Theoretische Teilchenphysik II

V: Prof. Kirill Melnikov, U: Dr. Ming-Ming Long, U: Dr. Andrey Pikelner

Exercise Sheet 5 | WS-2023 | Due date: 27.11.23

Non-Abelian gauge theories (50 Points)

Exercise 5.1: *(50 points)* Consider a non-Abelian gauge theory with group SU(N), a gauge field A_μ can be written as

$$A_\mu = T^a A^a_\mu, \tag{1}$$

where T^a are the generators of SU(N). In the lecture, we obtained the kinetic term of the Lagrangian for a non-Abelian gauge theory,

$$\mathcal{L}_{\text{kin}} = -\frac{1}{2}\text{Tr}\left[\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\right], \tag{2}$$

where $\hat{F}_{\mu\nu} = T^a F^a_{\mu\nu}$ with

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu \tag{3}$$

(a) *(5 points)* Prove the Bianchi identity,

$$\left(D_\rho \hat{F}_{\mu\nu}\right)^a + \left(D_\mu \hat{F}_{\nu\rho}\right)^a + \left(D_\nu \hat{F}_{\rho\mu}\right)^a = 0, \tag{4}$$

where explicitly

$$\left(D_\rho \hat{F}_{\mu\nu}\right)^a \equiv \left[\delta^{ac}\partial_\rho + g_s f^{abc} A^b_\rho\right] F^c_{\mu\nu}. \tag{5}$$

(b) *(20 points)* From the lecture, we know that the Lagrangian $\mathcal{L}_{\text{kin}}$ is invariant under the gauge transformations defined as follows,

$$A_\mu(x) \longrightarrow A'_\mu = U(x)A_\mu(x)U^\dagger(x) - \frac{i}{g_s}\left[\partial_\mu U(x)\right]U^\dagger(x) \tag{6}$$

for any $U(x) \in$ SU(N). In the case of SU(2), a gauge field A_μ can be decomposed as

$$A_\mu = \frac{\sigma^a}{2} A^a_\mu \tag{7}$$

where σ^a are the Pauli matrices. In the lecture, we argued that with infinitesimal transformations the obtained filed A'_μ also belongs to the corresponding Lie algebra, i.e. Eq. (7) still holds. Prove that under a generic transformation Eq. (6), the field A'_μ can also be written as Eq. (7).

Hint: Prove that A'_μ is traceless and hermitian given that A_μ does.

(c) *(10 points)* Now add a Dirac Lagrangian which reads

$$\mathcal{L}_{\text{D}} = \bar{\Psi}\left(i\gamma_\mu D^\mu - m\right)\Psi\,. \tag{8}$$

Compute the equations of motion for gauge fields and show that they can be written as

$$\left(D^\mu \hat{F}_{\mu\nu}\right)^a = g_s j^a_\nu, \tag{9}$$

where the left-hand-side is defined similarly to Eq. (5). What is the conserved currents j^a_ν? What is the conservation equation for j^a_ν?

(d) *(15 points)* Consider the dual field strength tensor

$$\widetilde{F}^a_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{a,\rho\sigma}. \tag{10}$$

Next, let us use it to construct a term as

$$\mathcal{L}_\theta = 2\theta \widetilde{F}^a_{\mu\nu}F^{a,\mu\nu}, \tag{11}$$

where θ is a constant[1]. Is $\mathcal{L}_\theta$ allowed in the Lagrangian of a non-Abelian gauge theory? Show that $\mathcal{L}_\theta$ is actually a total derivative

$$\mathcal{L}_\theta = \partial_\mu P^\mu, \tag{12}$$

and write down the P^μ. Does $\mathcal{L}_\theta$ change the equation of motion of gauge fields? Explain it.

SU(N) Group (50 Points)

Exercise 5.2: *(50 points)* A Lie group $\mathcal{G}$ can be parameterized by a set of continuous parameters,

$$\alpha^a, \qquad a = 1, 2, ..., M. \tag{13}$$

An element in $\mathcal{G}$ depends on these M parameters.

(a) *(5 points)* Show that any element of $\mathcal{G}$, say U, in a particular representation can be written as

$$U = e^{i\alpha^a T^a}, \tag{14}$$

where T^a are the generators of group $\mathcal{G}$.

(b) *(5 points)* For the SU(N) group, by definition,

$$UU^\dagger = \mathbb{1}, \qquad \det U = 1. \tag{15}$$

What constraints do these conditions impose on the generators T^a? Identify the number of generators.

(c) *(5 points)* In the case of SU(2) group, one can write its element as

$$U = e^{i\alpha^j \frac{\sigma^j}{2}}. \tag{16}$$

Here σ^j are the Pauli matrices. Show that one can also write

$$U = \cos\frac{\theta}{2}\mathbb{1} + i\sin\frac{\theta}{2}n^j\sigma^j. \tag{17}$$

What are the θ and $\vec{n}$ explicitly (in terms of α^j)?

(d) *(5 points)* The generators of a Lie group $\mathcal{G}$ form a Lie algebra which is a linear vector space. This algebra is defined by

$$\left[T^a, T^b\right] = if^{abc}T^c, \tag{18}$$

where f^{abc} are the structure constants, which are totally anti-symmetric. One can also define a totally symmetric group invariant as

$$d^{abc} = 2\mathrm{Tr}\left[T^a\{T^b, T^c\}\right]. \tag{19}$$

Note that this is only valid in fundamental representation. Prove that

$$d^{abe}f^{ecd} + d^{bce}f^{ead} + d^{cae}f^{ebd} = 0. \tag{20}$$

[1] In QCD, $\mathcal{L}_\theta$ is related to the strong CP problem and θ is known as the *strong CP phase*.

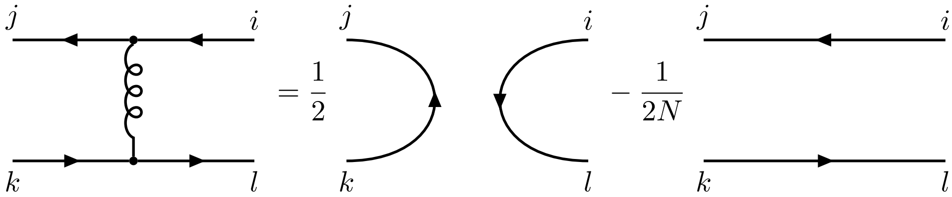
(e) *(10 points)* In the fundamental representation, the generators of SU(N) are normalized so that

$$\mathrm{Tr}\left[T^a T^b\right] = \frac{1}{2}\delta^{ab}. \tag{21}$$

Show that

$$T^a_{ij} T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{kj} - \frac{1}{N}\delta_{ij}\delta_{kl}\right). \tag{22}$$

This is known as Fierz identity which also has a grammatical representation,



(23)

(f) *(20 points)* The Casimir operators are invariants of the algebra. They commute with every single group element. The quadratic Casimir operator for a representation R is defined by

$$T^a_R T^a_R = C_2(R)\mathbb{1}. \tag{24}$$

Let's define

$$C_F \equiv C_2(\text{fund}) = \frac{N^2-1}{2N}, \qquad C_A \equiv C_2(\text{adj}) = N. \tag{25}$$

We can use the Fierz identity to calculate the so-called *color factors*. For example, summing the j, k indices in Eq. (22) would immediately give

$$\left(T^a T^a\right)_{il} = C_F \delta_{il}. \tag{26}$$

Use Fierz identity to calculate the following color factors.

- $\left(T^a T^b T^a\right)_{ij}$
- $\left(T^a T^b T^c T^a T^b\right)_{ij}$
- $\left(T^a T^b\right)_{ij} \left(T^a T^b\right)_{kl}$
- $\left[T^a, T^b\right]_{ij} \left[T^a, T^b\right]_{kl}$
- $\left(f^{abc} T^b T^c\right)_{ij}$
- $f^{acd} f^{bcd}$
- $\mathrm{Tr}\left(T^a T^b T^c\right) \mathrm{Tr}\left(T^a T^b T^c\right)$

You are encouraged to verify your results by using FORM[2] or FeynCalc[3].

[2] https://github.com/vermaseren/form
[3] https://feyncalc.github.io/