Theoretische Teilchenphysik II

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Exercise Sheet 11 WS-2023 Due date: 22.01.24

Axial-Vector-Vector (AVV) anomaly in Pauli-Villars and dimensional regularizations (100 Points)

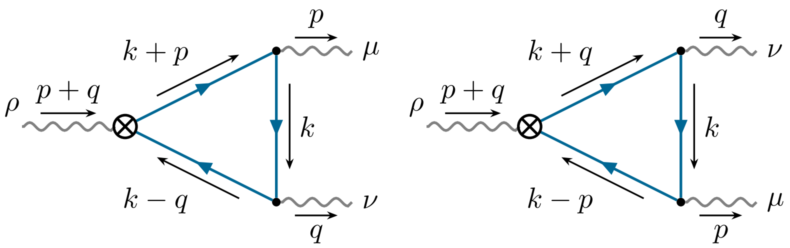

Figure 1: One-loop diagrams contributing to the AVV current correlator.

In lecture 14, we studied the diagrams shown in Fig. 1, that describe a one-loop contribution to the correlator of an axial current $\bar{\psi}\gamma_\rho\gamma_5\psi$ and two vector currents $\bar{\psi}\gamma_\mu\psi$, and argued that they violate the conservation of the axial current. To obtain these results, we used a particular way to deal with divergent integrals.

An alternative way is to compute the correlator using some regularization procedure. The goal of this exercise is to perform such a computation employing the Pauli-Villars and the dimensional regularization schemes.

The usage of regularization allows us to perform calculations in a more canonical way, as opposed to a method described in class. However, a choice of the regularization procedure breaks some internal symmetries of the theory the very moment a regulator is introduced. The only question is then whether the broken symmetry is restored when the regularization is lifted at the end of the calculation. We will show that for the axial-vector current, this is not the case.

Pauli-Villars regularization (40 Points)

Exercise 11.1: *(40 points)* Full contribution to the one-loop AVV anomaly contains two diagrams connected by permutation of two vector external legs:

$$\Gamma_{\rho|\mu\nu}(p,q) = T_{\rho|\mu\nu}(p,q) + T_{\rho|\nu\mu}(q,p). \tag{1}$$

From power counting arguments, one can see that these two diagrams are UV-divergent and to deal with these divergences we can introduce Pauli-Villars regularization. To do so, we add to diagrams in Fig. 1 two similar diagrams where massless fermions are replaced with fermions with the mass M and multiply the massive-fermion contribution with an additional minus sign. Symbolically the regularized version of the quantity Γ in Eq. (1) reads

$$\Gamma = \int \frac{\mathrm{d}^4k}{(2\pi)^4} I(k,p,q,m=0) \to \Gamma_{\text{reg}} = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\left[I(k,p,q,m=0) - I(k,p,q,m=M)\right]. \tag{2}$$

The Lagrangian of the theory with a new massive field lacks the original *chiral symmetry* and is the source of the anomalous behavior of the correlator.

(a) *(10 points)* Verify Ward identities for the vector current

$$p_\mu\Gamma_{\rho|\mu\nu,\text{reg}} = 0, \quad q_\nu\Gamma_{\rho|\mu\nu,\text{reg}} = 0. \tag{3}$$

(b) *(30 points)* Calculate the anomalous contribution of $(p_\rho + q_\rho)\Gamma_{\rho|\mu\nu,\text{reg}}$.
Hint: use substitution $\gamma_5(\not{p} + \not{q}) = \gamma_5(\not{k} + \not{p} - M) + (\not{k} - \not{q} - M)\gamma_5 + 2M\gamma_5$ to highlight the contribution of the anomaly explicitly.

Show that the final result for $(p_\rho + q_\rho)\Gamma_{\rho|\mu\nu,\text{reg}}$ is proportinal to $M^2 J(M)$, where $J(M)$ is convergent scalar integral with massive propagators. Calculate remaining integral keeping only leading term in the large mass M and verify that final result for anomaly is non-zero and is independent of regulator mass M.
Hint: use loop momentum rescaling $k \to Ml$ to extract leading term of expansion.

Dimensional regularization (60 Points)

Exercise 11.2: *(20 points)* A clear alternative to Pauli-Villars is the dimensional regularization. If dimensional regularization is used, the anomaly of the axial current arises in a different way, as we now discuss.

If we use the dimensional regularization to compute the AVV correlator, we need to define the matrix γ_5 matrix in $d = 4 - 2\varepsilon$ dimensions. Recall that in four dimensions, γ_5 is defined by two equations

$$\{\gamma_5, \gamma_\mu\} = 0, \quad \mathrm{tr}\,(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = 4i\epsilon_{\mu\nu\rho\sigma}. \tag{4}$$

Here $\epsilon_{\mu\nu\rho\sigma}$ is a strictly four-dimensional object.

(a) *(10 points)* Assume direct generalization of the four-dimensional γ_5 to the arbitrary d space-time dimensions case. Using anticomutativity of γ_5 with γ_μ and cyclicity of the trace, calculate the following traces and explain using the obtained results why the chosen procedure is incorrect.

$$d\,\mathrm{tr}\,(\gamma_5) \tag{5}$$
$$(d-2)\,\mathrm{tr}\,(\gamma_5\gamma_\mu\gamma_\nu) \tag{6}$$
$$(d-4)\,\mathrm{tr}\,(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) \tag{7}$$

Hint: use insertion of the $\gamma_\mu\gamma_\mu = d$ into the trace.

(b) *(10 points)* The above question reveals a conflict between the anticommutativity of γ_5 and the cyclicity property of the trace. One has to give up one of these two properties.

A very natural way is to declare that γ_5 is *not* continued to d-dimensional space time which means that γ_5 that appears in the axial and the axial-vector currents is given by

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{8}$$

In such a way, the anticommutativity doesn't hold anymore while the cyclicity does. Show that with γ_5 defined in this way the following equations hold,

$$\hat{\gamma}_\mu\hat{\gamma}_\nu\hat{\gamma}_\mu = (6-d)\hat{\gamma}_\nu, \tag{9}$$
$$\hat{\gamma}_\mu\bar{\gamma}_\nu\hat{\gamma}_\mu = (4-d)\bar{\gamma}_\nu, \tag{10}$$
$$\bar{\gamma}_\mu\hat{\gamma}_\nu\bar{\gamma}_\mu = -4\hat{\gamma}_\nu, \tag{11}$$
$$\mathrm{tr}\,(\gamma_5) = 0, \tag{12}$$
$$\mathrm{tr}\,(\gamma_5\gamma_\mu\gamma_\nu) = 0, \tag{13}$$
$$\mathrm{tr}\,(\gamma_5\hat{\gamma}_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = 0, \tag{14}$$
$$\mathrm{tr}\,(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = \begin{cases} 4i\epsilon_{\mu\nu\rho\sigma}, & \{\mu,\nu,\rho,\sigma\} \in \{0,1,2,3\} \\ 0, & \text{otherwise} \end{cases}, \tag{15}$$

where γ matrices are split into the four-dimensional (with a bar) and rest $(d-4)$-dimensional (with a hat) part,

$$\gamma_\mu = \bar{\gamma}_\mu + \hat{\gamma}_\mu, \tag{16}$$

with

$$\{\bar{\gamma}_\mu, \gamma_5\} = 0, \quad [\hat{\gamma}_\mu, \gamma_5] = 0. \tag{17}$$

And $\bar{\gamma}_\mu, \hat{\gamma}_\mu$ themselves satisfy

$$\{\bar{\gamma}_\mu, \bar{\gamma}_\mu\} = \bar{g}_{\mu\nu}, \quad \{\hat{\gamma}_\mu, \hat{\gamma}_\mu\} = \hat{g}_{\mu\nu}, \tag{18}$$

where the metric tensor is split as well,

$$\hat{g}_{\mu\nu} = \begin{cases} g_{\mu\nu} & \mu,\nu \geq 4 \\ 0 & \mu,\nu < 4 \end{cases}, \qquad \bar{g}_{\mu\nu} = \begin{cases} g_{\mu\nu} & \mu,\nu < 4 \\ 0 & \mu,\nu \geq 4 \end{cases}, \tag{19}$$

We also introduce the decomposition of momentum as follows for later usage,

$$p_\mu = \bar{p}_\mu + \hat{p}_\mu. \tag{20}$$

Exercise 11.3: *(40 points)* To complete the contribution of the anomalous correlator in dimensional regularization, we consider the sum of massless AVV diagrams in Fig. 1. By explicit calculation show that $(p_\rho + q_\rho)\Gamma_{\rho|\mu\nu}$ in dimensional regularisation is finite and non-vanishing. Note that the external momenta and Lorenz indices are assumed to be four-dimensional.

(a) *(20 points)* Compute the fermion traces using dimension splitting rules defined above.

(b) *(10 points)* Show that the vector Ward identities Eq. (3) hold. Can you use the Noether theorem to derive a conserved vector current in this case? Explain it.

(c) *(10 points)* Show that the final result for $(p_\rho + q_\rho)\Gamma_{\rho|\mu\nu}$ has no poles in dimensional regulator parameter ε but non-zero. Explain what type of symmetries are violated by the chosen scheme of regularization and the treatment of the γ_5 matrix.

You may find the following equations helpful.

$$\begin{aligned}
&\bar{g}^{\mu\nu}\bar{\gamma}_\nu = g^{\mu\nu}\bar{\gamma}_\nu = \bar{g}^{\mu\nu}\gamma_\nu = \bar{\gamma}^\mu, && \bar{g}^{\mu\nu}\bar{p}_\nu = g^{\mu\nu}\bar{p}_\nu = \bar{g}^{\mu\nu}p_\nu = \bar{p}^\mu, \\
&\hat{g}^{\mu\nu}\hat{\gamma}_\nu = g^{\mu\nu}\hat{\gamma}_\nu = \hat{g}^{\mu\nu}\gamma_\nu = \hat{\gamma}^\mu, && \hat{g}^{\mu\nu}\hat{p}_\nu = g^{\mu\nu}\hat{p}_\nu = \hat{g}^{\mu\nu}p_\nu = \hat{p}^\mu, \\
&\bar{g}^{\mu\nu}\hat{\gamma}_\nu = \hat{g}^{\mu\nu}\bar{\gamma}_\nu = 0, && \bar{g}^{\mu\nu}\hat{p}_\nu = \hat{g}^{\mu\nu}\bar{p}_\nu = 0, \\
&\bar{g}^{\mu\nu}\bar{g}_{\nu\rho} = g^{\mu\nu}\bar{g}_{\nu\rho} = \bar{g}^{\mu\nu}g_{\nu\rho} = \bar{g}^\mu_\rho && \bar{g}^{\mu\nu}\bar{g}_{\mu\nu} = g^{\mu\nu}\bar{g}_{\mu\nu} = \bar{g}^{\mu\nu}g_{\mu\nu} = 4, \\
&\hat{g}^{\mu\nu}\hat{g}_{\nu\rho} = g^{\mu\nu}\hat{g}_{\nu\rho} = \hat{g}^{\mu\nu}g_{\nu\rho} = \hat{g}^\mu_\rho && \hat{g}^{\mu\nu}\hat{g}_{\mu\nu} = g^{\mu\nu}\hat{g}_{\mu\nu} = \hat{g}^{\mu\nu}g_{\mu\nu} = d-4, \\
&\bar{g}^{\mu\nu}\hat{g}_{\nu\rho} = \hat{g}^{\mu\nu}\bar{g}_{\nu\rho} = 0, && \bar{g}^{\mu\nu}\hat{g}_{\mu\nu} = \hat{g}^{\mu\nu}\bar{g}_{\mu\nu} = 0, \\
&\bar{\gamma}^\mu\bar{\gamma}_\mu = \gamma^\mu\bar{\gamma}_\mu = \bar{\gamma}^\mu\gamma_\mu = 4, && \bar{p}^\mu\bar{p}_\mu = \bar{p}^\mu p_\mu = p^\mu\bar{p}_\mu = \bar{p}^2 \\
&\hat{\gamma}^\mu\hat{\gamma}_\mu = \gamma^\mu\hat{\gamma}_\mu = \hat{\gamma}^\mu\gamma_\mu = d-4, && \hat{p}^\mu\hat{p}_\mu = \hat{p}^\mu p_\mu = p^\mu\hat{p}_\mu = \hat{p}^2 \\
&\bar{\gamma}^\mu\hat{\gamma}_\mu = \hat{\gamma}^\mu\bar{\gamma}_\mu = 0, && \bar{p}^\mu\hat{p}_\mu = \hat{p}^\mu\bar{p}_\mu = 0.
\end{aligned}$$