Theoretische Teilchenphysik II

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Exercise Sheet 6 | WS-2023 | Due date: 04.12.23

Gaussian integral (30 Points)

Exercise 6.1: *(30 points)* In this exercise we will study Gaussian integrals. A Gaussian integral over one variable reads

$$\int_{-\infty}^{+\infty} \mathrm{d}x \, e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \tag{1}$$

where a is a positive real number.

(a) *(10 points)* We would like to generalize Eq. (1) to the multi-variable case. To this end, consider an N-dimensional vector $\mathbf{x}$ and show that

$$\int \mathrm{d}^N \vec{x} \, e^{-\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathrm{A}\mathbf{x}+\mathbf{b}^{\mathrm{T}}\mathbf{x}} = \frac{(2\pi)^{N/2}}{(\det \mathrm{A})^{1/2}} \, e^{\frac{1}{2}\mathbf{b}^{\mathrm{T}}\mathrm{A}^{-1}\mathbf{b}}. \tag{2}$$

In deriving the above result we assumed that A is a positive-definite symmetric $N \times N$ matrix and $\mathbf{b}$ is a real-valued N-dimensional vector.

(b) *(20 points)* Having computed the Gaussian integral over N real variables, we can generalize the calculation to the case of N complex variables $\mathbf{z}$. Show that

$$\int \left(\prod_{i=1}^{N} \mathrm{d}z_i \mathrm{d}z_i^* \right) \, e^{-\mathbf{z}^\dagger \mathrm{A}\mathbf{z}+\mathbf{b}^\dagger\mathbf{z}+\mathbf{z}^\dagger\mathbf{b}} = \frac{(2\pi)^N}{\det \mathrm{A}} \, e^{\mathbf{b}^\dagger \mathrm{A}^{-1}\mathbf{b}}, \tag{3}$$

where A is a Hermitian matrix and $\mathbf{b}$ is a complex N-dimensional vector.

Feynman rules from the path integral (70 Points)

Exercise 6.2: *(50 points)* Path integrals can be used to derive Feynman rules for an arbitrary quantum field theory in a mechanical way. The goal of this exercise is to do that for a scalar ϕ^4 theory. To this end, consider a real scalar field governed by the Lagrangian

$$\mathcal{L} = \underbrace{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2}_{\mathcal{L}_0} - \frac{\lambda}{4!}\phi^4. \tag{4}$$

We define the generating functional $W[J]$ as

$$W[J] = \frac{\int [\mathcal{D}\phi] e^{i\int \mathrm{d}^4x(\mathcal{L}+J\phi)}}{\int [\mathcal{D}\phi] e^{i\int \mathrm{d}^4x\mathcal{L}}}. \tag{5}$$

We will also denote by W_0 the generating functional of a free scalar theory, i.e.

$$W_0[J] = W[J]|_{\lambda=0}, . \tag{6}$$

(a) *(5 points)* Repeating the discussion in the script, show that

$$W_0[J] = e^{-\frac{1}{2}\int \mathrm{d}^4x_1 \mathrm{d}^4x_2 J(x_1)G(x_1,x_2)J(x_2)}, \tag{7}$$

where $G(x_1, x_2)$ is the two-point Green function

$$G(x_1, x_2) = \langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \frac{\delta^2}{i^2\delta J(x_1)\delta J(x_2)} W_0[J]\bigg|_{J=0} = \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{i\, e^{-ip(x_1-x_2)}}{p^2 - m^2 + i0}. \tag{8}$$

This result gives you a Feynman rule for the scalar-field propagator that you can use for computing Feynman diagrams. Write the propagator explicitly.

(b) *(10 points)* To complete the Feynman rules, we need an interaction term. Show that, up to order $\mathcal{O}(\lambda)$, $W[J]$ can be written as

$$W[J] = C\left[1 - \frac{i\lambda}{4!}\int \mathrm{d}^4y \left(\frac{\delta}{i\delta J(y)}\right)^4\right] W_0[J] \tag{9}$$

with

$$C = \frac{\int [\mathcal{D}\phi] e^{i\int \mathrm{d}^4x \mathcal{L}_0}}{\int [\mathcal{D}\phi] e^{i\int \mathrm{d}^4x \mathcal{L}}}. \tag{10}$$

(c) *(10 points)* Perform the functional derivative in Eq. (9), simplify the expression, and write $W[J]$ with $\mathcal{O}(\lambda)$ accuracy. Note that you should also expand the coefficient C. While doing that, it is important to remember that that $W[0] = 1$.

(d) *(10 points)* Putting everything together, compute the four-point Green function

$$G(x_1, x_2, x_3, x_4) = \frac{\delta^4}{i^4\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} W[J]\bigg|_{J=0}. \tag{11}$$

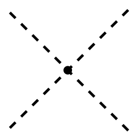
(e) *(15 points)* The above result involves both fully-connected and disconnected diagrams. The fully-connected diagrams are the ones that we are interested in since they involve the interaction term. There is a useful technical trick that allows us to project on the connected diagrams. To this end, we write

$$W[J] = e^{iP[J]}, \tag{12}$$

and note that $P[J]$ is the generating functional of *connected* Green functions. Hence,

$$G^C(x_1, ..., x_n) = \frac{i^{1-n}\delta^n}{\delta J(x_1)...\delta J(x_n)} P[J]\bigg|_{J=0}. \tag{13}$$

Using the above results, compute $P[J]$ with $\mathcal{O}(\lambda)$ accuracy and use it to compute the four-point connected Green function $G^C(x_1, x_2, x_3, x_4)$. Perform the Fourier transformations to get its form in momentum space $G^C(p_1, p_2, p_3, p_4)$. Upon amputating the four external propagators, you should immediately obtain the Feynman rule for the four-point vertex in the momentum space,



$$= \quad -i\lambda\,(2\pi)^4\delta(p_1 + p_2 + p_3 + p_4). \tag{14}$$

Exercise 6.3: *(20 points)* A peculiar way to re-write a given quantum field theory in a suitable manner is the introduction of *non-propagating field*. Non-propagating fields appear in the path integral but they do not possess kinetic terms.

As an example, consider a theory with the non-propagating field σ which is described by the Lagrangian

$$\mathcal{L}' = \mathcal{L} + \frac{3}{2\lambda}\left(\sigma + \frac{\lambda}{6}\phi^2\right)^2, \tag{15}$$

where $\mathcal{L}$ is the same as Eq. (4).

(a) *(5 points)* By performing a functional integral over the field σ, show that the new theory described by the Lagrangian $\mathcal{L}'$ is reduced to the original theory of interacting field ϕ only.

(b) *(15 points)* Draw lowest order diagrams for the scattering process $\phi\phi \to \phi\phi$ in the theory described by the Lagrangian (15). With a technique developed in the previous exercise derive the Feynman rules for ϕ^4 and $\sigma\phi^2$ vertices in that theory.