Theoretische Teilchenphysik II

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Exercise Sheet 8 | WS-2023 | Due date: 18.12.23

Four-gluon scattering amplitude (100 Points)

Exercise 8.1: *(100 points)* In this problem we will study the amplitude $\mathcal{M}$ for the four-gluon scattering $0 \to g(k_1) + g(k_2) + g(k_3) + g(k_4)$, using the spinor-helicity formalism and color ordering. The four Feynman diagrams contributing to this amplitude are shown in Fig. 1.

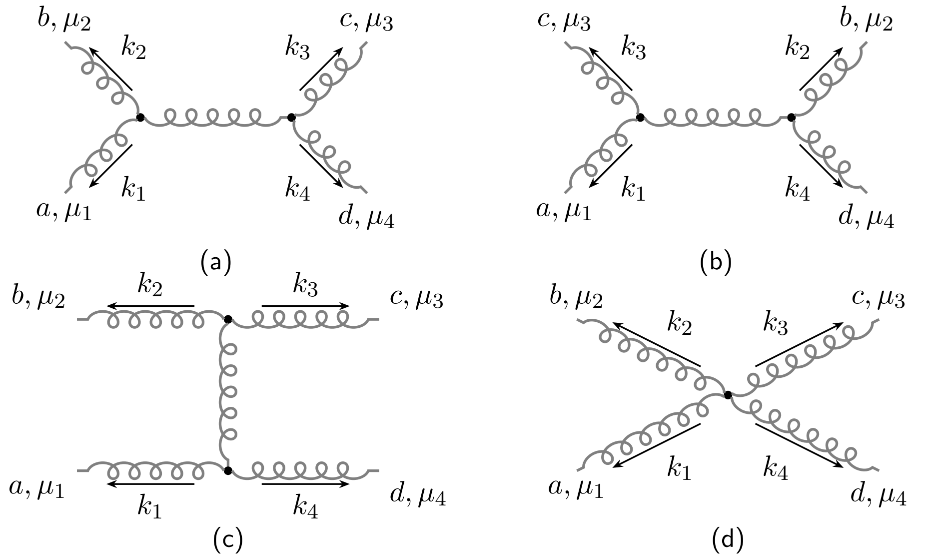

Figure 1: Feynman diagrams for four-gluon scattering process. All external momenta are outgoing.

(a) Show that the product of structure constants can be written as traces over T^a as follows:

$$f^{abe} f^{cde} = -\frac{1}{2} \left(\mathrm{Tr}\left[T^a T^b T^c T^d\right] - \mathrm{Tr}\left[T^b T^a T^c T^d\right] - \mathrm{Tr}\left[T^a T^b T^d T^c\right] + \mathrm{Tr}\left[T^b T^a T^d T^c\right] \right).$$

(b) Consider the color factors which appear in the amplitude $\mathcal{M}$ and, using the above relation, argue that $\mathcal{M}$ can be written in terms of color-ordered amplitudes $\mathcal{M}_i$

$$\begin{aligned} \mathcal{M} = &\mathcal{M}_1 \,\mathrm{Tr}\left[T^a T^b T^c T^d\right] + \mathcal{M}_2 \,\mathrm{Tr}\left[T^a T^b T^d T^c\right] + \mathcal{M}_3 \,\mathrm{Tr}\left[T^a T^c T^b T^d\right] \\ &+ \mathcal{M}_4 \,\mathrm{Tr}\left[T^a T^c T^d T^b\right] + \mathcal{M}_5 \,\mathrm{Tr}\left[T^a T^d T^b T^c\right] + \mathcal{M}_6 \,\mathrm{Tr}\left[T^a T^d T^c T^b\right], \end{aligned} \tag{1}$$

where $\mathcal{M}_i$ are functions of the gluon momenta and polarization vectors. Note that you **do not** need to calculate $\mathcal{M}_i$.

(c) It can be shown that the color-ordered amplitudes $\mathcal{M}_i$ can be written in terms of a single function $M(i_1, i_2, i_3, i_4)$ with different permutations of the gluon arguments, i.e. $\mathcal{M}_1 =$

$M(1,2,3,4)$, $\mathcal{M}_2 = M(1,2,4,3)$, etc. This function is

$$
\begin{aligned}
M(1,2,3,4) = -\frac{g^2}{2}\Bigg\{ & -\frac{i}{s_{14}}\left[\epsilon_1\cdot\epsilon_4(k_4-k_1)_\lambda + \epsilon_{1\lambda}(k_1+k_{14})\cdot\epsilon_4 + \epsilon_{4\lambda}(-k_4-k_{14})\cdot\epsilon_1\right] \\
& \times\left[\epsilon_2\cdot\epsilon_3(k_2-k_3)^\lambda + \epsilon_3^\lambda(k_3+k_{23})\cdot\epsilon_2 + \epsilon_2^\lambda(-k_2-k_{23})\cdot\epsilon_3\right] \\
& -\frac{i}{s_{12}}\left[\epsilon_1\cdot\epsilon_2(k_1-k_2)_\lambda + \epsilon_{2\lambda}(k_2+k_{12})\cdot\epsilon_1 + \epsilon_{1\lambda}(-k_1-k_{12})\cdot\epsilon_2\right] \\
& \times\left[\epsilon_3\cdot\epsilon_4(k_3-k_4)^\lambda + \epsilon_4^\lambda(k_4+k_{34})\cdot\epsilon_3 + \epsilon_3^\lambda(-k_3-k_{34})\cdot\epsilon_2\right] \\
& -i\left[2\epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4 - \epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4 - \epsilon_1\cdot\epsilon_4\epsilon_2\cdot\epsilon_3\right]\Bigg\},
\end{aligned}
\tag{2}
$$

where $k_{ij} = k_i + k_j$ and ϵ_i is the polarization vector of gluon i.

(d) Using the expressions for the gluon polarization vectors $\epsilon_{L,R}$ given in the lecture, show that

$$\epsilon_L(k_1,r)\cdot\epsilon_L(k_2,r) = \epsilon_R(k_1,r)\cdot\epsilon_R(k_2,r) = 0, \tag{3}$$

for any $k_{1,2}$, using the same reference vector r for both polarization vectors.

(e) Consider the case where the helicities of all four gluons are the same. Choose the same reference vector for all four polarization vectors, $r_1 = r_2 = r_3 = r_4 = r$, and show that these amplitudes vanish, i.e. $M(1_L,2_L,3_L,4_L) = M(1_R,2_R,3_R,4_R) = 0$. Generalize this to the scattering of an arbitrary number of gluons with the same helicity.

(f) Show that

$$\epsilon_L(k_1,k_2)\cdot\epsilon_R(k_2,r) = \epsilon_R(k_1,k_2)\cdot\epsilon_L(k_2,r) = 0. \tag{4}$$

(g) Using this result, choose the reference vectors for g_1, g_2, and g_3 to be $r_1 = r_2 = r_3 = k_4$, and show that $M(1_L,2_L,3_L,4_R) = M(1_R,2_R,3_R,4_L) = 0$. Argue that if the helicities of any three gluons are the same and one is different then the amplitude vanishes. Now generalize this to arbitrarily many gluons, where all but one have the same helicity.

(h) Thus the only helicity configuration that results in a non-vanishing amplitude for four-gluon scattering is that with two left-handed gluons and two right-handed gluons. We will consider $M(1_R,2_R,3_L,4_L)$. Show that, by choosing the reference vectors for $r_1 = r_2 = k_3$ and $r_3 = r_4 = k_2$,

$$M(1_R,2_R,3_L,4_L) = -ig^2\frac{[12]^3\langle 34\rangle}{[12][43]\langle 14\rangle[41]}. \tag{5}$$

Using momentum conservation, show that $\langle 34\rangle/\langle 14\rangle = [12]/[23]$ and hence

$$M(1_R,2_R,3_L,4_L) = ig^2\frac{[12]^4}{[12][23][34][41]}. \tag{6}$$

This is called a maximally helicity violating (MHV) amplitude, and in fact this can be generalized too. An n-gluon scattering amplitude with gluons i and j right-handed and all other gluons left-handed can be written as

$$M(1_L,\ldots,i_R,\ldots,j_R,\ldots,n_L) = \frac{[ij]^4}{[12][23]\ldots[(n-1)n][n1]}, \tag{7}$$

where we have dropped the coupling g. This is known as the *Parke-Taylor formula*. The result for $M(1_R,\ldots,i_L,\ldots,j_L,\ldots,n_R)$ is the same with the swap $\langle\,\rangle \leftrightarrow [\,]$.

(i) Use the Parke-Taylor formula, Eq. (7), to write down the expression for $M(1_R,2_L,3_R,4_L)$.