Theoretische Teilchenphysik II

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Exercise Sheet 4 | WS-2023 | Due date: 20.11.23

Divergencies in theories with gauge fields and fermions (100 Points)

Exercise 4.1: *(30 points)* Consider Yukawa theory describing the interactions of a scalar field ϕ with a fermion field ψ. The Lagrangian reads

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi - m_\psi^2\bar{\psi}\psi + \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{m_\varphi^2}{2}\varphi^2 - g\bar{\psi}\psi\varphi. \tag{1}$$

This Lagrangian may be employed to describe *tree-level* processes but it cannot be used as a consistent theory when effects in high orders of perturbation theory are considered. The goal of this exercise is to discuss this point in detail.

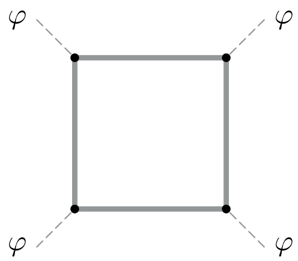


Figure 1: Divergent diagram generating new interaction.

(a) *(10 points)* Write down expression for the diagram in Fig.1 and use power counting arguments to show that this diagram has a superficial degree of divergence zero (we say the diagram is logarithmically divergent). Explain why this is a problem for the Lagrangian in Eq. (1). Modify the Lagrangian in Eq. (1) to ensure that it can be used as a Lagrangian of a consistent quantum field theory.

(b) *(20 points)* The diagram in Fig. 1 is logarithmically divergent. Suppose we calculate this diagram using Λ to cut off the integration over the loop momentum. Determine the dependence of the diagram in Fig. 1 on the cut-off Λ. How does the result depend on external momenta and masses of external and internal particles?

In dimensional regularization the dependence on the cut-off is, effectively, replaced with $1/\varepsilon$ divergences. Repeat the same calculation in dimensional regularization and determine the $1/\varepsilon$ divergence for the diagram in Fig. 1 for an arbitrary kinematics.

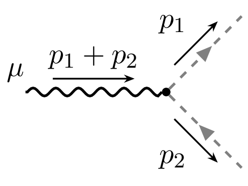
Exercise 4.2: *(40 points)* In this exercise, we consider scalar quantum electrodynamics(SQED). This is a theory that describes the interaction of the charged scalar field ϕ with the electromagnetic field. It is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\bar{\phi}D^\mu\phi - m^2\bar{\phi}\phi, \tag{2}$$

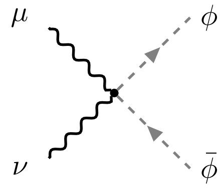
where $\bar{\phi}$ is the field conjugate to ϕ. The covariant derivatives read

$$D_\mu\phi = \partial_\mu\phi + igA_\mu\phi, \quad D_\mu\bar{\phi} = \partial_\mu\bar{\phi} - igA_\mu\bar{\phi}. \tag{3}$$

The perturbation theory follows from standard rules for the propagators of (charged) scalar fields and of photons, and the following interaction vertices



$$= ig\,(p_2 - p_1)_\mu\,,$$



$$= 2ig^2 g_{\mu\nu}. \tag{4}$$

Similar to the previous exercise, we are interested understanding whether or not the Lagrangian in Eq. (2) can be used a Lagrangian of a consistent theory.

(a) *(10 points)* Find all Green's functions with either positive or vanishing superficial degree of divergence. Draw diagrams that describes contributions to the divergent Green's functions at one loop

(b) *(30 points)* Calculate the divergent part of the one-loop amplitude $\mathcal{M}_{\phi\bar{\phi}\phi\bar{\phi}}$ for arbitrary kinematics of external ϕ-particles. What are the implications of this result for the the theory described by the Lagrangian Eq. (2). Which term needs to be added to this Lagrangian to make the theory renormalizable?

Exercise 4.3: *(30 points)* A superficial degree of divergence of the Green's function with four external electromagnetic fields is zero. This implies that the corresponding Green's function may be divergent. The goal of this exercise is to show that this is actually not the case.

(a) *(10 points)* Draw diagrams that contribute to light-by-light scattering in SQED and write down mathematical expressions for them in terms of Feynman integrals.

(b) *(20 points)* Only diagrams with scalar particles propagators contribute to the sum at one-loop level. To simplify the calculation, consider these diagrams with massive internal lines and put all external momenta to zero, keeping only the divergent part of each diagram. To calculate divergent parts of remaining integrals with loop momenta in the numerator, the following identities are useful:

$$\int \mathrm{d}^d k f(k^2) k^\mu k^\nu = \frac{g_{\mu\nu}}{d} \int \mathrm{d}^d k f(k^2) k^2, \tag{5}$$

$$\int \mathrm{d}^d k f(k^2) k^\mu k^\nu k^\rho k^\sigma = \frac{g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}}{d(d+2)} \int \mathrm{d}^d k f(k^2) k^4. \tag{6}$$