cf. Fig. 8.4. It is proportional to

$$\sim\frac{e_0^2}{q^2}\cdot\frac{1}{1-\Pi(q^2)}\ \text{for}\ \underset{=}{q^2\to 0}\ \frac{e_0^2}{1-\Pi(0)}\cdot\frac{1}{q^2}\equiv\frac{e^2}{q^2}\ . \tag{8.11}$$

We here have defined

$$\frac{e_0^2}{1-\Pi(0)}\equiv e_0^2\cdot Z_3=e^2\qquad\text{or}\qquad e_0^2(1+\delta Z_3)\equiv e^2\ . \tag{8.12}$$

The charge e_0 is called *bare charge* and e is called *physical charge*. The corresponding renormalisation constant Z_3 is defined through

$$\frac{1}{1-\Pi(0)}=Z_3=1+\delta Z_3\ . \tag{8.13}$$

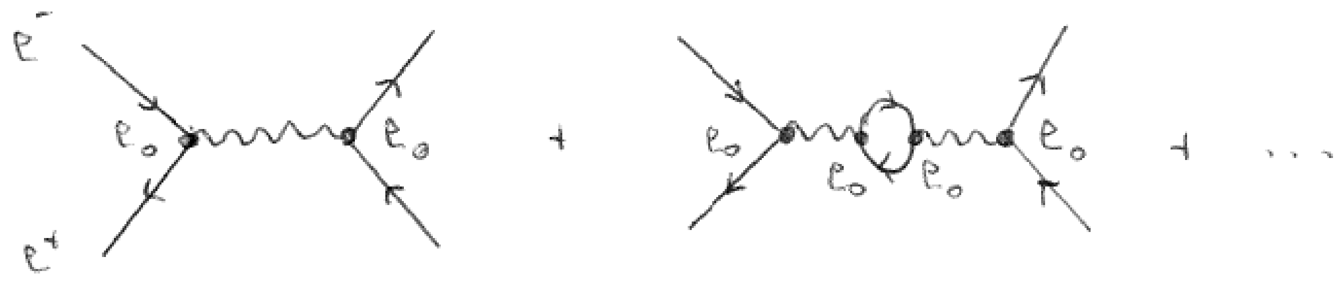


Figure 8.4: *ee*-Scattering.

At leading order we have $\delta Z_3 = \Pi(0)$. We now discuss the q^2-dependence of the scattering amplitude. We define

$$\hat{\Pi}(q^2) \equiv \Pi(q^2) - \Pi(0) \; . \tag{8.14}$$

Thereby we have modulo terms of higher order

$$\frac{e_0^2}{q^2(1-\Pi(q^2))} = \frac{e_0^2}{q^2(1-\Pi(0)-\hat{\Pi}(q^2))} = \frac{e^2}{q^2(1-\hat{\Pi}(q^2))} \; . \tag{8.15}$$

We here give without calculation the result of $\hat{\Pi}(q^2)$:

$$\hat{\Pi}(q^2) = -\frac{2\alpha}{\pi}\int_0^1 x(1-x)\left[-\log(\Delta) + \log(\Delta)|_{q^2=0}\right] dx \; , \tag{8.16}$$

with

$$\Delta = m^2 - x(1-x)\cdot q^2 \; . \tag{8.17}$$

Thereby we have

$$\hat{\Pi}(q^2) = +\frac{2\alpha}{\pi}\int_0^1 x(1-x)\cdot\log\left[1 - x(1-x)\cdot\frac{q^2}{m^2}\right] dx \; . \tag{8.18}$$

We first look at the behaviour for small q^2/m^2:

$$\hat{\Pi}(q^2) = -\frac{2\alpha}{\pi}\cdot\frac{q^2}{m^2}\int_0^1 x^2(1-x)^2 dx = -\frac{1}{15}\frac{\alpha}{\pi}\frac{q^2}{m^2} \; . \tag{8.19}$$

The contribution to the potential in the non-relativistic limit is obtained for $q^2 = -\vec{q}^2$. Hence

$$\frac{e^2}{\vec{q}^2}\left[1 + \frac{1}{15}\frac{\alpha}{\pi}\frac{\vec{q}^2}{m^2}\right] = \frac{e^2}{\vec{q}^2} + \frac{1}{15}\frac{\alpha}{\pi}\frac{e^2}{m^2} \; . \tag{8.20}$$

For small momentum transfer one hence obtains a change of the potential of the order α/π. In local space we have ($\alpha = e^2/(4\pi)$)

$$\frac{\alpha}{r} + \frac{\alpha^2}{15}\frac{4}{m^2}\,\delta(\vec{r}) \; . \tag{8.21}$$

One obtains an additional short-range contribution to the the potential and thereby a contribution to the Lamb-shift (splitting between s- and p-level).

In the following the behaviour for large positive/negative q^2/m^2 is discussed

$$\begin{aligned}\hat{\Pi}(q^2) &= \frac{2\alpha}{\pi}\int_0^1 x(1-x)\left[\ln\left(-\frac{q^2}{m^2-i\epsilon}\right)+\ln\left(x(1-x)-\frac{m^2-i\epsilon}{q^2}\right)\right]dx \\ &= \frac{2\alpha}{\pi}\int_0^1 x(1-x)\left[\ln\left(-\frac{q^2}{m^2}-i\epsilon\right)+\ln(x(1-x))+O\left(\frac{m^2}{q^2}\right)\right]dx \\ &= \frac{2\alpha}{\pi}\Big[\underbrace{\int_0^1 x(1-x)dx}_{=\frac{1}{6}}\ln\left(-\frac{q^2}{m^2}-i\epsilon\right)+\underbrace{\int_0^1 x(1-x)\ln(x(1-x))dx}_{=\frac{-5}{18}}\Big] \,. \end{aligned} \tag{8.22}$$

With

$$\ln(x+i\epsilon) = \ln|x| + i\pi\theta(x) \tag{8.23}$$

we have

$$\hat{\Pi}(q^2) = \frac{\alpha}{3\pi}\left(\ln\left(\frac{q^2}{m^2}\right)+\theta(q^2)\cdot i\pi-\frac{5}{3}\right) \,. \tag{8.24}$$

Thereby for the propagator holds for large q^2,

$$\frac{e^2}{q^2\left(1-\frac{\alpha}{3\pi}\ln\left(\frac{q^2}{m^2}\right)+...\right)} \,. \tag{8.25}$$

The effective charge hence grows with q^2. The expression diverges when the denominator of the fraction becomes zero. This is the case for

$$1-\frac{\alpha}{3\pi}\ln\left(\frac{q^2}{m^2}\right)=0 \;\Rightarrow\; q^2 = m^2\exp\left(\frac{3\pi}{\alpha}\right) \,. \tag{8.26}$$

This critical case is called Landau pole.

8.1.1 Meaning of the Imaginary Part

We want to investigate in the following the meaning of the imaginary part of the vacuum polarisation. For this we look at the case $q^2 > 0$. We have

$$\begin{aligned}\Im[\Pi(q^2)] &\sim \frac{2\alpha}{\pi}\Im\left[\int_0^1 x(1-x)\,\ln(m^2-i\epsilon-q^2x(1-x))\,dx\right] \\ &= \frac{2\alpha}{\pi}\left[\int_0^1 x(1-x)\,\pi\,\theta(q^2x(1-x)-m^2)\,dx\right] \,. \end{aligned} \tag{8.27}$$

Through the determination of the roots of the expression in the θ function on obtains as new integration range

$$\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{4m^2}{q^2}} < x < \frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4m^2}{q^2}} \,, \tag{8.28}$$

where $q^2 > 4m^2$ has to hold, as otherwise the integral vanishes. Thereby one obtains

$$\Im[\Pi(q^2)] = 2\alpha \int_{\frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{4m^2}{q^2}}}^{\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4m^2}{q^2}}} x(1-x)\,dx = \begin{cases} \frac{\alpha}{3}\left(1+\frac{2m^2}{q^2}\right)\sqrt{1-\frac{4m^2}{q^2}} & \text{für } q^2 \geq 4m^2 \\ 0 & \text{sonst} \end{cases} \tag{8.29}$$

The interpretation leads to the optical theorem. The total cross section is given by the imaginary part of the forward scattering amplitude, cf. Fig. 8.5. We remind that the cross

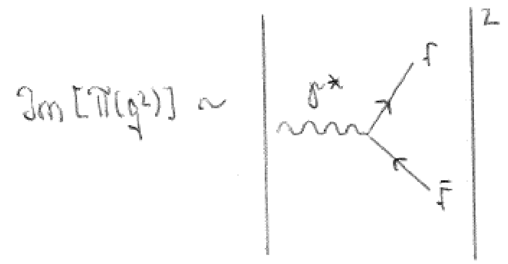


Figure 8.5: Relation between the imaginary part of the forward scattering amplitude and the total cross section.

section for $e^+e^- \to \mu^+\mu^-$ with $m_e = 0, m_\mu = m$ is given by

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi}{3}\frac{\alpha^2}{s}\left(1+\frac{2m^2}{s}\right)\sqrt{1-\frac{4m^2}{s}}\,. \tag{8.30}$$

8.1.2 Renormalisation of the External Photon Lines

If we couple a photon, starting from an external source, then also the vacuum polarisation appears, cf. Fig. 8.6. Either in external lines bubbles are not considered and $\epsilon_\mu \exp(-ikx)$

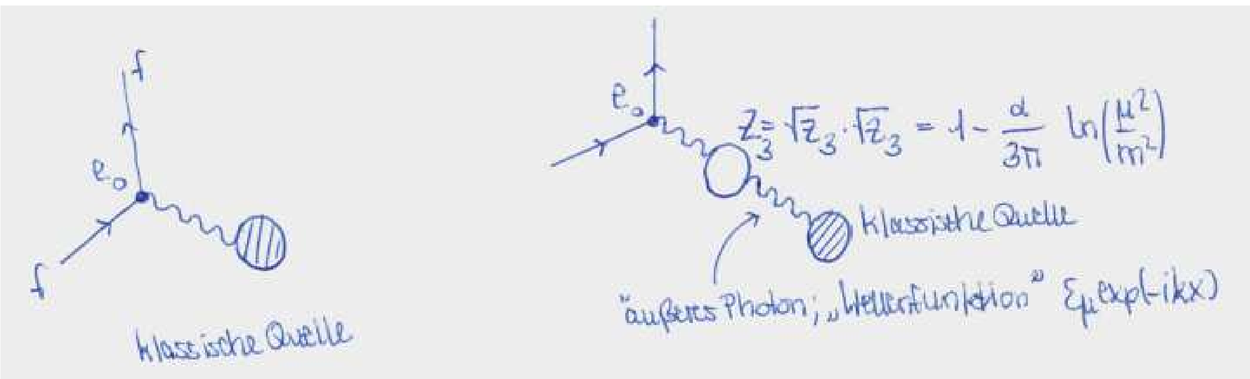


Figure 8.6: External photon line.

is replaced by $\sqrt{Z_3}\epsilon_\mu \exp(-ikx)$ (this then leads to $e_0 \cdot \sqrt{Z_3} = e_R$). Or they are considered, and one divides by $\sqrt{Z_3}$.

8.1.3 The Electron Propagator

In internal ($p^2 \neq m^2$) and external electron lines the self-energy diagram Fig. 8.7 appears. This leads in internal lines to the change of the electron propagator, in external lines it

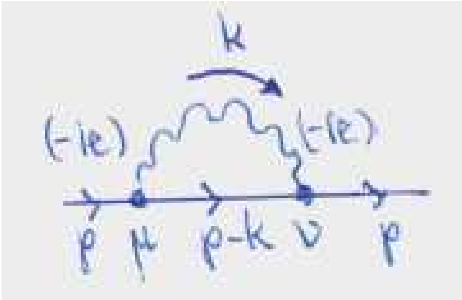


Figure 8.7: The self-energy of the electron.

contributes to the electron wave function renormalisation. We have

$$-i\Sigma(\not{p}) \equiv (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{(-i)}{k^2-\mu^2+i\epsilon}\gamma_\nu \frac{i}{\not{p}-\not{k}-m+i\epsilon}\gamma^\nu \ . \tag{8.31}$$

Here a small photon mass μ was introduced. And for the photon propagator the Feynman gauge was used, hence

$$\frac{-ig_{\mu\nu}}{k^2+i\epsilon} \ . \tag{8.32}$$

Furthermore holds

$$\frac{1}{\not{p}-\not{k}-m+i\epsilon} = \frac{\not{p}-\not{k}+m}{(p-k)^2-m^2+i\epsilon} \ . \tag{8.33}$$

We once again directly give the result. In the result there appears a logarithmic divergence, which is regularised by subtracting $\Sigma(p^2 \to \Lambda^2)$, where $\Lambda^2 \gg m^2$, $\Lambda^2 \gg \mu^2$ and $\Lambda^2 \gg p^2$ shall hold. We call the regularised $\Sigma(\not{p})$ in the following $\overline{\Sigma}(\not{p})$, and we have

$$\overline{\Sigma}(\not{p}) = \Sigma_{p^2} - \Sigma_{\Lambda^2} = \frac{\alpha}{2\pi}\int_0^1 [2m-\not{p}x] \ln\left(\frac{\Lambda^2 x}{m^2(1-x)+\mu^2 x - p^2 x(1-x) - i\epsilon}\right) dx \tag{8.34}$$

Here there appear terms $\sim \mathbf{1}$ and $\sim \not{p}$, which are independent of each other. In contrast, in the vacuum polarisation the terms $\sim g_{\mu\nu}$ and $k_\mu k_\nu$ are dependent on each other.

8.1.4 Renormalisation of the Electron Propagator

For the electron propagator (cf. Fig. 8.8) one obtains by means of the geometric series

$$\begin{aligned}\frac{i}{\not{p}-m_0} + \frac{i}{\not{p}-m_0}\cdot\left[-i\overline{\Sigma}(\not{p})\right]\cdot\frac{i}{\not{p}-m_0} + \ldots &= \frac{i}{\not{p}-m_0}\left[1+\left(\frac{\overline{\Sigma}(\not{p})}{\not{p}-m_0}\right)+\left(\frac{\overline{\Sigma}(\not{p})}{\not{p}-m_0}\right)^2+\ldots\right] \\ &= \frac{i}{(\not{p}-m_0)\cdot\left(1-\frac{\overline{\Sigma}(\not{p})}{\not{p}-m_0}\right)} = \frac{i}{\not{p}-m_0-\overline{\Sigma}(\not{p})} \ .\end{aligned} \tag{8.35}$$

Figure 8.8: The elecron propagator

We demand that the pole lies at the renormalised mass m_R, hence at $m = m_R$. Thereby, one obtains the condition

$$(\not{p} - m_0 - \overline{\Sigma}(\not{p})|_{\not{p}=m} = 0\ . \tag{8.36}$$

With

$$m \equiv m_0 + \delta m \tag{8.37}$$

we get from this condition

$$(\not{p} - m + \delta m - \overline{\Sigma}(\not{p})|_{\not{p}=m} = \delta m - \overline{\Sigma}(\not{p})|_{\not{p}=m} = 0 \tag{8.38}$$

and hence

$$\delta m = \overline{\Sigma}(\not{p})|_{\not{p}=m}\ . \tag{8.39}$$

We perform a Taylor expansion of $\overline{\Sigma}(\not{p})$ around $\not{p} \approx m$ and obtain

$$\begin{aligned}(\not{p} - m_0 - \overline{\Sigma}(\not{p})) &= \not{p} - m_0 - \underbrace{\overline{\Sigma}(\not{p})|_{\not{p}=m}}_{=\delta m} - \left.\frac{d\overline{\Sigma}}{d\not{p}}\right|_{\not{p}=m} (\not{p} - m) + O(\not{p} - m)^2 \\ &= (\not{p} - m)\left(1 - \left.\frac{d\overline{\Sigma}}{d\not{p}}\right|_{\not{p}=m} + O(\not{p} - m)\right)\ ,\end{aligned} \tag{8.40}$$

where we used that $m = m_0 + \delta m$. Close to the pole the propagator has the form (where Z_2 is the wave function renormalisation constant):

$$\frac{Z_2}{\not{p} - m} \quad \text{with} \quad Z_2^{-1} = 1 - \left.\frac{d\overline{\Sigma}}{d\not{p}}\right|_{\not{p}=m} \quad \text{and} \quad \delta Z_2 = \left.\frac{d\overline{\Sigma}}{d\not{p}}\right|_{\not{p}=m} \tag{8.41}$$

For the divergent part of the mass renormalisation one finds (without calculation)

$$\delta m = m_R - m_0 = \frac{3\alpha}{4\pi} m_0 \ln\left(\frac{\Lambda^2}{m_0^2}\right) + \text{ const.}\ . \tag{8.42}$$

Thereby, we can write

$$\begin{aligned}\frac{i}{\not{p} - m_0 - \overline{\Sigma}(\not{p})} &= \frac{i}{(\not{p} - m_0) - \delta m + (Z_2^{-1} - 1 + O((\not{p} - m)^2)) \cdot (\not{p} - m_0)} \\ &= \frac{i}{-\delta m + (Z_2^{-1} + O((\not{p} - m)^2)) \cdot (\not{p} - m_0)} \\ &= \frac{iZ_2}{(\not{p} - m_0) \cdot (1 + Z_2 \cdot O((\not{p} - m)^2)) - Z_2 \delta m} \\ &= \frac{iZ_2}{(\not{p} - m_0 - \delta m) \cdot (1 + O((\not{p} - m)^2))}\ .\end{aligned} \tag{8.43}$$

One calls

$$m_R = m_0 + \delta m \tag{8.44}$$

the physical mass m! The Z_2 plays a similar role as Z_3. However, Z_2 is not compensated by charge renormalisation but for internal lines through direct compensation with divergent vertices. For external lines Z_2 provides the wave function renormalisation.

8.1.5 Vertex Correction

We look at diagram Fig. 8.9. Neglecting the external spinors, we find

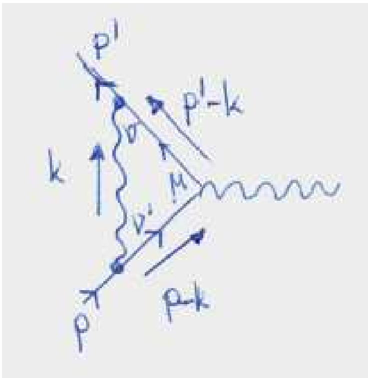


Figure 8.9: Vertex Correction.

$$\Lambda_\mu(p', p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{(-i)}{(k^2 - \mu^2 + i\epsilon)} \gamma_\nu \frac{i}{(\not p' - \not k - m + i\epsilon)} \gamma_\mu \frac{i}{\not p - \not k - m + i\epsilon} \gamma^{\nu'} \tag{8.45}$$

For the regularisation of the infrared divergences (divergences at small energies) a small photon mass μ was introduced. There are two linearly independent external momenta, which are given by p and $p^{'}$, where $p^2 = p^{'2} = m^2$. We look at the behaviour for $-p^{'} + p \equiv q \to 0$ with $q^2 < 0$. We have for $\Lambda_\mu(p, p)$ a matrix in spinor space with one Lorentz index, which only depends on p. Possible forms are $m\gamma_\mu f_1(p^2)$ or $p_\mu f_2(p^2)\mathbf{1}$. These two possibilities are not linearly independent, because in between spinors $\bar{u}(p)$ and $u(p)$ we have

$$m\gamma_\mu \overset{\text{Dirac}}{=} \not p \gamma_\mu \overset{\text{anti commutation}}{=} 2p_\mu - \gamma_\mu \not p = 2p_\mu - \gamma_\mu m \ . \tag{8.46}$$

Hence $\Lambda_\mu(p, p) \sim \gamma_\mu$ in the limit $q \to 0$. The Λ_μ is calculated after renormalisation, hence

$$\bar{\Lambda}_\mu = \Lambda_\mu(\text{photon mass} = \mu) - \Lambda_\mu(\text{photon mass (regulator mass)} \ = \Lambda_R) \ . \tag{8.47}$$

We define

$$\bar{u}(p)\, \bar{\Lambda}_\mu(p, p)\, u(p) = (Z_1^{-1} - 1)\bar{u}\gamma_\mu u \ [+ \text{ anomal magnetic moment } \ \sim q_\mu] \ . \tag{8.48}$$

The reasoning ist: Born$[\gamma_\mu]$ + 1-loop correction $\Rightarrow Z_1^{-1}$. Furthermore, the Ward identiy

$$\Lambda_\mu(p, p) = -\frac{\partial \Sigma(p)}{\partial p^\mu} \ . \tag{8.49}$$

holds. The Ward identity is of crucial importance for numerous checks of the renormalisation theory. The identity follows from the comparison of the integrands. Furthermore, we have

$$-\frac{\partial}{\partial p^\mu}\left(\frac{1}{\not p - \not k - m + i\epsilon}\right) = \frac{1}{\not p - \not k - m + i\epsilon}\gamma_\mu \frac{1}{\not p - \not k - m + i\epsilon} . \tag{8.50}$$

This is shown diagrammatically in Fig. 8.10. The derivation of the self-energy w.r.t. the

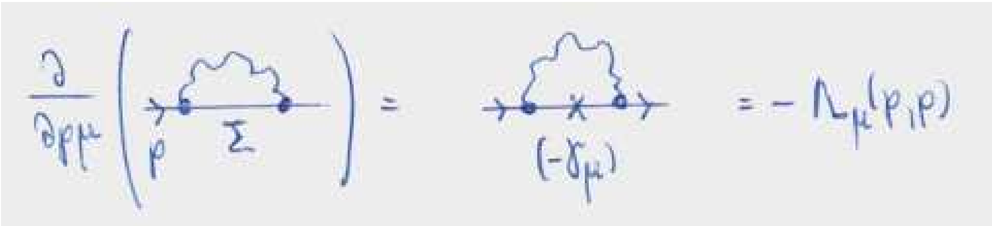


Figure 8.10: Diagrammatic Representation of the derivation w.r.t. the four-momentum.

external momentum hence corresponds to the replacement of $-\gamma_\mu$ in the internal fermion propagator. This also holds after renormalisation. Because of

$$\overline{\Sigma} = \delta m - [Z_2^{-1} - 1 + O((\not p - m)^2)](\not p - m) \tag{8.51}$$

we have

$$-\frac{\partial \overline{\Sigma}}{\partial p^\mu} = \overline{\Lambda}_\mu = [Z_2^{-1} - 1]\gamma_\mu . \tag{8.52}$$

From the Ward identity hence follows that

$$Z_1 = Z_2 . \tag{8.53}$$

8.1.6 Greens Function

At the end of this chapter, let us have a few additional remarks on the Greens function and the scattering matrix element.

Be $\phi(x)$ an interacting field. The c-number "functions" (distributions)

$$G(x_1, ..., x_n) = \langle 0|T[\phi(x_1)...\phi(x_n)]|0\rangle \tag{8.54}$$

are called n-point Greens functions of this field theory (cf. e.g. the 2-point Greens function of the free theory).

Meaning:

- From the knowledge of all $G(x_1, ..., x_n)$ $(n = 1, ...)$ one could reconstruct the corresponding field theory, i.e. the Fock space and the field operators. In the free theory we have

$$n - \text{point-function} = \sum(\text{products of the free 2-point function}) . \tag{8.55}$$

- The Greens functions hold for off-shell matrix elements. By going on-shell, one obtains the S-matrix elements S_{fi}.
- These statements hold analoguously for interacting fields (with or without spin).

8.2 The Lehmann-Symanzik-Zimmermann (LSZ) Reduction Formula

Let us start by a few considerations on the "asymptotic theory", i.e. the "in" and "out" states. The consequence of the adiabatic turning on and off of the interaction is

$$\begin{array}{lccl} x_0 \to -\infty: & \phi(x) & \to & \sqrt{Z}\phi_{\text{in}}(x) \\ x_0 \to +\infty: & \phi(x) & \to & \sqrt{Z}\phi_{\text{out}}(x) \end{array} \tag{8.56}$$

This is a convergence in the weak sense, i.e. it holds only for matrix elements. The fields $\phi_{\text{in}}(x)$, $\phi_{\text{out}}(x)$ have the same properties as the free scalar field, i.e. they are solutions of the free Klein-Gordon equation,

$$(\Box + m^2)\phi_{\text{in,out}}(x) = 0 \; . \tag{8.57}$$

Hence

$$\phi_{\text{in,out}}(x) = \int d\tilde{k}\, [a_{\text{in,out}}(k)e^{-ikx} + a^\dagger_{\text{in,out}}(x)e^{ikx}] \; , \tag{8.58}$$

with

$$k^\mu = \begin{pmatrix} +\sqrt{\vec{k}^2 + m^2} \\ \vec{k} \end{pmatrix} \; . \tag{8.59}$$

This means that the "in" and "out" states are[1]

$$|k_1, ..., k_n\rangle_{\text{in}} = a^\dagger_{\text{in}}(k_1)...a^\dagger_{\text{in}}(k_n)|0\rangle \tag{8.60}$$

and

$$|k_1, ..., k_n\rangle_{\text{out}} = a^\dagger_{\text{out}}(k_1)...a^\dagger_{\text{out}}(k_n)|0\rangle \; . \tag{8.61}$$

Remark that the "in" and "out" states, respectively fields, are not identical. Because

$$S|\beta, \text{out}\rangle = |\beta, \text{in}\rangle \; , \; \text{also} \; S a^\dagger_{\text{out}} S^{-1} S|0\rangle = a^\dagger_{\text{in}}|0\rangle \; . \tag{8.62}$$

Since $S|0\rangle = |0\rangle$, we have

$$S\phi_{\text{out}}S^{-1} = \phi_{\text{in}} \; . \tag{8.63}$$

The "in" and "out" fields fulfill the usual commutation relations, hence

$$[\phi_{\text{in}}(x), \phi_{\text{in}}(y)] = i\Delta(x - y) \; . \tag{8.64}$$

The commutator

$$[\phi_{\text{in}}(x), \phi_{\text{out}}(y)] \; , \tag{8.65}$$

however, is a priori not known. We now look at the meaning of $\sqrt{Z}$. We have

$$\langle n|\phi_{\text{in,out}}(x)|0\rangle = e^{ipx}\delta_{n1} \; . \tag{8.66}$$

[1]Note that the states still have to be normalised properly, i.e. they have to be smeared by a function in momentum space. We leave this out here for convenience. Note also that the commutation relations between the in and out operators, $[a_{\text{out}}(k), a^\dagger_{\text{in}}(q)]$, are not known a priori.

This means that $\phi_{\text{in,out}}(x)$ creates/annihilates 1-particle states. For an interacting field, however, in general

$$\langle n - \text{Teilchenzustand } |\phi(x)|0\rangle \neq 0 \ . \tag{8.67}$$

Therefore we expect that

$$\langle 1|\phi(x)|0\rangle = \underbrace{\sqrt{Z}}_{<1}\langle 1|\phi_{\text{in,out}}|0\rangle \ . \tag{8.68}$$

In the following the LSZ reduction formula for the scalar field will be derived. Let us start with a few preliminary remarks:

- The S-matrix element can be written in the "in"- and "out"-basis as:

$$\begin{aligned} S_{fi} &= \underbrace{\langle \text{out}, p_1, ..., p_m}_{\text{final state}} | \underbrace{q_1, ..., q_n, \text{in}\rangle}_{\text{initial state}} \\ &= \langle 0|a_{\text{out}}(p_1)...a_{\text{out}}(p_n)a^\dagger_{\text{in}}(q_1)...a^\dagger_{\text{in}}(q_n)|0\rangle \ . \end{aligned} \tag{8.69}$$

- Amplitudes of the form

$$\begin{aligned} &\underbrace{\langle \text{out}, p_1...}_{\langle 0|a_{\text{out}}(p_1)...a_{\text{out}}(p_m)} \quad |a^\dagger_{\text{out}}(q_1)|q_2, ..., \text{in}\rangle \\ = &\sum_{j=1}^{m}(2\pi)^3 2p_{0j}\delta^{(3)}(\vec{p}_j - \vec{q}_1)\langle \text{out}, p_1...\hat{p}_j...|q_2, ..., \text{in}\rangle \ . \end{aligned} \tag{8.70}$$

  are called disconnected. Here, we have used that

$$a_{\text{out}}(p_m)a^\dagger{}_{\text{out}}(q_1) = a^\dagger_{\text{out}}(q_1)a_{\text{out}}(p_m) + (2\pi)^3 2p_{0m}\delta^{(3)}(\vec{p}_m - \vec{q}_1) \ . \tag{8.71}$$

  Furthermore $\hat{p}_j$ means, that this particle is missing in the matrix element. Disconnected means that at least one particle does not participate in the scattering process ("runs through"), cf. Fig. 8.11.

We use that from Glg. (8.58) follows (at an arbitrary time t)

$$a^\dagger_{\text{in}}(q) = \frac{1}{i}\int_t d^3x e^{-iqx}\overleftrightarrow{\partial_0}\phi_{\text{in}}(x) \tag{8.72}$$

and analoguously

$$a_{\text{out}}(q) = \frac{1}{i}\int_t d^3x e^{iqx}\overleftrightarrow{\partial_0}\phi_{\text{out}}(x) \ . \tag{8.73}$$

We now look at

$$\begin{aligned} S_{fi} &\equiv \langle \text{out}, p_1...p_m|q_1...q_n, \text{in}\rangle \\ &= \langle \text{out}, p_1...p_m|a^\dagger_{\text{in}}(q_1)|q_2...q_n, \text{in}\rangle \ . \end{aligned} \tag{8.74}$$

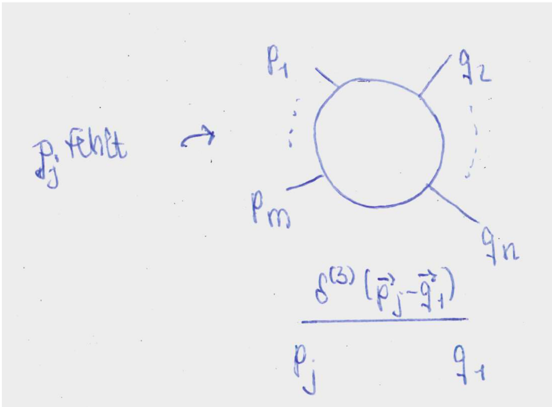


Figure 8.11: Example for a disconnected diagram in the scattering process.

We replace (8.72) in (8.74). Furthermore, we choose the time $t \to -\infty$, so that in the matrix element (8.74) the field ϕ_{in} can be replaced by $Z^{-1/2}\phi$. Thereby, we have

$$S_{fi} = \lim_{t\to-\infty} Z^{-1/2} \langle \text{out}, p_1 ... p_m | \frac{1}{i} \int_t d^3x e^{-iq_1x} \overleftrightarrow{\partial}_0 \phi(x) | q_2 ... q_n, \text{in}\rangle \; . \tag{8.75}$$

We use that (for abritrary integrand) holds

$$\int_t d^3x e^{-iq_1x} \overleftrightarrow{\partial}_0 \phi(x) = \int_{t'} d^3x e^{-iq_1x} \overleftrightarrow{\partial}_0 \phi(x) - \int_t^{t'} dx_0 \partial_0 \int d^3x e^{-iq_1x} \overleftrightarrow{\partial}_0 \phi(x) \; . \tag{8.76}$$

We now let $t' \to \infty$ (we have $t \to -\infty$) and use that in the matrix element $Z^{-1/2}\phi \overset{t'\to+\infty}{\to} \phi_{\text{out}}$. Insertion in Eq. (8.75) leads to

$$\begin{aligned} S_{fi} &= \underbrace{\langle \text{out}, p_1 ... p_m | a^\dagger_{\text{out}}(q_1) | q_2 ... q_m, \text{in}\rangle}_{\text{discon. amplitude}} \\ &\quad + iZ^{-1/2} \int d^4x \partial_0 [e^{-iq_1x} \overleftrightarrow{\partial}_0 \langle \text{out}, p_1 ... p_m | \phi(x) | q_2 ... q_n, \text{in}\rangle] \end{aligned} \tag{8.77}$$

In the second term we use that

$$-\partial_0^2 e^{-iq_1x} = (-\Delta + m^2) e^{-iq_1x} \; , \tag{8.78}$$

sind $q_{01}^2 - \vec{q}_1^{\,2} = m^2$. Thereby, we have

$$\int d^4x \partial_0 [...] = \int d^4x \left[(-\Delta e^{-iq_1x} + m^2 e^{-iq_1x}) \langle ... \rangle + e^{-iq_1x} \partial_0^2 \langle ... \rangle \right] \; . \tag{8.79}$$

$\Delta ...$ is transferred to $\langle ... \rangle$ by partial integration and the boundary terms for $\vec{x} \to \infty$ are neglected. Thereby we obtain

$$\begin{aligned} S_{fi} &= \text{disconnected amplitude} \\ &\quad + iZ^{-1/2} \int d^4x e^{-iq_1x} (\Box + m^2) \langle \text{out}, p_1 ... p_m | \phi(x) | q_2 ... q_n, \text{in}\rangle \; . \end{aligned} \tag{8.80}$$

This means, we have "reduced out" one particle from the "in" state. We can continue with the reduction and for example "reduce out" the particle with momentum p_1 from the "out" state. Then

$$\begin{aligned}
&\langle \mathrm{out}, p_1...p_m|\phi(x)|q_2...q_n|\mathrm{in}\rangle \\
&= \langle \mathrm{out}, p_2...p_m|a_{\mathrm{out}}(p_1)|q_2...q_n, \mathrm{in}\rangle \\
&= \lim_{y_0\to+\infty} \frac{i}{\sqrt{Z}} \int_{y_0} d^3y e^{ip_1y} \overleftrightarrow{\partial}_{y_0} \langle \mathrm{out}, p_2...p_m| \underbrace{\phi(y)\phi(x)}_{(*)} |q_2...q_n, \mathrm{in}\rangle \ . \qquad (8.81)
\end{aligned}$$

The $(*)$ can be replaced by the time-ordered product $T[\phi(x)\phi(y)]$, as $y_0 = +\infty$. As long as $x_0 < \infty$ the additional term $\partial_{y_0}\theta(y_0 - x_0) = \delta(y_0 - x_0)$ does not contribute. We do this trick so that the operator $a_{\mathrm{in}}(p_1)$, which will appear, is placed right of $\phi(x)$ and acts on the "in" state. We use analoguously to Eq. (8.76)

$$\begin{aligned}
&\int_{y_0=+\infty} e^{ip_1y}\overleftrightarrow{\partial}_0 T[\phi(y)\phi(x)] \\
=\ &\phi(x) \underbrace{\int_{y_0'=-\infty} d^3y e^{ip_1y}\overleftrightarrow{\partial}_{y_0}\phi(y)}_{\frac{1}{\sqrt{Z}}\phi\to\phi_{\mathrm{in}}\Rightarrow -ia_{\mathrm{in}}} + \underbrace{\int_{-\infty}^{+\infty} dy_0\partial_0 \int d^3y e^{ip_1y}\overleftrightarrow{\partial}_0 T[\phi(y)\phi(x)]}_{\text{The 4-dim integral is treated analoguously to (8.80).}} \qquad (8.82)
\end{aligned}$$

Thereby we obtain for Eq. (8.81)

$$\begin{aligned}
&\langle \mathrm{out}, p_1...p_m|\phi(x)|q_2...q_n|\mathrm{in}\rangle \\
=\ &\underbrace{\langle \mathrm{out}, p_2...p_m|\phi(x)a_{\mathrm{in}}|q_2...q_n, \mathrm{in}\rangle}_{\text{disconnected amplitude}} \\
&+\frac{i}{\sqrt{Z}} \int d^4y e^{ip_1y}(\Box_y + m^2)\langle \mathrm{out}, p_2...p_m|T[\phi(y)\phi(x)]|q_2...q_n, \mathrm{in}\rangle \ . \qquad (8.83)
\end{aligned}$$

One can continue doing this and reduce out all particles. One finally obtains the LSZ reduction formula for the scalar field theory:

$$\begin{aligned}
&\langle \mathrm{out}, p_1...p_m|q_1...q_n, \mathrm{in}\rangle = \langle \mathrm{in}, p_1...p_m|S|q_1...q_n, \mathrm{in}\rangle \\
&= \text{disconnected amplitudes} \\
+\ &\left(\frac{i}{\sqrt{Z}}\right)^{n+m} \int d^4y_1....d^4x_n e^{i\sum_{j=1}^m p_jy_j - i\sum_{r=1}^n q_rx_r} \\
\times\ &(\Box_{y_1} + m^2)...(\Box_{x_n} + m^2)\langle 0|T[\underbrace{\phi(y_1)....\phi(x_n)}_{m+n \text{ fields}}]|0\rangle \qquad (8.84)
\end{aligned}$$

Remarks:

- The LSZ reduction formula is exact: on the right we have the complete (=exact) Greens function, on the left the exact S-matrix element.
- The disconnected amplitudes contain at least one δ function, e.g. $\delta^{(3)}(\vec{p}_1 - \vec{q}_j)$ etc.
- If all particles participate in the scattering process, i.e. if $p_i^\mu \neq q_j^\mu$ for all i, j, then there are not disconnected amplitudes.

We continue with the discussion of the formula and simplify the notation: $y_1, ..., y_m, x_1, ..., x_n \to x_1, ..., x_{m+n}$. We perform the Fourier transformation of the Greens function:

$$G(x_1, ..., x_{m+n}) = \int \Pi_{j=1}^{m+n} \frac{d^4 l_j}{(2\pi)^4} e^{i \sum_{r=1}^{m+n} l_r x_r} \tilde{G}(l_1, ..., l_{m+n}) \, . \tag{8.85}$$

We replace this in Eq. (8.84) and use that

$$i(\Box_{x_j} + m^2) \to i(-l_j^2 + m^2) = \frac{l_j^2 - m^2}{i} \, . \tag{8.86}$$

We here have obtained the inverse scalar propagator in momentum space and thereby obtain

$$\begin{aligned} & \langle \text{out}, p_1 ... p_m | q_1 ... q_n, \text{in} \rangle = \langle \text{in}, p_1 ... p_m | S | q_1 .... q_n, \text{in} \rangle \\ = \; & \text{disconnected amplitudes} \\ & + \left(\frac{1}{\sqrt{Z}} \right)^{m+n} \frac{q_1^2 - m^2}{i} ... \frac{p_m^2 - m^2}{i} \tilde{G}(q_1, ..., q_n, -p_1, ..., -p_m)|_{p_j^2 = m^2, ..., q_r^2 = m^2} \, . \end{aligned} \tag{8.87}$$

The minus sign in front of $p_1, ..., p_m$ results from the fact that the momenta are outgoing. Since all particles are "on-shell", the $\tilde{G}$ has to be evaluated at $p_j^2 = m^2, q_r^2 = m^2$ for all j, r. Equation (8.87) means, that $\tilde{G}(...)$ has to have $(n+m)$ poles at m^2 in the variables p_j^2, q_r^2. The S-matrix element hence is, modulo a normalisation factor $Z^{-(m+n)/2}$, the residuum of this multiple pole. Less singular terms in $\tilde{G}$ do not contribute to S_{fi}. See also Fig. 8.12 for the meaning of the LSZ reduction formula.

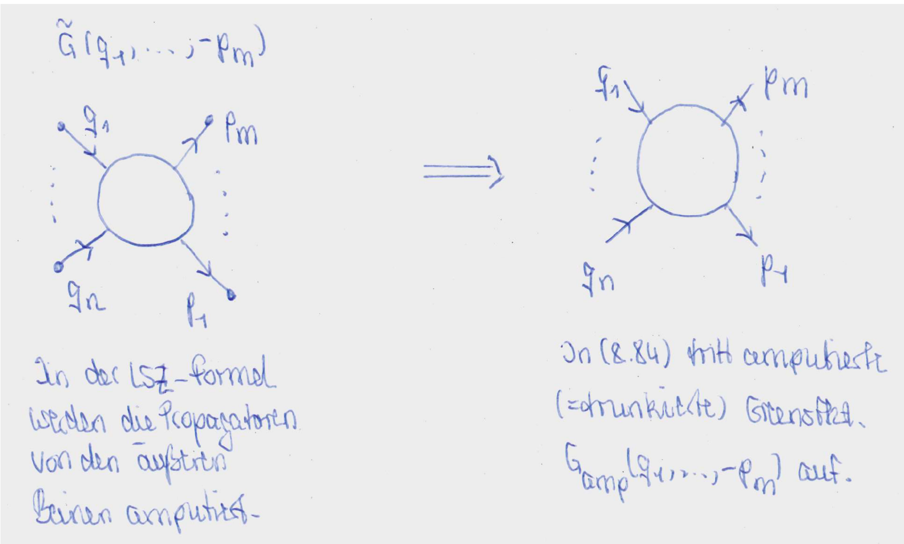


Figure 8.12: Graphic representation of the LSZ reduction formula. Note that in the figure caption is should not read (8.84) but (8.87).

A small kinematic remark: The inversion of the Fourier transformation is

$$\tilde{G}(l_1, ..., l_{m+n}) = \int \Pi_j d^4 x_j e^{-i \sum_r l_r x_r} G(x_1, x_2, ..., x_{m+n}) \, . \tag{8.88}$$

Because of translation invariance the G only depends on differences between the coordinates, e.g. $G = G(x_2 - x_1, ..., x_{m+n} - x_1)$. We write $l_1 x_1 + ... + l_{m+n} x_{m+n} = x_1 \sum_{r=1}^{n+m} l_r + l_2(x_2 - x_1) + ... + l_{m+n}(x_{m+n} - x_1)$ and thereby obtain

$$\tilde{G}(l_1, ...l_{m+n}) = (2\pi)^4 \delta^{(4)}(l_1 + ... + l_{m+n}) G(l_1, ..., l_{m+n}) \ . \tag{8.89}$$

Chapter 9

On the Path to the Standard Model - Gauge Symmetries

The principle of local gauge invariance is essential for quantum field theory. We start by looking at the example of QED. The Dirac Lagrangian for a free fermion field Ψ of mass m reads

$$\mathcal{L}_0 = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi \; . \tag{9.1}$$

It is invariant under a transformation with a unitary matrix $U = e^{-i\alpha} \in U(1)$. This means that applying the transformation

$$\Psi(x) \to \exp(-i\alpha)\Psi(x) = \Psi - i\alpha\Psi + \mathcal{O}(\alpha^2) \tag{9.2}$$

and for the addjoint spirnor $\bar{\Psi} = \Psi^\dagger \gamma^0$,

$$\bar{\Psi}(x) \to \exp(i\alpha)\bar{\Psi}(x). \tag{9.3}$$

the Lagrangian $\mathcal{L}_0$ goes over into itself. We distinguish

- global gauge transformations: $\alpha =$const.
- local gauge transformations: $\alpha = \alpha(x)$.

The Noether current of the above global gauge symmetry reads

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} \frac{\delta \Psi}{\delta \alpha} + \frac{\delta \bar{\Psi}}{\delta \alpha} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Psi})} = i\bar{\Psi}\gamma^\mu(-i\Psi) = \bar{\Psi}\gamma^\mu \Psi \; , \tag{9.4}$$

with

$$\partial_\mu j^\mu = 0 \; . \tag{9.5}$$

It implies charge conservation.

9.1 Coupling to a Photon

When we include the coupling to a photon, the Lagrangian reads

$$\mathcal{L} = \bar{\Psi}\gamma^\mu(i\partial_\mu - qA_\mu)\Psi - m\bar{\Psi}\Psi = \mathcal{L}_0 - qj^\mu A_\mu \ , \tag{9.6}$$

with j^μ given in Eq. (9.4). Applying the following gauge transformation to the external photon field A_μ,

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu\Lambda(x) \tag{9.7}$$

the Lagrangian goes over into

$$\mathcal{L} \to \mathcal{L} = \mathcal{L}_0 - qj^\mu A_\mu - \underbrace{qj^\mu\partial_\mu\Lambda}_{q\bar{\Psi}\gamma^\mu\Psi\partial_\mu\Lambda} \ . \tag{9.8}$$

This means that $\mathcal{L}$ is not gauge invariant. The transformations of the fields Ψ and $\bar{\Psi}$ have to be changed such that the Lagrangian becomes gauge invariant. This is done by introducing an x-dependent parameter α, hence $\alpha = \alpha(x)$. Thereby

$$i\partial_\mu\Psi \to i\exp(-i\alpha)(\partial_\mu\Psi) + (\partial_\mu\alpha)\exp(-i\alpha)\Psi \ , \tag{9.9}$$

so that

$$\mathcal{L}_0 \to \mathcal{L}_0 + \bar{\Psi}\gamma^\mu\Psi\partial_\mu\alpha \ . \tag{9.10}$$

This term cancels the additional term in Eq. (9.8) if

$$\alpha(x) = q\Lambda(x) \ . \tag{9.11}$$

Thereby the complete gauge transformation reads

$$\Psi \ \to \ \Psi'(x) = U(x)\Psi(x) \qquad \text{with} \qquad U(x) = exp(-iq\Lambda(x)) \quad (U \text{ unitary}) \tag{9.12}$$

$$\bar{\Psi} \ \to \ \bar{\Psi}'(x) = \bar{\Psi}(x)U^\dagger(x) \tag{9.13}$$

$$A_\mu(x) \ \to \ A_\mu(x) + \partial_\mu\Lambda(x) = U(x)A_\mu(x)U^\dagger(x) - \frac{i}{q}U(x)\partial_\mu U^\dagger(x) \ . \tag{9.14}$$

The Lagrangian transforms according to

$$\begin{aligned}\mathcal{L} \to \mathcal{L}' &= \bar{\Psi}\gamma^\mu U^{-1} i\partial_\mu(U\Psi) - q\bar{\Psi}U^{-1}\gamma^\mu\left(UA_\mu U^{-1} - \frac{i}{q}U\partial_\mu U^{-1}\right)U\Psi - m\bar{\Psi}U^{-1}U\Psi \\ &= \bar{\Psi}\gamma^\mu i\partial_\mu\Psi + \bar{\Psi}\gamma^\mu(U^{-1}i(\partial_\mu U))\Psi - q\bar{\Psi}\gamma^\mu\Psi A_\mu + \bar{\Psi}\gamma^\mu(i(\partial_\mu U^{-1})U)\Psi - m\bar{\Psi}\Psi \\ &= \mathcal{L} + i\bar{\Psi}\gamma^\mu\partial_\mu(U^{-1}U)\Psi = \mathcal{L} \ .\end{aligned} \tag{9.15}$$

Minimal substitution $p_\mu \to p_\mu - qA_\mu$ leads to

$$i\partial_\mu \to i\partial_\mu - qA_\mu \equiv iD_\mu \ . \tag{9.16}$$

Here $D_\mu(x)$ is the *covariant derivative*. The expression *covariant* means, that it transforms exactly as the field

$$\Psi(x) \to U(x)\Psi(x) \qquad \text{and} \qquad D_\mu\Psi(x) \to U(x)(D_\mu\Psi(x)) \ . \tag{9.17}$$

This means

$$(D_\mu\Psi)' = D'_\mu\Psi' = D'_\mu U\Psi \stackrel{!}{=} UD_\mu\Psi \ , \tag{9.18}$$

so that the covariant derivative transforms according to

$$\begin{aligned} D'_\mu &= UD_\mu U^{-1} = \exp(-iq\Lambda)(\partial_\mu + iqA_\mu)\exp(iq\Lambda) = \partial_\mu + iq\partial_\mu\Lambda + iqA_\mu \\ &= \partial_\mu + iqA'_\mu \ . \end{aligned} \tag{9.19}$$

Thereby

$$\mathcal{L} = \bar{\Psi}\gamma^\mu iD_\mu\Psi - m\bar{\Psi}\Psi \tag{9.20}$$

is obviously gauge invariant.

The kinetic energy of the photons is given by

$$\mathcal{L}_{kin} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad \text{mit} \qquad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \ . \tag{9.21}$$

The field strength tensor $F^{\mu\nu}$ can be expressed through the covariant derivative. We choose the following ansatz for the tensor of rang 2,

$$[D_\mu, D_\nu] = [\partial_\mu - iqA_\mu, \partial_\nu - iqA_\nu] = -iq[\partial_\mu, A_\nu] - iq[A_\mu, \partial_\nu] = -iq(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{9.22}$$

Thereby, we have for the field strength tensor

$$F^{\mu\nu} = \frac{i}{q}[D^\mu, D^\nu] \ . \tag{9.23}$$

Its transformation behaviour is given by

$$\frac{i}{q}[UD^\mu U^{-1}, UD^\nu U^{-1}] = \frac{i}{q}U[D^\mu, D^\nu]U^{-1} = UF^{\mu\nu}U^{-1} \ . \tag{9.24}$$

The unitary group $U(1)$ is an Abelian gauge group as for $f, g \in U(1)$ it holds that $f \circ g = g \circ f$.

9.2 Representation of Non-Abelian Groups

Be G a group with the elements $g_1, g_2... \in G$. An n-dimensional representation of G is given by the map $G \to C^{(n,n)}$, $g \to U(g)$. It is a map of abstract elements of the group onto complex $n \times n$ matrices, so that $U(g_1 g_2) = U(g_1)U(g_2)$ holds and hence the group properties are preserved.

A $U \in SU(N)$ can be written as $U = \exp(i\theta^a T^a)$. In general, each group element, which can be obtained from the identity element through continuous transformation of the parameters, can be written has $\exp(i\theta^a T^a)$, where θ^a are real parameters and T^a are linearly independent operators. The set of all linear combinations of $\theta^a T^a$ forms a vector space with the basis elements $\theta^a T^a$. They are also called generators of the group. In the case of the $SU(N)$ the generators are hermitian. For the $SU(2)$ we have $U = \exp(i\vec{\omega} \cdot \vec{J})$. The group $SU(N)$ has $N^2 - 1$ generators T^a. For the $SU(2)$ these are the angular momentum operators J_i. The $N^2 - 1$ real parameters θ^a are given by $\vec{\omega}$ in the $SU(2)$. The fundamental representation of the $SU(2)$ reads $J_i = \sigma_i/2$ and in the general case $T^a = \lambda^a/2$.

Independent of the representations the generators fulfill the following commutator relation

$$[T^a, T^b] = if^{abc}T^c \; . \tag{9.25}$$

The f^{abc} are the structure constants of the $SU(N)$ Lie algebra. The commutation relation hence defines the algebra, which is associated with the group. The generators are not uniquely normalized. We have

$$\mathrm{Tr}(T^aT^b) = T_R\delta^{ab} \; , \tag{9.26}$$

where T_R is the Dynkin index. It depends on the representation. For the fundamental representation it is mostly chosen as

$$T_R \equiv T_F = 1/2 \; . \tag{9.27}$$

From Eq. (9.25) follows

$$[T^a, T^b]T^c = if^{abd}T^dT^c \quad \Rightarrow \quad if^{abc}T_R = \mathrm{Tr}([T^a, T^b]T^c) \; . \tag{9.28}$$

The structure constants f^{abc} are hence totally anti-symmetric and define $(N^2-1)(N^2-1)$-dimensional matrices $T^a_{lk} \equiv -if^a_{lk} \equiv -if^{alk}$. For the $SU(2)$ we have

$$[J_i, J_j] = \epsilon_{ijk}J_k \; . \tag{9.29}$$

The generators of Lie groups fulfill the Jacobi identity

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0 \; . \tag{9.30}$$

Using (9.25), one obtains

$$0 = (-if^b_{cl})(-if^a_{lk}) + (-if^a_{lc})(-if^b_{lk}) + if^{abl}(-if^l_{ck}) \; . \tag{9.31}$$

And thereby

$$0 = (T^bT^a)_{ck} - (T^aT^b)_{ck} + if^{abl}(T^l)_{ck} \; . \tag{9.32}$$

We thus have obtained an $N^2 - 1$-dimensional representation of the $SU(N)$ Lie algebra,

$$[T^a, T^b] = if^{abc}T^c \; . \tag{9.33}$$

This is the *adjoint representation.* There are the following $SU(N)$ representations,

- $d = 1$: trivial representation (singulet).
- $d = N$: fundamental representation ($\lambda^a/2$), anti-fundamental representation ($-\lambda^{*a}/2$). The generators are $N \times N$ matrices.
- $d = N^2 - 1$: adjoint representatioin. The generators are $(N^2-1) \times (N^2-1)$-matrices. The indices of the representation run over the same range as the number of generators, which forms the dimension of the group. In the adjoint representation hence the dimension of the vector space, in which the matrices act, is equal to the dimension of the group.

If a representation r and its complex conjugate representation $\bar{r}$ with

$$T^a_{\bar{r}} = -(T^a_r)^* \; , \tag{9.34}$$

are equivalent, hence $T^a_{\bar{r}} = UT^a_rU^\dagger$, then the representation is called real. The fundamental representation of $SU(2)$ is real, but not the one of $SU(3)$. This is why the anti-quarks have an anti-colour. The adjoint representation of the $SU(3)$ is real.

Casimir operators Casimir operators allow to characterise representations independently of the chosen basis. The quadratic Casimir operator is defined by

$$\sum_a T^a T^a = C_2(R) 1 \ , \tag{9.35}$$

where $C_2(R)$ depends on the representation, but not on the basis of the generators T^a.

9.3 The Matrices of the $SU(N)$

The elements of the $SU(N)$ in general are represented through

$$U = \exp\left(i\theta^a \frac{\lambda^a}{2}\right) \qquad \text{with} \qquad \theta^a \in \mathbb{R} \ . \tag{9.36}$$

Here the $\lambda^a/2$ are the generators of the group $SU(N)$. For the $SU(2)$ the λ^a are given by the Pauli matrices σ^i $(i = 1, 2, 3)$ and θ^a is a 3-component vector. For an element of the group $SU(2)$ we hence have

$$U = \exp\left(i\vec{\omega}\frac{\vec{\sigma}}{2}\right) \ . \tag{9.37}$$

For a general U we have

$$U^\dagger = \exp\left(-i\theta^a \left(\frac{\lambda^a}{2}\right)^\dagger\right) \stackrel{!}{=} U^{-1} = \exp\left(-i\theta^a \frac{\lambda^a}{2}\right) \ . \tag{9.38}$$

The generators hence have to be hermitian, i.e.

$$(\lambda^a)^\dagger = \lambda^a \ . \tag{9.39}$$

In addition, for the $SU(N)$ it has to hold that

$$\det(U) = 1 \ . \tag{9.40}$$

With

$$\det(\exp(A)) = \exp(\mathrm{Tr}(A)) \tag{9.41}$$

we get

$$\det\left(\exp\left(i\theta^a \frac{\lambda^a}{2}\right)\right) = \exp\left(i\theta^a \mathrm{Tr}\left(\frac{\lambda^a}{2}\right)\right) \stackrel{!}{=} 1 \ . \tag{9.42}$$

From this follows that

$$\mathrm{Tr}(\lambda^a) = 0 \ . \tag{9.43}$$

The generators of the $SU(N)$ have to be traceless. The group $SU(N)$ has N^2-1 generators λ^a with $\mathrm{Tr}(\lambda^a)=0$. For the $SU(3)$ these are the Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} . \tag{9.44}$$

The matrices $\lambda^a/2$ are normalised as

$$\mathrm{Tr}\left(\frac{\lambda^a}{2}\frac{\lambda^b}{2}\right) = \frac{1}{2}\delta^{ab} . \tag{9.45}$$

For the Pauli matrices ($i=1,2,3$) we have

$$\mathrm{Tr}(\sigma_i^2) = 2 \qquad \text{und} \qquad \mathrm{Tr}(\sigma_1\sigma_2) = \mathrm{Tr}(i\sigma_3) = 0 . \tag{9.46}$$

Multiplied by 1/2 they form the generators of the group $SU(2)$. The generator matrices fulfill the completeness relation

$$\frac{\lambda^a_{ij}}{2}\frac{\lambda^a_{kl}}{2} = \frac{1}{2}\left(\delta_{il}\delta_{kj} - \frac{1}{N}\delta_{ij}\delta_{kl}\right) , \tag{9.47}$$

because

$$0 \stackrel{!}{=} \frac{\lambda^a_{ii}}{2}\frac{\lambda^a_{kl}}{2} = \frac{1}{2}\delta_{il}\delta_{ki} - \frac{1}{2N}\delta_{ii}\delta_{kl} = \frac{1}{2}\delta_{kl} - \frac{1}{2}\delta_{kl} = 0 . \tag{9.48}$$

The gauge group underlying quantum chromo dynamics (QCD) is the $SU(3)$. The QCD describes the strong interaction between colour charged particles. The quarks are in the fundamental representation of the $SU(3)$. The Feynman rule for the interaction between one gluon and two quarks contains the $T^a_{ij} = \lambda^a_{ij}/2$, with $i,j = 1,...,N_c$ ($N_c=3$) and $a=1,...,8$. N_c denotes the number of the quark colours. The gluons are in the adjoint representation of the $SU(3)$, which is expressed through the matrices $(F^a)_{bc} = -if^{abc}$.

9.4 Non-Abelian Gauge Theories, $SU(N)$ Symmetries

In the following we consider a Lagrangian which is invariant under transformations of the group $SU(N)$, where

$$SU(N) = \{U \in \mathbb{C}^{N\times N} | UU^\dagger = 1 \wedge \det U = 1\} . \tag{9.49}$$

Each $U \in SU(N)$ can be written as

$$U = \exp(i\theta_a T^a) , \; \theta_a \in \mathbb{R} . \tag{9.50}$$

From $UU^\dagger = 1$ follows that $T^a = (T^a)^\dagger$, from $\det U = 1$ follows with $\det U = e^{\mathrm{Tr}(\ln U)}$ that $\mathrm{Tr}(T^a) = 0$.

Fermion Fields Starting point is the Lagrangian for N Dirac fields $\psi_i(x)$ $(i = 1, .., N)$,

$$\mathcal{L} = \sum_{i=1...N} \bar{\psi}_i(i\gamma^\mu\partial_\mu - m)\psi_i = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi \qquad \text{with} \quad \bar{\Psi} = (\bar{\psi}_1, \bar{\psi}_2, ..., \bar{\psi}_N) \ . \quad (9.51)$$

The Lagrangian is invariant under a global $SU(N)$ gauge transformation (the index a runs over $a = 1, ..., N^2 - 1$)

$$\Psi \to \Psi' = \exp\left(i\theta^a T^a\right)\Psi = \left(1 + i\theta^a T^a + \mathcal{O}((\theta^a)^2)\right)\Psi = U\Psi = \quad \text{and} \quad \bar{\Psi} \to \bar{\Psi}' = \bar{\Psi}U^{-1} \quad (9.52)$$

respectively, $(i, j = 1, ..., N)$

$$\psi_i(x) \to U_{ij}\psi_j(x) \ . \quad (9.53)$$

The generators T^a are

$$\begin{array}{lll} \text{fundamental representation:} & (T^a)_{ij} = \left(\frac{\lambda^a}{2}\right)_{ij} & d = N \\ \text{adjoint representation} & (T^a)_{bc} = -if^{abc} & d = N^2 - 1 \\ \text{trivial representation} & T^a = 0 \Leftrightarrow U(\theta) = 1 \ . & \end{array} \quad (9.54)$$

Examples:

- $\Psi = \begin{pmatrix} p \\ n \end{pmatrix}$: $SU(2)$ transformations in the isospin space, proton-neutron doublet.
- $\Psi = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$: $SU(2)_L$, weak interaction on left-handed fermions.
- $\Psi = (q_1, q_2, q_3)^T$, quarks, $SU(3)_C$. Here, each q_i $(i = 1, 2, 3)$ is a four-component spinor. The QCD Lagrangian is invariant under $SU(3)_C$ transformations.

Representation of the Gauge Fields The gauge fields are in the adjoint representation of the $SU(N)$. Thereby, we have $N^2 - 1$ gauge fields $G^a_\mu(x)$ $(a = 1, ..., N^2 - 1)$. In a non-Abelian gauge theory also the gauge fields carry charge (e.g. in the QCD the colour charge), in an Abelian gauge theory, however, not (the photon does not have an electric charge). The adjoint representation of the $SU(N)$ is given by the matrices $(T^a)_{bc}$, which are obtained from the structure constants of the group,

$$(T^a)_{bc} = -if^{abc} \ , \quad a, b, c = 1, ...N^2 - 1 \ . \quad (9.55)$$

Fermion Gauge Boson Interaction In analogy to QED we can write the interaction between fermions and gauge bosons as

$$\mathcal{L}_{\text{int}} = \sum_{i,j=1}^{N} \bar{\psi}_i(i\gamma_\mu(D^\mu[G])_{ij} - m_j\delta_{ij})\psi_j \ . \quad (9.56)$$

The covariant derivative is given by

$$(D^\mu[G])_{ij} = \delta_{ij}\partial^\mu - ig\sum_{a=1}^{N^2-1} G^\mu_a(x)T^a_{ij} \equiv \delta_{ij}\partial^\mu - ig(\mathcal{G}^\mu)_{ij} \ . \quad (9.57)$$

The T^a can be different, but G^a_μ is identical in all D_μ. For example in supersymmetry (SUSY),

$$\begin{array}{lll} \text{squark, quark} & T^a = \frac{\lambda^a}{2} & (d = N) \\ \text{gluino, gluon} & (T^a)_{bc} = -if^{abc} & (d = N^2 - 1) \end{array} \quad (9.58)$$

Gauge-Invariant Lagrangian Let us now look at local symmetries, hence $\theta^a = \theta^a(x)$. The transformation of Ψ is given by $\Psi' = U\Psi$. We want to achieve that the Lagrangian is invariant under these gauge transformations. This is fulfilled if the covariant derivative transforms exactly as Ψ, hence $(D_\mu \Psi)' = U(D_\mu \Psi)$. Thereby

$$(D_\mu \Psi)' = D'_\mu \Psi' = D'_\mu U\Psi \Rightarrow D'_\mu U = U D_\mu \; . \tag{9.59}$$

This if fulfilled, because

$$\partial_\mu - ig\mathcal{G}'_\mu \;=\; D'_\mu = U D_\mu U^{-1} = U(\partial_\mu - ig\mathcal{G}_\mu)U^{-1} = UU^{-1}\partial_\mu + U(\partial_\mu U^{-1}) - igU\mathcal{G}_\mu U^{-1} \Rightarrow \tag{9.60}$$

$$\mathcal{G}'_\mu \;=\; \frac{i}{g} U(\partial_\mu U^{-1}) + U\mathcal{G}_\mu U^{-1} \; . \tag{9.61}$$

Important: G'^a_μ is independent of the representation U. With infinitesimal

$$U = \exp(iT^a\theta^a) = 1 + iT^a\theta^a + \mathcal{O}(\theta^{a2}) \tag{9.62}$$

we have

$$\begin{aligned}\mathcal{G}'_\mu \;&=\; G'^b_\mu T^b = \frac{i}{g} U(-i)T^a\,(\partial_\mu\theta^a)\,U^{-1} + \underbrace{(1+i\theta^a T^a)G^c_\mu T^c(1-i\theta^b T^b)}_{G^c_\mu T^c + iG^c_\mu \underbrace{(T^aT^c - T^cT^a)}_{if^{acb}T^b}\theta^a + \mathcal{O}(\theta^2)} \\ &=\; T^b\,\underbrace{(\frac{1}{g}\partial_\mu\theta^b + G^b_\mu + i(-if^{abc})\theta^a G^c_\mu)}_{G'^b_\mu} \; .\end{aligned} \tag{9.63}$$

The field strength tensor is defined as $\mathcal{F}^{\mu\nu} \sim [D^\mu, D^\nu]$. Let us look at the commutator,

$$\begin{aligned}[D^\mu, D^\nu] \;&=\; [\partial_\mu - igT^aG^a_\mu, \partial_\nu - igT^bG^b_\nu] = -igT^b\partial_\mu G^b_\nu - igT^a(-\partial_\nu G^a_\mu) + (-ig)^2 G^a_\mu G^b_\nu \underbrace{[T^a, T^b]}_{if^{abc}T^c} \\ &=\; -igT^a\,\underbrace{(\partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g\,\underbrace{f^{bca}}_{f^{abc}}\,G^b_\mu G^c_\nu)}_{=:F^a_{\mu\nu}} = -igT^aF^a_{\mu\nu} \equiv -ig\mathcal{F}_{\mu\nu} \; .\end{aligned} \tag{9.64}$$

The $F^a_{\mu\nu}$ are independent of the representation of the T^a. We have for the transformation behaviour

$$\mathcal{F}'_{\mu\nu} \;=\; \frac{i}{g}[D'^\mu, D'^\nu] = \frac{i}{g}[UD_\mu U^{-1}, UD_\nu U^{-1}] = U\mathcal{F}_{\mu\nu}U^{-1} \tag{9.65}$$

homogenuous transformation

And with Eq. (9.63)

$$(F^a_{\mu\nu})' = F^a_{\mu\nu} + i(-if^{bac})\theta^b F^c_{\mu\nu} + ... \tag{9.66}$$

Furthermore, from this follows that

$$F^{a\mu\nu}F^a_{\mu\nu} \;=\; 2\mathrm{Tr}(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu})\left(= 2\mathrm{Tr}(F^{a\mu\nu}T^aF^b_{\mu\nu}T^b) = 2F^{a\mu\nu}F^b_{\mu\nu}\underbrace{\mathrm{Tr}(T^aT^b)}_{\frac{1}{2}\delta^{ab}} = F^{\mu\nu a}F^a_{\mu\nu}\right) \tag{9.67}$$

is gauge invariant

Thereby we have for the kinetic Lagrangian

$$\mathcal{L}_{kin,A} = -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} = -\frac{1}{2}\mathrm{Tr}(\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}) \ . \qquad (9.68)$$

This Lagrangian for the gauge fields is also called Yang-Mills Lagrangian. It contains cubic and quartic terms in the the gauge fields. This leads in QCD to the 3-gluon and the 4-gluon vertices. Remark that the gauge fields as in the case of the photon have to be massless. A mass term bilinear in the G^a_μ would break the $SU(N)$ gauge invariance.

9.5 The QCD Lagrangian

Example: QCD is invariant under the colour $SU(3)$. The 6 quark fields carry colour charge and are in the fundamental representation,

$$\Psi_q = \begin{pmatrix} \psi_{q1} \\ \psi_{q2} \\ \psi_{q3} \end{pmatrix} \qquad q = u, d, c, s, t, b \ . \qquad (9.69)$$

They form triplets. The 8 gluons G^μ are in the adjoint representation. The QCD Lagrangian reads

$$\mathcal{L}_{QCD} = -\frac{1}{4}G^{a\mu\nu}G^a_{\mu\nu} + \sum_{q=1...6}\bar{\Psi}_q(i\gamma^\mu D_\mu - m_q)\Psi_q \ , \qquad (9.70)$$

with

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g f^{abc} G^b_\mu G^c_\nu \ . \qquad (9.71)$$

The quark masses have the values

$$m_u \approx 1.7...3.1 \text{ MeV} \qquad m_d \approx 4.1...5.7 \text{ MeV} \qquad m_s \approx 100 \text{ MeV} \qquad (9.72)$$

$$m_c \approx 1.29 \text{ GeV} \qquad m_b \approx 4.19 \text{ GeV} \qquad m_t \approx 173 \text{ GeV} \ . \qquad (9.73)$$

9.6 Chiral Gauge Theories

Let us look at

$$\mathcal{L}_f = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi \ . \qquad (9.74)$$

In the chiral representation the 4×4 γ matrices are given by

$$\gamma^\mu = \left(\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} & -\vec{\sigma} \\ \vec{\sigma} & \mathbf{0} \end{pmatrix} \right) = \begin{pmatrix} 0 & \sigma^\mu_- \\ \sigma^\mu_+ & 0 \end{pmatrix} \qquad (9.75)$$

$$\gamma^5 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \ , \qquad (9.76)$$

where σ_i $(i = 1, 2, 3)$ are the Pauli matrices. With

$$\Psi = \begin{pmatrix} \chi \\ \varphi \end{pmatrix} \quad \text{and} \quad \bar{\Psi} = \Psi^\dagger\gamma^0 = (\chi^\dagger, \varphi^\dagger)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (\varphi^\dagger, \chi^\dagger) \qquad (9.77)$$

we get

$$\bar{\Psi} i\gamma^\mu D_\mu \Psi = i(\varphi^\dagger, \chi^\dagger) \underbrace{\begin{pmatrix} 0 & \sigma_-^\mu \\ \sigma_+^\mu & 0 \end{pmatrix} \begin{pmatrix} D_\mu \chi \\ D_\mu \varphi \end{pmatrix}}_{\begin{pmatrix} \sigma_-^\mu D_\mu \varphi \\ \sigma_+^\mu D_\mu \chi \end{pmatrix}} = \varphi^\dagger i \sigma_-^\mu D_\mu \varphi + \chi^\dagger i \sigma_+^\mu D_\mu \chi \,. \tag{9.78}$$

The gauge interaction holds independently bouth for

$$\Psi_L = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \frac{1}{2}(1-\gamma_5)\Psi \quad \text{and} \quad \Psi_R = \begin{pmatrix} \chi \\ 0 \end{pmatrix} = \frac{1}{2}(1+\gamma_5)\Psi \,. \tag{9.79}$$

The Ψ_L and Ψ_R can have different gauge representations. But

$$m\bar{\Psi}\Psi = m(\varphi^\dagger, \chi^\dagger) \begin{pmatrix} \chi \\ \varphi \end{pmatrix} = m(\varphi^\dagger \chi + \chi^\dagger \varphi) = m(\bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L) \,. \tag{9.80}$$

The mass term mixes Ψ_L and Ψ_R. This implies *symmetry breaking* if Ψ_L and Ψ_R have different representations.

What about a mass term for gauge bosons? Let us look at

$$\mathcal{L} = -\frac{1}{4} \underbrace{F^{a\mu\nu} F^a_{\mu\nu}}_{\text{gauge invariant}} + \frac{m^2}{2} \underbrace{A^{a\mu} A^a_\mu}_{\text{not gauge invariant}} \,. \tag{9.81}$$

For example for the $U(1)$

$$(A_\mu A^\mu)' = (A_\mu + \partial_\mu \theta)(A^\mu + \partial^\mu \theta) = A_\mu A^\mu + 2A_\mu \partial^\mu \theta + (\partial_\mu \theta)(\partial^\mu \theta) \,. \tag{9.82}$$

The mass term for A^μ breaks the gauge symmetry.

Chapter 10

Spontaneous Symmetry Breaking

Die Symmetrie einer Lagrangedichte ist *spontan gebrochen*, wenn die Lagrangedichte symmetrisch ist, aber das physikalische Vakuum *nicht* der Symmetrie gehorcht. Wir werden sehen, daß, wenn die Lagrangedichte einer Theorie invariant unter einer exakten kontinuierlichen Symmetrie ist, welche nicht die Symmetrie des physikalischen Vakuums ist, eines oder mehrere masselose Spin-0 Teilchen auftreten. Diese werden Goldstone Bosonen genannt. Wenn die spontan gebrochene Symmetrie eine lokale Eichsymmetrie ist, führt das Zusammenspiel (induziert durch den Higgsmechanismus) zwischen den Möchtegern-Goldstone Bosonen und den masselosen Eichbosonen zu den Massen der Eichbosonen und entfernt die Goldstone Bosonen aus dem Spektrum.

10.1 Beispiel: Ferromagnetismus

Es handelt sich um ein System wechselwirkender Spins,

$$H = -\sum_{i,j} J_{ij} \vec{S}_i \cdot \vec{S}_j \; . \tag{10.1}$$

Das Skalarprodukt der Spinoperatoren ist unter Rotation ein Singulett, ist also rotationsinvariant. Im Grundzustand des Ferromagneten (bei genügend niedriger Temperatur, unterhalb der Curie-Temperatur) zeigen alle Spins in dieselbe Richtung. Dies ist der Zustand niedrigster Energie. Der Grundzustand ist nicht mehr rotationsinvariant. Bei Drehung des Systems entsteht ein neuer Grundzustand derselben Energie, der sich aber vom vorigen unterscheidet. Der Grundzustand ist also entartet. Die Auszeichnung einer bestimmten Richtung bricht die Symmetrie. Es liegt spontane Symmetriebrechung (SSB) vor.

10.2 Beispiel: Feldtheorie für ein komplexes Feld

Wir betrachten die Lagrangedichte für ein komplexes Skalarfeld

$$\mathcal{L} = (\partial_\mu \phi)^* (\partial^\mu \phi) - \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2 \quad \text{mit dem Potential} \quad V = \mu^2 \phi^* \phi + \lambda (\phi^* \phi)^2 \tag{10.2}$$

(Hinzufügen höherer Potenzen in ϕ führt zu einer nicht-renormierbaren Theorie.) Die Lagrangedichte ist invariant unter einer $U(1)$-Symmetrie,

$$\phi \to \exp(i\alpha)\phi \; . \tag{10.3}$$

Figure 10.1: Das Higgspotential.

Wir betrachten den Grundzustand. Dieser ist gegeben durch das Minimum von V,

$$0 = \frac{\partial V}{\partial \phi^*} = \mu^2 \phi + 2\lambda(\phi^*\phi)\phi \quad \Rightarrow \quad \phi = \begin{cases} 0 & \text{für} \quad \mu^2 > 0 \\ \phi^*\phi = -\frac{\mu^2}{2\lambda} & \text{für} \quad \mu^2 < 0 \end{cases} \tag{10.4}$$

Der Parameter λ muß positiv sein, damit das System nicht instabil wird. Für $\mu^2 < 0$ nimmt das Potential die Form eines Mexikanerhutes an, siehe Fig. 10.1. Bei $\phi = 0$ liegt ein lokales Maximum, bei

$$|\phi| = v = \sqrt{-\frac{\mu^2}{2\lambda}} \tag{10.5}$$

ein globales Minimum. Teilchen entsprechen harmonischen Oszillatoren für die Entwicklung um das Minimum des Potentials. Fluktuationen in Richtung der (unendlich vielen degenerierten) Minima besitzen Steigung null und entsprechen masselosen Teilchen, den Goldstone Bosonen. Fluktuationen senkrecht zu dieser Richtung entsprechen Teilchen mit Masse $m > 0$. Die Entwicklung um das Maximum bei $\phi = 0$ würde zu Teilchen negativer Masse (Tachyonen) führen, da die Krümmung des Potentials hier negativ ist.

Entwicklung um das Minimum bei $\phi = v$ führt zu (wir haben für das komplexe skalare Feld zwei Fluktuationen φ_1 und φ_2)

$$\phi = v + \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2) = \left(v + \frac{1}{\sqrt{2}}\varphi_1\right) + i\frac{\varphi_2}{\sqrt{2}} \quad \Rightarrow \tag{10.6}$$

$$\phi^*\phi = v^2 + \sqrt{2}v\varphi_1 + \frac{1}{2}(\varphi_1^2 + \varphi_2^2) \, . \tag{10.7}$$

Damit erhalten wir für das Potential

$$V = \lambda(\phi^*\phi - v^2)^2 - \frac{\mu^4}{4\lambda} \qquad \text{mit} \qquad v^2 = -\frac{\mu^2}{2\lambda} \quad \Rightarrow \tag{10.8}$$

$$V = \lambda\left(\sqrt{2}v\varphi_1 + \frac{1}{2}(\varphi_1^2 + \varphi_2^2)\right)^2 - \frac{\mu^4}{4\lambda} \, . \tag{10.9}$$

Vernachlässige den letzten Term in V, da es sich nur um eine konstante Nullpunktsverschiebung handelt. Damit ergibt sich für die Lagrangedichte

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi_1)^2 + \frac{1}{2}(\partial_\mu \varphi_2)^2 - 2\lambda v^2 \varphi_1^2 - \sqrt{2}v\lambda\varphi_1(\varphi_1^2 + \varphi_2^2) - \frac{\lambda}{4}(\varphi_1^2 + \varphi_2^2)^2 \, . \tag{10.10}$$

Die in den Feldern quadratischen Terme liefern die Massen, die in den Feldern kubischen und quartischen Terme sind die Wechselwirkungsterme. Es gibt ein massives und ein masseloses Teilchen,

$$m_{\varphi_1} = 2v\sqrt{\lambda} \quad \text{und} \quad m_{\varphi_2} = 0 \, . \tag{10.11}$$

Bei dem masselosen Teilchen handelt es sich um das Goldstone Boson.

10.3 Das Goldstone Theorem

Seien

$$\begin{aligned} N &= \text{Dimension der Algebra der Symmetriegruppe der vollständigen Lagrangedichte.} \\ M &= \text{Dimension der Algebra der Gruppe, unter welcher das Vakuum nach der} \\ & \quad\ \text{spontanen Symmetriebrechung invariant ist.} \end{aligned}$$

$\Rightarrow$ Es gibt N-M Goldstone Bosonen ohne Masse in der Theorie.

Das Goldstone Theorem besagt, daß es für jeden spontan gebrochenen Freiheitsgrad der Symmetrie ein masseloses Goldstone Boson gibt.

Sei $\phi_i(x)$ ein Satz von Operatoren mit nichttrivialem Transformationsverhalten unter einer Symmetriegruppe G. Das Transformationsverhalten ist gegeben durch

$$[Q^a, \phi_i(x)] = T^a_{ij}\phi_j(x) \quad \text{mit} \quad Q^a = \int d^3y j^{0a}(y) \quad \text{und} \quad \partial_\mu j^{\mu a} = 0 \, . \tag{10.12}$$

Die T^a sind die Darstellungsmatrizen der Generatoren. Ist der Vakuumerwartungswert (VEV) $< 0|\phi_j(x)|0 >$ eines dieser nichttrivial transformierenden Felder ungleich null, dann existieren masselose Anregungen.

Beweis:
Falls $< 0|\phi_j|0 > \neq 0$ gibt es ein T^a mit $0 \neq < 0|T^a_{ij}\phi_j|0 >$, da der Satz von Generatoren T^a linear unabhängig ist. Damit gilt

$$0 \neq < 0|T^a_{ij}\phi_j|0 > = < 0|[Q^a, \phi_i]|0 > \ . \tag{10.13}$$

Es existiert also ein Ladungsoperator Q und ein Feld $\phi(x)$, für die $< 0|[Q, \phi(x)]|0 > \neq 0$. Daraus ergibt sich $Q|0 > \neq 0$. Das Vakuum hat also eine Ladung $\neq 0$. Es transformiert sich nichttrivial unter Symmetrietransformationen. Da es auf x nicht ankommt, wähle den Nullpunkt. Damit

$$\frac{dQ}{dt} = 0 \quad \Rightarrow \quad \frac{d}{dt}[Q(t), \phi(0)] = 0 \, . \tag{10.14}$$

Und damit also

$$< 0|[Q(t), \phi(0)]|0 > = C \neq 0 \, . \tag{10.15}$$

Es ist

$$j^0(y) = \exp(-iP \cdot y) j^0(0) \exp(iP \cdot y) \, , \tag{10.16}$$

wobei P^μ der Impulsoperator ist. Sei die Translationssymmetrie spontan gebrochen,

$$
\begin{aligned}
C &= \sum_n \int d^3y \, \big\{ < 0|\exp(-iP\cdot y) j^0(0) \exp(iP\cdot y)|n >< n|\phi(0)|0 > \\
&\quad - < 0|\phi(0)|n >< n|\exp(-iP\cdot y) j^0(0)\exp(iP\cdot y)|0 > \big\} \\
&= \sum_n \int d^3y \, \big\{ [< 0|j^0(0)\exp(iP_n\cdot y)|n >< n|\phi(0)|0 > \\
&\quad - < 0|\phi(0)|n >< n|\exp(-iP_n\cdot y) j^0(0)|0 >] \big\} \\
&= (2\pi)^3 \sum_n \big\{ [< 0|j^0(0)|n >< n|\phi(0)|0 > \exp(iE_n t) \\
&\quad - < 0|\phi(0)|n >< n|j^0(0)|0 > \exp(-iE_n t)] \delta^{(3)}(\vec{P}_n) \big\} \ .
\end{aligned}
\tag{10.17}
$$

Hier wurde

$$
\int d^3y \exp(\pm i\vec{P}_n\vec{y}) = (2\pi)^3 \delta^{(3)}(\vec{P}_n) \tag{10.18}
$$

verwendet. Da

$$
\exp(-iE_n t) = \exp(-iM_n t) \tag{10.19}
$$

kann der Beitrag zu $C = const.$ nur von $M_n = 0$ kommen. Der Vakuumzustand $|n >= |0 >$ trägt nicht bei, da sich beide Terme wegheben. Das heisst

$$
\text{Es gibt einen Zustand } |n > \neq |0 > \quad \text{mit } M_n = 0 \quad \text{und} < n|\phi(0)|0 > \neq 0 \neq < n|j^0(0)|0 > \tag{10.20}
$$

Der Beweis verlangt

- Manifeste Lorentz-Kovarianz
- Vollständigkeit der physikalischen Zustände.

Diese Bedingung kann von Eichtheorien nicht erfüllt werden. Um beispielsweise die Elektrodynamik zu quantisieren, muß zwischen dem Lorentz-kovarianten Gupta-Bleuler Formalismus mit unphysikalischen indefiniten metrischen Zuständen oder der Quantisierung in einer physikalischen Eichung, wo manifeste Lorentz-Kovarianz verloren geht, gewählt werden.

Für Eichtheorien gilt das Goldstone Theorem nicht: Masselose skalare Freiheitsgrade werden von den Eichbosonen absorbiert, um ihnen Masse zu geben. Das Goldstone Phänomen führt zum Higgs Phänomen.

10.4 Chirale Symmetriebrechung in der QCD

Die Masse der Pionen ist sehr klein, $0 \approx m_\pi \approx 10^{-1} m_P$. Es stellt sich die Frage, warum. Da Pionen nur u- und d-Quarks enthalten, betrachten wir nur diese beiden Quark-Flavours. Für die Masse der u- und d-Quarks haben wir $m_{u,d} \approx \mathcal{O}(5 \text{ MeV}) \ll \Lambda_{QCD}$. Betrachten wir nun die Lagrangedichte für verschwindende u- und d-Quarkmassen,

$$
\mathcal{L} = \bar{u} i \not{D} u + \bar{d} i \not{D} d \ . \tag{10.21}
$$

Mit $\psi = (u,d)^T$ können wir die Lagrangedichte ich einen links- und rechtshändigen Anteil aufspalten,

$$\psi_L = \frac{1}{2}(\mathbb{1} - \gamma_5)\psi \qquad \text{und} \qquad \psi_R = \frac{1}{2}(\mathbb{1} + \gamma_5)\psi \; . \tag{10.22}$$

Also

$$\mathcal{L} = \sum_{s=L,R} \bar{u}_s i \not{D} u_s + \bar{d}_s \not{D} i d_s \; . \tag{10.23}$$

Sie ist invariant unter einer $SU(2)$-Symmetrie

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix} \to \exp\left(i\theta_L \cdot \frac{\vec{\sigma}}{2}\right) \begin{pmatrix} u_L \\ d_L \end{pmatrix} \qquad \begin{pmatrix} u_R \\ d_R \end{pmatrix} \to \exp\left(i\theta_R \cdot \frac{\vec{\sigma}}{2}\right) \begin{pmatrix} u_R \\ d_R \end{pmatrix} . \tag{10.24}$$

Die Lagrangedichte ist separat symmetrisch für die links- und rechtschiralen Terme. Sie ist also symmetrisch unter einer $SU(2)_L \times SU(2)_R$. Es gibt die erhaltenen Ströme

$$(\bar{u}, \bar{d})\gamma^\mu \frac{\vec{\sigma}}{2}\frac{1}{2}(\mathbb{1} + \gamma_5) \begin{pmatrix} u \\ d \end{pmatrix} \qquad \text{und} \qquad (\bar{u}, \bar{d})\gamma^\mu \frac{\vec{\sigma}}{2}\frac{1}{2}(\mathbb{1} - \gamma_5) \begin{pmatrix} u \\ d \end{pmatrix} \tag{10.25}$$

Addition und Subtraktion der Ströme führt auf den Vektor- (V_μ) und den Axialvektorstrom (A_μ)

$$V_\mu^i = (\bar{u}, \bar{d})\gamma_\mu \frac{\sigma^i}{2} \begin{pmatrix} u \\ d \end{pmatrix} \qquad \text{und} \qquad A_\mu^i = (\bar{u}, \bar{d})\gamma_\mu\gamma_5 \frac{\sigma^i}{2} \begin{pmatrix} u \\ d \end{pmatrix} \qquad i = 1, 2, 3 \; . \tag{10.26}$$

Damit verbunden sind 6 erhaltene Ladungen. Die Felder selbst können keinen von null verschiedenen VEV haben. (Farbneutralität des QCD-Vakuums). Allerdings kann das Kondensat aus Quark und Antiquark einen nichtverschwindenden VEV besitzen,

$$< 0|\bar{u}(x)u(x)|0 >=< 0|\bar{d}(x)d(x)|0 >\neq 0 \; . \tag{10.27}$$

Dieser erhält zwar die $SU(2)$-Symmetrie, bricht aber die axiale Symmetrie spontan. Diese SSB führt auf drei masselose Goldstonebosonen, die Pionen π^+, π^- und π^0. Es handelt sich um pseudoskalare Mesonen. Dabei

$$\pi^+ = \frac{1}{\sqrt{2}}(\pi_1 + i\pi_2) \qquad \text{und} \qquad \pi^- = \frac{1}{\sqrt{2}}(\pi_1 - i\pi_2) \; . \tag{10.28}$$

Ferner $(i, j = 1, 2, 3)$

$$\begin{aligned} < 0|A_\mu^i(y)|\pi^j(k) >\neq 0 \;\; &= \;\; < 0|\exp(iP\cdot y)A_\mu^i(0)\exp(-iP\cdot y)|\pi^j(k) > \\ &= \;\; < 0|A_\mu^i(0)\exp(-ik\cdot y)|\pi^j(k) > \\ &= \;\; \exp(-ik\cdot y) < 0|A_\mu^i(0)|\pi^j(k) > \\ &= \;\; i f_\pi \delta^{ij}\exp(-iky)k_\mu \; . \end{aligned} \tag{10.29}$$

Dies ist die Grundlage, um die Lebensdauer für Pionen auszurechnen. Dabei ist f_π die Zerfallskonstante des Pions.

Die Dimension von $SU(2)_L \times SU(2)_R$ ist $d_i = 6$. Diese wurde heruntergebrochen auf die $SU(2)$ mit Dimension $d_f = 3$. Die Anzahl der Goldstone Bosonen entspricht der Anzahl der spontan gebrochenen Generatoren $d_i - d_f = 3$. Wir haben also drei masselose Pionen.

In Wirklichkeit sind die u- und d-Quarks nicht masselos. Die chirale Symmetrie ist also nicht nur spontan, sondern auch explizit gebrochen. Da die betroffenen Quarkmassen jedoch sehr klein sind, ist auch die Masse der Pionen recht klein.

10.5 Spontane Brechung einer $O(N)$ Symmetrie

Wir betrachten die Lagrangedichte für N reelle skalare Felder

$$\mathcal{L} = \sum_{i=1...N} \frac{1}{2}(\partial_\mu \phi_i)(\partial^\mu \phi_i) - V\left(\sum_{i=1...N} \phi_i^2\right) \qquad \text{mit} \qquad V\left(\sum_{i=1...N} \phi_i^2\right) = V(\vec{\phi}\cdot\vec{\phi}) \tag{10.30}$$

Die Lagrangedichte ist symmetrisch bezüglich einer $O(N)$ Transformation $\phi_i \to O_{ij}\phi_j$ ($i, j = 1...N$). Bei O handelt es sich um orthogonale $N \times N$ Matrizen. Das Minimum von V sei bei $|\vec{\phi}| = v \neq 0$, z.B. $V = \lambda(\phi^2 - v^2)^2$. Wir entwickeln $\vec{\phi}$ um das Minimum. O.B.d.A.,

$$\vec{\phi} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ v \end{pmatrix} + \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_{N-1} \\ \varphi_N \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_{N-1} \\ v + \varphi_N \end{pmatrix} \tag{10.31}$$

Und

$$\vec{\phi}^2 = v^2 + 2v\varphi_N + \sum_{i=1...N} \varphi_i^2 \ . \tag{10.32}$$

Die Richtung N bzw. φ_N ist damit ausgezeichnet. Die restlichen $N-1$ Felder sind nach wie vor invariant unter einer $N-1$-dimensionalen Rotation. Für das Potential erhalten wir

$$V = \lambda\left(2v\varphi_N + \sum_{i=1...N} \varphi_i^2\right)^2 = 4\lambda v^2 \varphi_N^2 + 4\lambda v \varphi_N \sum_{i=1...N} \varphi_i^2 + \lambda\left(\sum_{i=1...N} \varphi_i^2\right)^2 \ . \tag{10.33}$$

Die in den Feldern kubischen und quartischen Terme beschreiben die Wechselwirkungen. Der in φ_N quadratische Term ist der mit φ_N assoziierte Massenterm. Die Masse zum Quadrat ist

$$m_{\varphi_N}^2 = 8\lambda v^2 \ . \tag{10.34}$$

Es handelt sich hier um ein massives *Higgs Boson*, welches einen nichtverschwindenden VEV v besitzt. Die übrigen $N-1$ Felder sind masselos, $m_i = 0$ für $i = 1...N-1$. Es handelt sich um die Goldstone Bosonen. Die ursprüngliche Symmetrie $O(N)$ mit $N(N-1)/2$ Generatoren wurde heruntergebrochen auf die Symmetrie $O(N-1)$ mit $(N-1)(N-2)/2$ Generatoren. Die Anzahl der Goldstone Bosonen d_G entspricht der Anzahl der gebrochenen Generatoren, also

$$\frac{1}{2}\left[N(N-1) - (N-1)(N-2)\right] = N - 1 \ . \tag{10.35}$$

Wir haben also $N-1$ Goldstone Bosonen.

10.6 Spontan gebrochene Eichsymmetrien

Wir betrachten als Beispiel die Lagrangedichte eines komplexen skalaren Feldes Φ, welches an ein Photonfeld A_μ koppelt, die invariant ist unter $U(1)$. Die lokalen Transvormationen sind gegeben durch

$$\Phi \to \exp(-ie\Lambda(x))\Phi(x) \qquad \text{und} \qquad A_\mu \to A_\mu + \partial_\mu\Lambda\ . \tag{10.36}$$

Die Lagrangedichte lautet

$$\mathcal{L} = [(\partial_\mu - ieA_\mu)\Phi^*][(\partial^\mu + ieA^\mu)\Phi] \underbrace{-\mu^2\Phi^*\Phi - \lambda(\Phi^*\Phi)^2}_{-V(\Phi)} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\ . \tag{10.37}$$

(Bemerkung: Um die Lagrangedichte zu quantisieren muß noch ein Eichfixierungsterm eingeführt werden.) Für $\mu^2 < 0$ kommt es zu spontaner Symmetriebrechung der $U(1)$. Dann hat das Feld einen nichtverschwindenden VEV,

$$< 0|\Phi|0 >= v = \sqrt{\frac{-\mu^2}{2\lambda}}\ . \tag{10.38}$$

Die Fluktuationen um das Minimum (Entwicklung um das Minimum) sind gegeben durch

$$\Phi = v + \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2) = \left(v + \frac{H(x)}{\sqrt{2}}\right)\exp\left(\frac{i}{\sqrt{2}}\frac{\chi(x)}{v}\right)\left(\approx v + \frac{1}{\sqrt{2}}(H(x) + i\chi(x))\right) \tag{10.39}$$

Damit

$$\begin{aligned} D_\mu\Phi = (\partial_\mu + ieA_\mu)\Phi(x) &= \frac{1}{\sqrt{2}}(\partial_\mu\varphi_1 + i\partial_\mu\varphi_2) + ieA_\mu v + \frac{e}{\sqrt{2}}A_\mu(-\varphi_2 + i\varphi_1) \\ &= \exp\left(i\frac{\chi}{\sqrt{2}v}\right)\left[\partial_\mu + ie\left(A_\mu + \frac{\partial_\mu\chi}{\sqrt{2}ev}\right)\right]\left(v + \frac{H}{\sqrt{2}}\right) \end{aligned} \tag{10.40}$$

Um bilineare Mischterme in den Feldern zu vermeiden, führen wir folgende Eichtransformation durch,

$$A'_\mu = A_\mu + \partial_\mu\left(\frac{\chi}{\sqrt{2}ev}\right)\ . \tag{10.41}$$

Damit ergibt sich für die kinetische Energie (nenne A' ab jetzt A)

$$\begin{aligned} (D_\mu\Phi)^*(D^\mu\Phi) &= \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + e^2A_\mu A^\mu\left(v + \frac{H}{\sqrt{2}}\right)^2 = \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + \underbrace{(e^2v^2)}_{\frac{1}{2}m_A^2}A_\mu A^\mu \\ &\quad \underbrace{+e^2A_\mu A^\mu\left(\sqrt{2}vH + \frac{H^2}{2}\right)}_{\text{Wechselwirkungsterme}}\ . \end{aligned} \tag{10.42}$$

Und die gesamte Lagrangedichte lautet

$$\begin{aligned} \mathcal{L} &= = \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + \frac{1}{2}m_A^2A_\mu A^\mu + e^2A_\mu A^\mu\left(\sqrt{2}vH + \frac{H^2}{2}\right) \\ &\quad -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \underbrace{2\lambda v^2}_{\frac{1}{2}m_H^2}H^2 - \sqrt{2}v\lambda H^3 - \frac{\lambda}{4}H^4\ . \end{aligned} \tag{10.43}$$

Hierbei wurde der konstante Term λv^4, welcher lediglich den Nullpunkt des Vakuums verschiebt, weggelassen. Die Massen des Higgsteilchens H und des Photons ergeben sich zu

$$m_A^2 = 2e^2v^2 \tag{10.44}$$
$$m_H^2 = 4\lambda v^2 . \tag{10.45}$$

Es tritt also ein massives Photon (Eichboson) und ein massives skalares Feld, das Higgsteilchen, auf. Das Goldstone Boson tritt als Freiheitsgrad nicht in Erscheinung. Die Anzahl der Freiheitsgrade ist aber erhalten geblieben. Denn bei ungebrochener $U(1)$-Symmetrie ist das Photon masselos und besitzt 2 physikalische Freiheitsgrade, die zwei transversalen Polarisationen. Die unphysikalische skalare und longitudinale Polarisation tragen im Gupta-Bleuler-Formalismus nicht bei. Das komplexe skalare Feld (entspricht einem geladenen Teilchen) Φ besitzt 2 Freiheitsgrade. Bei gebrochener $U(1)$-Symmetrie haben wir ein massives Photon mit 3 Freiheitsgraden (mit longitudinaler Polarisation) und ein massives reelles Higgs Boson mit einem Freiheitsgrad. Das Goldstone Boson wurde *aufgegessen*, um dem Photon Masse zu geben, d.h. um den longitudinalen Freiheitsgrad des massiven Eichteilchens zu liefern.

Nochmal: In Eichtheorien treten die Goldstone Bosonen nicht in Erscheinung. Sie sind *Möchtegern* (im Englischen *would-be*) Goldstone Bosonen. Bei SSB werden sie direkt in die longitudinalen Freiheitsgrade der massiven Eichbosonen absorbiert. Es gilt bei Eichtheorien: Seien

N = Dimension der Algebra der Symmetriegruppe der vollständigen Lagrangedichte.
M = Dimension der Algebra der Gruppe, unter welcher das Vakuum nach der spontanen Symmetriebrechung invariant ist.
n = Die Anzahl der skalaren Felder

$$\Rightarrow$$

Es gibt M masselose Vektorfelder. (M ist die Dimension der Symmetrie des Vakuums.)
Es gibt $N-M$ massive Vektorfelder. ($N-M$ ist die Anzahl der gebrochenen Generatoren.)
Es gibt $n-(N-M)$ skalare Higgsfelder.

10.7 Addendum: Goldstone Theorem - klassische Feldtheorie

<u>Proof of the Goldstone theorem in classical field theory:</u>
The Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - V(\varphi) \tag{10.46}$$

is invariant under the rotation

$$\varphi \to e^{-i\alpha_a R_a}\varphi \qquad a = 1, ..., N , \tag{10.47}$$

which can infinitesimally be written as

$$\varphi \to \varphi - i\alpha R\varphi \tag{10.48}$$

From the invariance it follows that

$$\delta V = \frac{\partial V}{\partial \varphi}\delta\varphi = -i\alpha\frac{\partial V}{\partial \varphi}R\varphi = 0 \qquad \forall \alpha, \varphi \tag{10.49}$$

so that

$$\frac{\partial^2 V}{\partial\varphi\partial\varphi}R\varphi + \frac{\partial V}{\partial \varphi}R = 0 \tag{10.50}$$

After spontaneous symmetry breaking we have the ground state

$$\frac{\partial V}{\partial \varphi} = 0 \qquad \text{for} \quad \varphi = v \neq 0 \tag{10.51}$$

from which follows the Goldstone equation:

$$\boxed{\frac{\partial^2 V}{\partial\varphi\partial\varphi} = 0 \qquad \text{for} \quad \varphi = v} \tag{10.52}$$

and

$$\frac{\partial^2 V}{\partial\varphi\partial\varphi} \equiv M^2 \tag{10.53}$$

is the mass matrix of the system. Expanding φ about the ground state

$$\varphi = v + \varphi' \tag{10.54}$$

we have

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial\varphi)^2 - [V(v) + \overbrace{\frac{\partial V}{\partial \varphi}}^{0}\varphi' + \frac{1}{2}\varphi'\frac{\partial^2 V}{\partial\varphi\partial\varphi}\varphi' + ...] \\ &= \frac{1}{2}(\partial\varphi')^2 - \frac{1}{2}\varphi'\frac{\partial^2 V}{\partial\varphi\partial\varphi}\varphi' + ... \end{aligned} \tag{10.55}$$

The Goldstone equation is thus the condition equation for the masses

$$\underline{\underline{M^2 R v = 0}} \tag{10.56}$$

- The equation is fulfilled if the generators R^a, $a = 1, 2, ..., M$ leave the vacuum invariant: $R^a v = 0$.
- The remaining generators R^a, $a = M+1, ..., N$ form a set of linearly independent vectors $R^a v$. These are eigen-vectors of the zero-eigenvalues of the mass matrix M^2. The zero-eigenvalue is hence $N - M$ times degenerated. Q.e.d.

Appendix A

Addendum: Mathematische Hintergrundinformationen

A.1 Gruppen

Sei ein Paar $(G, *)$ mit einer Menge G und einer inneren zweistelligen Verküpfung/Gruppenmultiplikation. $* : G \times G \to G, (a, b) \mapsto a * b$ heißt Gruppe, wenn folgende Axiome erfüllt sind

1. Die Gruppe ist *abgeschlossen*. D.h. wenn $g, h \,\epsilon\, G \Rightarrow g * h \,\epsilon\, G$.
2. *Assoziativität:* $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$.
3. $\exists$ *Einselement* e mit der Eigenschaft $g * e = e * g = g \ \ \forall\, g \,\epsilon\, G$.
4. Zu jedem g gibt es ein *Inverses* g^{-1} mit $g^{-1} * g = g * g^{-1} = e$.

Abelsche Gruppe: Eine Gruppe heißt *abelsch*, wenn $g * h = h * g$.

Kontinuierliche Gruppen: Sie besitzen unendlich viele Elemente und werden durch n Parameter beschrieben. Bei *Liegruppen* ist n endlich. Alle einparametrigen Liegruppen sind abelsch. Typisches Beispiel: $U(1)$ mit den Elementen $e^{i\phi}$ und ϕ als Parameter.

A.2 Algebra

Ein linearer Raum (Vektorraum) wird zu einer *Algebra* $\mathbf{A}$, wenn eine binäre Operation (Multiplikation) zweier Elemente m, n existiert, so daß $mn \ \in \mathbf{A}$. Es gelten die Linearitätsbeziehungen $(k, m, n \in \mathbf{A})$

$$\begin{aligned} k(c_1 m + c_2 n) &= c_1 km + c_2 kn \\ (c_1 m + c_2 n)k &= c_1 mk + c_2 nk \,. \end{aligned} \tag{A.1}$$

Dabei sind c_1, c_2 reelle (komplexe) Zahlen. Man spricht je nach Fall von reeller (komplexer) Algebra.

Eine Algebra heißt *kommutativ*, wenn

$$mn = nm \,. \tag{A.2}$$

Sie heißt *assoziativ*, wenn

$$k(mn) = (km)n \; . \tag{A.3}$$

Sie heißt *Algebra mit Einselement*, wenn sie ein Einselement **1** besitzt mit

$$\mathbf{1}m = m\mathbf{1} = m \; . \tag{A.4}$$

Sei **A** eine *assoziative* Algebra mit Einselement und $B \subset \mathbf{A}$ eine Menge von Elementen b^1, b^2 etc. Die Algebra heißt von B *erzeugt*, wenn jedes $m \epsilon \mathbf{A}$ durch ein Polynom endlichen Grades in den Elementen b^i geschrieben werden kann,

$$m = c\mathbf{1} + \sum_{k=1}^{p} \sum_{i_1, i_2, \ldots, i_k} c_{i_1 i_2 \ldots i_k} b^{i_1} b^{i_2} \ldots b^{i_k} \; , \tag{A.5}$$

wobei die Koeffizienten $c_{i_1 i_2 \ldots i_k}$ komplexe Zahlen sind. Die Elemente der Menge B heißen *Generatoren* von **A**. Das Einselement gehört nicht zu den Generatoren.

A.2.1 Clifford-Algebren

Eine Clifford-Algebra C_N wird von N Generatoren $\xi^1, \xi^2, \ldots, \xi^N$ erzeugt, für die

$$\boxed{\xi^a \xi^b + \xi^b \xi^a = 2\delta^{ab}}$$

mit $a, b = 1, \ldots, N$.

Die Dimension der Clifford-Algebra C_N ist 2^N. Es existiert ein enger Zusammenhang zwischen Clifford-Algebren und den Quantisierungsbedingungen für Fermionen.

Im allgemeinen lassen sich Clifford-Algebren für beliebige *symmetrische* Metriken g^{mn} definieren. So gilt insbesondere für die pseudoeuklidische Metrik

$$g_{ab} = \mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{N}, \underbrace{-1, \ldots, -1}_{M}) \; , \tag{A.6}$$

Clifford-Algebra $C_{N,M}$: $\{\Gamma^m, \Gamma^n\} = 2g^{mn}\mathbf{1}$.

Die Anzahl der Generatoren ist $d = N + M$.

A.3 Liealgebren

Eine Algebra ist ein Vektorraum, der von den Generatoren $A, B, C, \ldots$ aufgespannt wird: beliebige Linearkombinationen von Generatoren ergeben wieder Generatoren. Eine Algebra verfügt über ein *Produkt* zwischen den Generatoren. Im Fall der Liealgebra ist das Produkt der Kommutator

$$A \circ B := [A, B] \; , \tag{A.7}$$

mit den folgenden Eigenschaften

$$A \circ B = -B \circ A \tag{A.8}$$

$$(A \circ B) \circ C + (C \circ A) \circ B + (B \circ C) \circ A = 0 \; . \tag{A.9}$$

Liealgebren sind nicht assoziativ. Die Beziehung (A.9) heißt *Jacobi-Identität*.