Theoretische Teilchenphysik II

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Exercise Sheet 9 | WS-2023 | Due date: 25.12.23

Cross section for gluon scattering (100 Points)

In previous exercise, we derived an expression for the scattering amplitude for $0 \to 4$ gluon scattering, as a sum of color factors times partial amplitudes:

$$\mathcal{M}(p_{1}{}_{a_1}^{(h_1)}, p_{2}{}_{a_2}^{(h_2)}, p_{3}{}_{a_3}^{(h_3)}, p_{4}{}_{a_4}^{(h_4)}) = \sum_{\{i,j,k\} \in \text{permutations of } \{2,3,4\}} \text{Tr}\big(T_{a_1} T_{a_i} T_{a_j} T_{a_k}\big) \mathcal{M}^{h_1 h_i h_j h_k}(p_1, p_i, p_j, p_k) \quad (1)$$

where p_i refers to the momenta, a_i to the colors, and h_i to the helicities.

We also saw that the partial amplitudes for processes with all (and all except one) helicities being identical, vanish:

$$\mathcal{M}^{llll} = \mathcal{M}^{rrrr} = \mathcal{M}^{rlll} = \mathcal{M}^{lrrr} = 0 \quad (2)$$

and that the partial amplitude for two left-handed and two right-handed gluons is given as

$$ig^2 \frac{[ij]^4}{[12][23][34][41]} \quad (3)$$

where i and j refer to the right-handed ones.

In this exercise we will calculate the differential cross-section, through the squared amplitude:

$$|\mathcal{A}|^2 = \sum_{\text{helicities}} \big|\mathcal{M}(p_1^{a_1}, p_2^{a_2}, p_3^{a_3}, p_4^{a_4})\big|^2 \quad (4)$$

Inserting eq. (1) into eq. (4), each term will be a sum over 36 terms, each of which can be calculated with the help of the identity

$$\sum_{a=1}^{8} T^a_{ij} T^a_{kl} = \delta_{il}\delta_{kj} - \frac{1}{N_c}\delta_{ij}\delta_{kl} \quad (5)$$

where $N_c = 3$ is the number of colors. We will not do that here. Instead we are going to use a trick known as $U(1)$ decoupling.

Exercise 9.1: *(100 points)* Let us imagine adding to the Lagrangian of $SU(3)$ Yang Mills theory, a photon with field strength tensor $\Phi^{\mu\nu}$, so the total Lagrangian is $\mathcal{L} = -\frac{1}{4} G^{a\mu\nu} G^a_{\mu\nu} - \frac{1}{4}\Phi^{\mu\nu}\Phi_{\mu\nu}$. We will now construct a unified description of the gluon and the extra photon, by re-expressing the Lagrangian as $\mathcal{L} = -\frac{1}{4}\,\text{Tr}(F^{\mu\nu}F_{\mu\nu})$ where $F_{\mu\nu} = \sum_{a=1}^{8} T^a G^a_{\mu\nu} + T^9 \Phi_{\mu\nu}$.

Consider the rules of color algebra, that we enforce to hold also in the extended theory:

$$[T^a, T^b] = i f^{abc} T^c \qquad \text{Tr}(T^a T^b) = \delta^{ab} \quad (6)$$

1. Show that a valid representation of the ninth generator is

$$T^9 = \tfrac{1}{\sqrt{3}} \mathbf{I}_3 \quad (7)$$

   where $\mathbf{I}_3$ is the 3×3 unit matrix.

2. Show that $f^{ij9} = 0$ for all i, j from 1 to 9.

3. What gauge group is generated by these nine generators?

4. Show that the identity given by eq. (5) becomes

$$\sum_{a=1}^{9} T^a_{ij} T^a_{kl} = \delta_{il}\delta_{kj} \tag{8}$$

for the extended theory. We will use this expression to simplify the derivation of the cross section.

5. Let us consider $0 \to 4$ "gluon" scattering in the **extended** theory. Argue that when none of the "gluons" carry the ninth color index the result is identical to ordinary Yang-Mills theory, and if one of the "gluons" carry the ninth color index, the result will be zero.

6. Eq. (1) was derived without any assumptions about the gauge-group, and therefore it will hold in the extended theory too. For the case where the first gluon carry the ninth index, eq. (1) becomes

$$0 \;=\; \mathcal{M}(p{1}_9^{(h_1)}, p{2}_{a_2}^{(h_2)}, p{3}_{a_3}^{(h_3)}, p{4}_{a_4}^{(h_4)}) \;=\sum_{\{i,j,k\}\in\text{permutations of }\{2,3,4\}} \mathsf{Tr}\big(T_9 T_{a_i} T_{a_j} T_{a_k}\big)\mathcal{M}^{h_1 h_i h_j h_k}(p_1, p_i, p_j, p_k) \tag{9}$$

Show that this implies

$$\sum_{\{i,j,k\}\in\text{permutations of }\{2,3,4\}} \mathcal{M}^{h_1 h_i h_j h_k}(p_1, p_i, p_j, p_k) = 0 \tag{10}$$

This is a manifestation of the $U(1)$ decoupling identity. Please note that this identity only involves color-stripped amplitudes, which are independent of the gauge group, and therefore eq. (10) is valid also in ordinary Yang-Mills theory. We introduced the extra photon in order to derive an identity valid for theories without that photon!

7. Let us consider the color factor of the first of the 36 terms in each term of eq. (4). Show that

$$\mathrm{Tr}(T_a T_b T_c T_d)\,\mathrm{Tr}(T_a T_b T_c T_d)^* = N_c^4. \tag{11}$$

(hint: First show that $\mathrm{Tr}(T_a T_b T_c T_d)^* = \mathrm{Tr}(T_d T_c T_b T_a)$, then apply eq. (8)).

8. Show that

$$\mathrm{Tr}(T_a T_b T_c T_d)\,\mathrm{Tr}(T_a T_b T_d T_c)^* = N_c^2. \tag{12}$$

9. The remaining combinations are given as

$$\begin{aligned} \mathrm{Tr}(T_a T_b T_c T_d)\,\mathrm{Tr}(T_a T_c T_b T_d)^* &= N_c^2\,, \qquad & \mathrm{Tr}(T_a T_b T_c T_d)\,\mathrm{Tr}(T_a T_c T_d T_b)^* &= N_c^2\,, \\ \mathrm{Tr}(T_a T_b T_c T_d)\,\mathrm{Tr}(T_a T_d T_b T_c)^* &= N_c^2\,, \qquad & \mathrm{Tr}(T_a T_b T_c T_d)\,\mathrm{Tr}(T_a T_d T_c T_b)^* &= N_c^2\,. \end{aligned} \tag{13}$$

Show that the squared amplitude may be expressed as

$$|\mathcal{A}|^2 = \sum_{\text{helicities}} \left(N_c^4 \sum_{i=1}^{6} |\mathcal{M}_i|^2 + N_c^2 \sum_{i\neq j} \mathcal{M}_i \mathcal{M}_j^* \right) \tag{14}$$

$$= N_c^2 (N_c^2 - 1) \sum_{\text{helicities}} \sum_{\text{perm.}} |\mathcal{M}_i|^2 \tag{15}$$

where the permutation sum is the same as in eq. (1). (hint: for the second step, use eq. (10)).

10. We now define the Mandelstam variables

$$s_{12} = (p_1 + p_2)^2 \qquad s_{13} = (p_1 + p_3)^2 \qquad s_{14} = (p_1 + p_4)^2 \tag{16}$$

Show that

$$|\mathcal{M}(p_1^R, p_2^R, p_3^L p_4^L)|^2 = g^4 \frac{s_{12}^2}{s_{14}^2} \qquad \text{and}$$
$$|\mathcal{M}(p_1^R, p_2^L, p_3^R p_4^L)|^2 = g^4 \frac{s_{13}^4}{s_{12}^2 s_{14}^2} \tag{17}$$

11. Show that the squared amplitude may be expressed as

$$|\mathcal{A}|^2 = 2g^4 N_c^2 (N_c^2 - 1) \sum_{\text{perm.}} \frac{s_{1i}^4 + s_{1j}^4 + s_{1k}^4}{s_{1i}^2 s_{1k}^2} \tag{18}$$

12. The final result for the differential cross section for $2 \to 2$ gluon scattering is

$$\frac{d\sigma_{gg \to gg}}{d\cos\theta} = \frac{9\pi\alpha_s^2}{4s} \left(3 - \frac{st}{u^2} - \frac{tu}{s^2} - \frac{su}{t^2} \right). \tag{19}$$

with $t = -s(1 - \cos(\theta))/2$, and $u = -s(1 + \cos(\theta))/2$. In the limit of small θ, this becomes

$$\frac{d\sigma_{gg \to gg}}{d\cos\theta} \approx \frac{9\pi\alpha_s^2}{16s} \frac{1}{\sin^4(\theta/2)} \tag{20}$$

What does this expression remind you of, and why?