Theoretische Teilchenphysik II

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Exercise Sheet 10 | WS-2023 | Due date: 15.01.24

Renormalization group equations (RGEs) and the resummation of large logarithms (100 Points)

Exercise 10.1: *(30 points)* Recall from lecture 12 (c.f. Eq. (12.33)) that the QCD coupling constant α_s satisfies the following differential equation

$$\frac{\mathrm{d}\alpha_s}{\mathrm{d}\ln\mu} = \frac{-2\epsilon\,\alpha_s}{1+\alpha_s\frac{1}{Z_{\alpha_s}}\frac{\partial Z_{\alpha_s}}{\partial\alpha_s}}, \tag{1}$$

where

$$Z_{\alpha_s} = 1 + \sum_{n=1}^{\infty}\frac{A_n}{\epsilon^n}, \tag{2}$$

and A_n are functions of α_s.

(a) Explain steps that are needed to rewrite Eq. (1) in the following way

$$\begin{aligned}
&\mu\frac{\mathrm{d}\alpha_s}{\mathrm{d}\mu} = -2\epsilon\alpha_s + 2\alpha_s^2 A_1',\\
&A_2' = \alpha_s (A_1')^2 + A_1 A_1',\\
&A_3' = \alpha_s A_1' A_2' + A_2 A_1',\\
&A_4' = \alpha_s A_3' A_1' + A_3 A_1'.\\
&\ldots
\end{aligned} \tag{3}$$

Assume that

$$A_1 = \sum_{n=1}^{\infty} a_{1n}\alpha_s^n. \tag{4}$$

Taking into account of a_{11} *only* write explicitly the solution to α_s and determine $A_{2,3,4,..}$ in Eq. (3). Repeat the same by accounting for a_{11} and a_{12}.

(b) We defined the anomalous dimension in the lecture 13 as (c.f. Eq. (13.6), the subscript is dropped)

$$\gamma = \frac{1}{Z}\frac{\mathrm{d}}{\mathrm{d}\ln\mu}Z, \tag{5}$$

where Z is also the same form as Z_{α_s},

$$Z = 1 + \sum_{n=1}^{\infty}\frac{B_n}{\epsilon^n}. \tag{6}$$

Find a similar structure as in Eq. (3) in this case.

Exercise 10.2: *(40 points)* Renormalization group equations provide a powerful way to study the structure of perturbation theory and, especially, large logarithms of kinematic variables that appear in arbitrary Green's functions. Such logarithms typically arise in high-order perturbative calculation.

To be more specific, consider renormalized four-point Green's function at one-loop in ϕ^4 theory,[1]

$$G^{(4)}(\{p_i\}) = -i\lambda(m)\Big(1 + \frac{\lambda(m)}{32\pi^2}\Big[3L + f_1\Big]\Big) + \mathcal{O}(\lambda(m)^3), \quad E = \frac{\sqrt{s}}{2}, \tag{7}$$

where $L = \ln E^2/m^2$. This equation is valid for $s \sim |t| \sim |u| \gg m^2$ and the function f_1 does not contain the large logarithm L.

It is clear from the above equation that the perturbative expansion parameter for $G^{(4)}(\{p_i\})$ is not $\lambda(m)$ but rather $\lambda(m)L \gg \lambda(m)$ provided that $E \gg m$. Generally, it has the following form,

$$G^{(4)}(\{p_i\}) = -i\lambda(m)\sum_{n=0}^{\infty}\left(\frac{\lambda(m)}{32\pi^2}\right)^n \sum_{k=0}^{n} c_{n,k}L^k. \tag{8}$$

In each order of the perturbative expansion, the leading logarithm (LL) has the largest power followed by the next-to-leading (or sub-leading) one (NLL), etc (N^{k}LL). It is possible to use renormalization group analysis to re-sum these large logarithms to all orders in the perturbative expansion. Below we discuss how to do this.

(a) Compute the β-function and the field anomalous dimensions in ϕ^4 theory at one loop in the $\overline{\rm MS}$ scheme. Write an equation for the running coupling constant $\lambda(\mu)$ and solve it using $\lambda(\mu) = \lambda(m)$ at $\mu = m$ as the boundary condition.

(b) Use the above result to write $\lambda(m)$ as a series expansion in $\lambda(E)$. Substitute it back to Eq. (7) and show that the Green's function assumes the following form

$$G^{(4)}(\{p_i\}) = -i\lambda(E)\left(1 + \frac{\lambda(E)}{32\pi^2}f_1\right) + \mathcal{O}(\lambda(m)^3). \tag{9}$$

Why it is advantageous to use this expression for the Green's function as compared to the expression in Eq. (7)?

(c) Use the results that you derived in item (a) to set up a RGE for the Green's function and explain how to solve it using the one-loop expressions for the β-function and the field anomalous dimension. Use the solution to explain how the LLs are re-summed to all orders.

(d) To re-sum the NLLs, what ingredients do you need? Explain it.

Exercise 10.3: *(30 points)* In the QCD and ϕ^4 theory, there is only one coupling constant. Let's generalize to the case with multiple coupling constants. Consider a theory defined by the Lagrangian,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{\lambda_1}{4!}\phi_1^4 - \frac{\lambda_2}{4!}\phi_2^4 - \frac{\lambda_3}{4}\phi_1^2\phi_2^2. \tag{10}$$

(a) Draw all the Feynman diagrams that are needed to compute the three β-functions of the coupling constants, $\lambda_{1,2,3}$ at leading order.

(b) Compute these β-functions in the $\overline{\rm MS}$ scheme.

(c) Try to solve the RGEs of $\lambda_{1,2,3}$ numerically and make stream plots to illustrate the solutions.

[1] Note that the physical coupling constant $\lambda(m)$ is defined by the scattering amplitude of four particles at rest

$$\lambda(m) = i\mathcal{M}_{\phi\phi\to\phi\phi}(s = 4m^2, t = u = 0).$$

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