Theoretical Particle Physics II

Summer Term 2025

Karlsruhe Institute of Technology (KIT)

given by

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Contents

Chapter 1

Preliminary Remarks

1.1 Organization

- Webpage of the lecture: `https://ilias.studium.kit.edu/ilias.php?baseClass=ilrepositorygui&ref_id=2607451`

- Lecture times: Tuesday, 14-15h30, and Friday, 9h45-11h15; Room: Otto-Lehmann Hörsaal;
  Exercises: Thursday, 9h45-11h15 (10/1) and 11h30-13h00 (8/2).

- Exercise responsibles: Dr. Marco Bonetti, Dr. Duarte Fontes

- Criteria for successfully passing the exercise class and to obtain the exercise certificate are (both of them must be satisfied): 40% of the total points for all exercise sheets in the semester and presentation of at least one full exercise at the blackboard during an exercise class. You can submit your solutions in teams of at most two students.

- New exercise sheets will be published on the website on Mondays by 12:00. The solutions can be submitted until the following Monday at 12:00, either as a digital upload via ILIAS, or by putting it into the mailbox at the entrance of building 30.23 with the label "Theoretical Particle Physics II 2025 Exercise Solutions". The discussion of the exercise sheet will take place in the exercise classes in the week of the submission.

- Due to public holidays, the following exercise classes will be moved to the previous Tuesdays, same time slot, in building 30.23 (Physikhochhaus), room 11/12: 01.05.25 to Tuesday 29.04.25; 29.05.25 to Tuesday 27.05.25; 19.06.25 to Tuesday 17.06.25.

1.2 Literature

- M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, 1995)

- T.-P. Cheng, L.-F. Li, *Gauge Theory of Elementary Particle Physics* (Oxford University Press)

- C. Itzykson, J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill)

- P. Ramon, *Field Theory: a modern primer*
- M. Böhm, A. Denner and H. Joos, *Gauge Theories of the Strong and Electroweak Interaction* (Teubner, 2001)
- Chris Quigg *Gauge Theories of the Strong, Weak and Electromagnetic Interactions* (Benjamin/Cummings, 1983)
- G. Dissertori, I. Knowles, M. Schmeling, *Quantum Chromodynamics* (Oxford University Press)
- O. Nachtmann, *Elementary Particle Physics* (Springer 1990)
- L. H. Ryder, *Quantum Field Theory* (2nd ed., Cambridge University Press, 1996)
- R. K. Ellis, W. J. Stirling and B. R. Webber, *QCD and Collider Physics* (Cambridge University Press 1996)
- P. H. Frampton, *Gauge Field Theories* (Benjamin/Cummings)

Here literature on (among others) renormalization is:

- C. Itzykson, J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill)
- P. Ramon, *Field Theory: a modern primer*
- M. Böhm, A. Denner and H. Joos, *Gauge Theories of the Strong and Electroweak Interaction* (Teubner, 2001)
- W.J.P. Beenakker, *Electroweak corrections: techniques and applications*

And literature about path integrals e.g. is:

- Gert Roepstorff, *Path Integral Approach to Quantum Physics* (Springer)

Further literature for interested readers:

- Martinus Veltman *Facts and Mysteries in Elementary Particle Physics* (World Scientific, 2003)
- V. D. Barger and R. J. N. Phillips, *Collider Physics* (Addison-Wesley, 1997)
- Eds. Roger Cashmore, Luciano Maiani, Jean-Pierre Revol *Prestigious Discoveries at CERN* (Springer, 2004)

1.3 Disclaimer

The present script, Theoretical Particle Physics II, summer term 2025, KIT, by Prof. Dr. M.M. Mühlleitner, is not free of mistakes.

The script is intended only for KIT-internal use to accompany the lectures. Redistribution, processing and other use of the script is prohibited.

Chapter 2

Introduction

Elementary particle physics means physics at the smallest scales, respectively at the highest (relativistic) energies. Look e.g. at the wave-particle duality and the de Broglie relation,

$$E = h\nu \rightsquigarrow E\uparrow \;\Leftrightarrow\; \nu\uparrow \;\Leftrightarrow\; \lambda\downarrow \quad \text{smallest scales .} \tag{2.1}$$

The basis of the description of high-energy physics is quantum field theory. It is the synthesis of quantum mechanics and special relativity. In quantum mechanics, we use wave equations. These cannot describe processes where the number or the type of the particles change. Moreover, relativistic wave equations exhibit inconsistencies (e.g. negative energy solutions). In quantum field theory we identify particles with modes of a field, and the field itself is quantised ("2nd quantisation"). This allows us to describe the creation and annihilation of particles. Particles are excitations of relativistic fields. Photons e.g. are the excitations of electromagnetic fields. In the description of the fundamental interactions between the particles symmetries play an important role. Symmetries mean the invariance under certain transformations. The Standard Model of particle physics is based on gauge symmetries.

Why do we do high-energy physics? - We want to find answers to our basic questions about the universe:

1. What is the universe made of?
2. How did the universe evolve?
3. What are the fundamental building blocks of matter, and which forces hold them together?

What is the status of elementary particle physics today?

1. The known matter can be described by a few fundamental particles.
2. The diverse interactions are described by fundamental forces between the particles.
3. The physics laws can be described mathematically using a few fundamental principles (except for gravity).

2.1 Conventions

Natural units: In theoretical particle physics we use natural units (Planck units). We set the speed of light c and the Planck constant h equal to 1. The energy unit (which is not fixed by this choice) which is used, is the electron volt: $1\,\text{eV} = 1.6 \cdot 10^{-19}\,\text{J}$.

1. We set the speed of light c equal to 1:

$$c = 3 \cdot 10^8 \, \frac{\text{m}}{\text{s}} \equiv 1 \;\Rightarrow\; 1 \text{ s} = 3 \cdot 10^8 \text{ m} \tag{2.2}$$

2. The Planck constant is set equal to 1:

$$\hbar = \frac{h}{2\pi} = 6.6 \cdot 10^{-25}\,\text{GeV s} \equiv 1 \;\Rightarrow\; 1\,\text{s} = 1.5 \cdot 10^{24}\,\text{GeV}^{-1} \; . \tag{2.3}$$

And

$$\hbar c = 1 \;\Rightarrow\; 1\,\text{m} = 5.1 \cdot 10^{15}\,\text{GeV}^{-1} \; . \tag{2.4}$$

Furthermore,

$$m = \frac{E_{\text{rest}}}{c^2} = E_{\text{rest}} \tag{2.5}$$

$$m = \frac{1\,\text{eV}}{c^2} = \frac{1.6 \cdot 10^{-19}}{(3 \cdot 10^8)^2}\,\text{kg} = 1.78 \cdot 10^{-36}\,\text{kg} \stackrel{!}{=} 1\,\text{eV} \;\Rightarrow\; 1\,\text{kg} = 5.6 \cdot 10^{26}\,\text{GeV} \tag{2.6}$$

3. The elementary electric charge $e > 0$ is given by the Sommerfeld fine-structure constant α:

$$\frac{e^2}{4\pi} = \alpha \approx \frac{1}{137....} \;\Rightarrow\; e = 0.3. \tag{2.7}$$

The charge e is dimensionless.

All physics units are hence given in terms of powers of energy. The exponent is the (mass) dimension. He therefore have

$$[\text{Length}] = [\text{Time}] = -1 \;, \quad [\text{Mass}] = 1 \;, \quad [e] = 0 \; . \tag{2.8}$$

Minkowski Metric A metric space is a vector space with a metric. We have the contravariant four-vector

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} t \\ \vec{x} \end{pmatrix} \quad \text{(contravariant)} \; . \tag{2.9}$$

The dual space of the vector space contains as elements the covariant four-vectors

$$x_\mu = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ -\vec{x} \end{pmatrix} \quad \text{(covariant)} \; . \tag{2.10}$$

The transition between contra- and covariant is mediated by the *Minkowski metric* $g_{\mu\nu}$,

$$x_\mu = g_{\mu\nu} x^\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} t \\ \vec{x} \end{pmatrix} = \begin{pmatrix} t \\ -\vec{x} \end{pmatrix} \; . \tag{2.11}$$

The scalar product (which is invariant under Lorentz transformations - see next section) is given by

$$x \cdot y = x_\mu y^\mu = x^\mu g_{\mu\nu} y^\nu = x^0 y^0 - \vec{x} \cdot \vec{y} \; . \tag{2.12}$$

For the length of a Lorentz vector,

$$x^2 = x_0^2 - \vec{x}^2 \; , \tag{2.13}$$

we have the classifications

$$\begin{array}{ll} x^2 > 0 & \text{time-like} \\ x^2 = 0 & \text{light-like} \\ x^2 < 0 & \text{space-like} \; . \end{array} \tag{2.14}$$

It is

$$g_{\mu\nu} = g^{\mu\nu} \quad \text{und} \quad g_\mu^\nu = \delta_\mu^\nu \; . \tag{2.15}$$

Derivatives can also be written in a Lorentz-covariant way. We have

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \; , \tag{2.16}$$

$$\frac{\partial}{\partial x^\mu} = \begin{pmatrix} \frac{\partial}{\partial t} \\ \vec{\nabla} \end{pmatrix} \; . \tag{2.17}$$

We will use in the following the "slash" notation for the contraction with γ matrices,

$$\not{p} = p_\mu \gamma^\mu \; , \not{\partial} = \partial_\mu \gamma^\mu \; . \tag{2.18}$$

Levi-Civita-Tensor The Levi-Civita tensor is defined through

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{for even permutations} \\ -1 & \text{for uneven permutations} \\ 0 & \text{sonst} \end{cases} \tag{2.19}$$

We have here

$$\epsilon^{0123} = +1 \quad \Rightarrow \quad \epsilon_{0123} = g_{0\mu} g_{1\nu} g_{2\rho} g_{3\sigma} \epsilon^{\mu\nu\rho\sigma} = g_{00} g_{11} g_{22} g_{33} \epsilon^{0123} = -\epsilon^{0123} = -1 \; . \tag{2.20}$$

We also have (σ_2 is the second Pauli matrix)

$$\epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2 \ , \qquad \text{d.h. } \epsilon^{12} = 1 \ . \tag{2.21}$$

<u>Einstein Sum Convention</u> We sum over doubly appearing indicies, i.e.

$$a_i b_i = \sum_i a_i b_i \ . \tag{2.22}$$

Mostly, we have

$$a_\mu b_\mu = \sum_{\mu=0}^{3} a_\mu b^\mu \ . \tag{2.23}$$

For four-vectors, the Greek indices run from 0 to 3, and the Latin indices frun from 1 to 3.

2.2 Lorentz Group and Poincaré Group

2.2.1 The Lorentz Transformation

In classical physics and special relativity the tensor concept plays a central role. According to the covariance principle, physics laws can be expressed through tensor equations:

$$\text{Physics laws} \ \Leftrightarrow \ \text{Tensor equations} \ . \tag{2.24}$$

Physics laws are invariant under coordinate transformations. A tensor equations relates vectors (tensors of rank 1) and tensors of higher rank. In quantum field theory we also deal with fermions. They have spin of half unit and are fundamentally different from bosons with unit spin. They are described through spinors. The covariance principle for fermions is

$$\text{Physics laws} \ \Leftrightarrow \ \text{Spinor equations} \ . \tag{2.25}$$

A typical example is the Dirac equation. Once the transformation properties of objects like tensors, spinors are known, we can construct invariant quantities, i.e. Lorentz invariants, from them. The Lagrangian density e.g. is a Lorentz-invariant quantity. From the Lagrangian density we can then derive the equations of motion.

All linear transformations in Minkowski space,

$$x^\mu \to x'^\mu = \Lambda^\mu_{\ \nu} x^\nu \tag{2.26}$$
$$\text{with } \ x'_\mu y'^\mu = x_\mu y^\mu \quad \text{for all } x, y \ , \tag{2.27}$$

are called Lorentz transformations. They form the Lorentz group. It corresponds to the pseudo-orthogonal group $O(3,1)$. This means for the 4×4 matrices that $\Lambda \in O(3,1)$. From (2.27) it follows that

$$g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu} \Lambda^\mu_{\ \rho} x^\rho \Lambda^\nu_{\ \sigma} x^\sigma = g_{\rho\sigma} x^\rho x^\sigma \quad \Rightarrow \tag{2.28}$$
$$g_{\rho\sigma} = g_{\mu\nu} \Lambda^\mu_{\ \rho} \Lambda^\nu_{\ \sigma} \ . \tag{2.29}$$

And hence

$$\Lambda^T g \Lambda = g \quad \Rightarrow \quad \det g = \det(\Lambda^T g \Lambda) \quad \Rightarrow \quad \det \Lambda = \pm 1 \ . \tag{2.30}$$

We hence have

$$x'^2 = x^2 \ . \tag{2.31}$$

The "length" $\sqrt{x^2} = \sqrt{t^2 - \vec{x}^2}$ hence remains invariant in Minkowski space. The d'Alembert operator is a Lorentz-invariant differential operator, given by

$$\partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \Delta = \Box \ . \tag{2.32}$$

Another important Lorentz-invariant quantity is the product of the four-momentum

$$p^\mu = \begin{pmatrix} p_0 \\ \vec{p} \end{pmatrix} = \begin{pmatrix} E \\ \vec{p} \end{pmatrix} \ , \tag{2.33}$$

hence

$$p'^2 = p^2 = E^2 - \vec{p}^2 \ . \tag{2.34}$$

We say, that a particle is on its mass shell, when we have

$$p^2 = m^2 \ . \tag{2.35}$$

We have used here the convention introduced above that $c = 1$. In the non-relativistic limit, we have $|\vec{p}| \ll m$ so that we can perform an expansion in $|\vec{p}|/m$,

$$E = \sqrt{m^2 + \vec{p}^2} = m \left(1 + \frac{\vec{p}^2}{m^2}\right)^{\frac{1}{2}} = m + \frac{\vec{p}^2}{2m} + \mathcal{O}\left(\frac{\vec{p}^4}{m^3}\right) \ . \tag{2.36}$$

For velocities close to the speed of light, in the ultra-relativistic limit, we have $|\vec{p}| \gg m$. Then the mass can be neglected, and we have $E \approx |\vec{p}|$.

The Lorentz group can be classified following two properties: the sign of the determinant, $\det \Lambda$, and the sign of $\Lambda^0_{\ 0}$. The Lorentz transformations

1. $L_+^\uparrow = \{\Lambda \in L : \det \Lambda = +1, \ \Lambda^0_{\ 0} > 0\}$ are called proper orthochronous.

2. $L_+^\downarrow = \{\Lambda \in L : \det \Lambda = +1, \ \Lambda^0_{\ 0} < 0\}$ are called proper non-orthochronous.

3. $L_-^\uparrow = \{\Lambda \in L : \det \Lambda = -1, \ \Lambda^0_{\ 0} > 0\}$ are called improper orthochronous.

4. $L_-^\downarrow = \{\Lambda \in L : \det \Lambda = -1, \ \Lambda^0_{\ 0} < 0\}$ are called improper non-orthochronous.

They form the Lorentz group

$$L = L_+^\uparrow \cup L_+^\downarrow \cup L_-^\uparrow \cup L_-^\downarrow \ . \tag{2.37}$$

The proper orthochronous Lorentz group does not allow for space space reflection ($\det \Lambda = +1$) and the time coordinate cannot change the sign.

The proper orthochronous Lorentz group

$$L_+^\uparrow = \{\Lambda \in O(1,3) | \det\Lambda = 1, \Lambda^0_{\ 0} > 0\} \tag{2.38}$$

contains rotations and boosts. The rotations are given by

$$\Lambda(0,\vec{\varphi}) \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R(\vec{\varphi}) & \\ 0 & & & \end{pmatrix} \tag{2.39}$$

with the axis $\frac{\vec{\varphi}}{|\vec{\varphi}|}$ and the angle $\varphi = |\vec{\varphi}|$ and the rotation matrix elements $R(\vec{\varphi})_{ij}$.

A pure boost into a reference system which moves with a relative velocity v in the direction of the $x^i = x$-axis is given by ($\nu = \mathrm{artanh} v$)

$$\Lambda(\vec{\nu},0) = \begin{pmatrix} \cosh\nu & -\sinh\nu & 0 & 0 \\ -\sinh\nu & \cosh\nu & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{2.40}$$

2.2.2 The Poincaré Group

Tensors or (relativistic) bosons are objects which transform according to the tensor representation of the Lorentz group. Spinors or (relativistic) fermions are objects which transform according to the spinor represenation of the Lorentz group. Hence, by studying the Lorentz group, we can distinguish between bosons and fermions and assign particles to one of the two categories. But to completely treat the world of elementary particles we need to study the Poincaré group.

The Poincaré group is the group of Lorentz transformations and translations in Minkowski space. It describes the structure of our space-time, and all its irreducible representations are characterised by mass and spin, hence by the fundamental properties of the elementary particles.

Poincaré transformations in Minkowski space are composed of a Lorentz transformation with $\Lambda^\mu_{\ \nu}$ and a translation by a^μ. We hence have the translation group T and the Poincaré group P given by

$$T \;=\; \{x^\mu \to x'^\mu = x^\mu + a^\mu \;:\; a^\mu \in \mathbb{R}^4\} \tag{2.41}$$

$$P \;=\; \{x^\mu \to x'^\mu = \Lambda^\mu_{\ \nu} x^\nu + a^\mu \;:\; \Lambda^\mu_{\ \nu} \in L,\ a^\mu \in \mathbb{R}^4\} \tag{2.42}$$

All generators of symmetries relevant for physics have to be invariant under Poincaré transformations. There is only one extension of the space-time symmetry that is compatible with relativistic quantum field theory. This is superymmetry, which relates fermions and bosons.

Chapter 3

Gauge Symmetries

The principle of local gauge invariance is essential for quantum field theory. We start by looking at the example of quantum electrodynamics (QED). The Dirac Lagrangian for a free fermion field Ψ of mass m reads

$$\mathcal{L}_0 = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi \ . \tag{3.1}$$

It is invariant under a transformation with a unitary matrix $U = e^{-i\alpha} \in U(1)$. This means that applying the transformation

$$\Psi(x) \to \exp(-i\alpha)\Psi(x) = \Psi - i\alpha\Psi + \mathcal{O}(\alpha^2) \tag{3.2}$$

and for the adjoint spinor $\bar{\Psi} = \Psi^\dagger \gamma^0$,

$$\bar{\Psi}(x) \to \exp(i\alpha)\bar{\Psi}(x). \tag{3.3}$$

the Lagrangian $\mathcal{L}_0$ goes over into itself. We distinguish

- global gauge transformations: α =const.

- local gauge transformations: $\alpha = \alpha(x)$.

The Noether current of the above global gauge symmetry reads

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} \frac{\delta \Psi}{\delta \alpha} + \frac{\delta \bar{\Psi}}{\delta \alpha} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Psi})} = i\bar{\Psi}\gamma^\mu(-i\Psi) = \bar{\Psi}\gamma^\mu\Psi \ , \tag{3.4}$$

with

$$\partial_\mu j^\mu = 0 \ . \tag{3.5}$$

It implies charge conservation.

3.1 Coupling to a Photon

When we include the coupling to a photon, the Lagrangian reads

$$\mathcal{L} = \bar{\Psi}\gamma^\mu(i\partial_\mu - qA_\mu)\Psi - m\bar{\Psi}\Psi = \mathcal{L}_0 - qj^\mu A_\mu \ , \tag{3.6}$$

with j^μ given in Eq. (3.4). Appyling the following gauge transformation to the external photon field A_μ,

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x) \tag{3.7}$$

the Lagrangian goes over into

$$\mathcal{L} \to \mathcal{L} = \mathcal{L}_0 - qj^\mu A_\mu - \underbrace{qj^\mu \partial_\mu \Lambda}_{q\bar{\Psi}\gamma^\mu\Psi\partial_\mu\Lambda} \; . \tag{3.8}$$

This means that $\mathcal{L}$ is not gauge invariant. The transformations of the fields Ψ and $\bar{\Psi}$ have to be changed such that the Lagrangian becomes gauge invariant. This is done by introducing an x-dependent parameter α, hence $\alpha = \alpha(x)$. Thereby

$$i\partial_\mu \Psi \to i \exp(-i\alpha)(\partial_\mu \Psi) + (\partial_\mu \alpha) \exp(-i\alpha)\Psi \; , \tag{3.9}$$

so that

$$\mathcal{L}_0 \to \mathcal{L}_0 + \bar{\Psi}\gamma^\mu \Psi \partial_\mu \alpha \; . \tag{3.10}$$

This term cancels the additional term in Eq. (3.8) if

$$\alpha(x) = q\Lambda(x) \; . \tag{3.11}$$

Thereby the complete gauge transformation reads

$$\Psi \to \Psi'(x) = U(x)\Psi(x) \qquad \text{with} \qquad U(x) = \exp(-iq\Lambda(x)) \quad (U \text{ unitary}) \tag{3.12}$$

$$\bar{\Psi} \to \bar{\Psi}'(x) = \bar{\Psi}(x) U^\dagger(x) \tag{3.13}$$

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x) = U(x) A_\mu(x) U^\dagger(x) - \frac{i}{q} U(x) \partial_\mu U^\dagger(x) \; . \tag{3.14}$$

The Lagrangian transforms according to

$$\begin{aligned} \mathcal{L} \to \mathcal{L}' &= \bar{\Psi}\gamma^\mu U^{-1} i\partial_\mu (U\Psi) - q\bar{\Psi}U^{-1}\gamma^\mu \left(U A_\mu U^{-1} - \frac{i}{q} U \partial_\mu U^{-1} \right) U\Psi - m\bar{\Psi}U^{-1}U\Psi \\ &= \bar{\Psi}\gamma^\mu i\partial_\mu \Psi + \bar{\Psi}\gamma^\mu (U^{-1} i(\partial_\mu U))\Psi - q\bar{\Psi}\gamma^\mu \Psi A_\mu + \bar{\Psi}\gamma^\mu (i(\partial_\mu U^{-1})U)\Psi - m\bar{\Psi}\Psi \\ &= \mathcal{L} + i\bar{\Psi}\gamma^\mu \partial_\mu (U^{-1}U)\Psi = \mathcal{L} \; . \end{aligned} \tag{3.15}$$

Minimal substitution $p_\mu \to p_\mu - qA_\mu$ leads to

$$i\partial_\mu \to i\partial_\mu - qA_\mu \equiv iD_\mu \; . \tag{3.16}$$

Here $D_\mu(x)$ is the *covariant derivative*. The expression *covariant* means, that it transforms exactly as the field,

$$\Psi(x) \to U(x)\Psi(x) \qquad \text{and} \qquad D_\mu \Psi(x) \to U(x)(D_\mu \Psi(x)) \; . \tag{3.17}$$

This means

$$(D_\mu \Psi)' = D'_\mu \Psi' = D'_\mu U\Psi \stackrel{!}{=} UD_\mu \Psi \; , \tag{3.18}$$

so that the covariant derivative transforms according to

$$\begin{aligned} D'_\mu &= UD_\mu U^{-1} = \exp(-iq\Lambda)(\partial_\mu + iqA_\mu)\exp(iq\Lambda) = \partial_\mu + iq\partial_\mu \Lambda + iqA_\mu \\ &= \partial_\mu + iqA'_\mu \; . \end{aligned} \tag{3.19}$$

Thereby

$$\mathcal{L} = \bar{\Psi}\gamma^\mu i D_\mu \Psi - m\bar{\Psi}\Psi \tag{3.20}$$

is obviously gauge invariant.

The kinetic energy of the photons is given by

$$\mathcal{L}_{kin} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad \text{with} \qquad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \ . \tag{3.21}$$

The field strength tensor $F_{\mu\nu}$ can be expressed through the covariant derivative. We choose the following ansatz for the tensor of rang 2,

$$[D_\mu, D_\nu] = [\partial_\mu - iqA_\mu, \partial_\nu - iqA_\nu] = -iq[\partial_\mu, A_\nu] - iq[A_\mu, \partial_\nu] = -iq(\partial_\mu A_\nu - \partial_\nu A_\mu) \ . \tag{3.22}$$

Thereby, we have for the field strength tensor

$$F^{\mu\nu} = \frac{i}{q}[D^\mu, D^\nu] \ . \tag{3.23}$$

Its transformation behaviour is given by

$$\frac{i}{q}[UD^\mu U^{-1}, UD^\nu U^{-1}] = \frac{i}{q}U[D^\mu, D^\nu]U^{-1} = UF^{\mu\nu}U^{-1} \ . \tag{3.24}$$

The unitary group $U(1)$ is an Abelian gauge group as for $f, g \in U(1)$ it holds that $f \circ g = g \circ f$.

3.2 Groups

Be a pair $(G, *)$ with a set G and an inner binary connection/group multiplication. $* : G \times G \to G, (a, b) \mapsto a * b$ is called group if the following axioms are fulfilled

1. The group is *closed*. This means, if $g, h \epsilon G \Rightarrow g * h \epsilon G$.
2. *Associativity:* $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$.
3. $\exists$ *Identity element* e with the property $g * e = e * g = g \ \ \forall g \epsilon G$.
4. For each g there is an *inverse* g^{-1} with $g^{-1} * g = g * g^{-1} = e$.

Abelian group: A group is called *Abelian*, if $g * h = h * g$.

Continuous groups: They contain an infinite number of elements and are described by n parameters. The elements depend in a continuous and differentiable way on a set of real parameters θ^a, $a = 1, ..., n$, where n is the dimension of the group. For *Lie groups* n is finite. We chose $g(\theta = 0) = e$. All one-parameter Lie groups are Abelian. A typical example is $U(1)$ with the elements $e^{i\phi}$ and ϕ as parameter.
Examples of Lie groups:
(i) $O(N)$: orthogonal group, dimension $\frac{N(N-1)}{2}$. We have $MM^T = 1$ so that $\det M = \pm 1$. We have the $SO(N)$ for $\det M = 1$.
(ii) $U(N)$: unitary group, dimension N^2. We have $UU^\dagger = 1$. We have the $SU(N)$ for $\det U = 1$. Its dimension is $N^2 - 1$.
(iii) $SL(N, \mathbb{C})$: complex matrices A, $\det A = 1$, dimension $2N^2 - 2$. E.g. the symplectic group $\mathrm{Sp}(2n, \mathbb{C})$.

3.3 Representations of Non-Abelian Groups

Be G a group with the elements $g_1, g_2... \in G$. An n-dimensional representation of G is given by the map $G \to C^{(n,n)}$, $g \to U(g)$. It is a map of abstract elements of the group onto complex $n \times n$ matrices, so that $U(g_1 g_2) = U(g_1)U(g_2)$ holds and hence the group properties are preserved.

A $U \in SU(N)$ can be written as $U = \exp(i\theta^a T^a)$. In general, each group element, which can be obtained from the identity element through continuous transformation of the parameters, can be written as $\exp(i\theta^a T^a)$, where θ^a are real parameters and T^a are linearly independent operators. The set of all linear combinations of $\theta^a T^a$ forms a vector space with the basis elements $\theta^a T^a$. They are also called generators of the group. In the case of the $SU(N)$ the generators are hermitian. For the $SU(2)$ we have $U = \exp(i\vec{\omega}\cdot\vec{J})$. The group $SU(N)$ has N^2-1 generators T^a. For the $SU(2)$ these are the angular momentum operators J_i, $i = 1, 2, 3$. The N^2-1 real parameters θ^a are given by $\vec{\omega}$ in the $SU(2)$. The fundamental representation of the $SU(2)$ reads $J_i = \sigma_i/2$ and in the general case $T^a = \lambda^a/2$.

Independent of the representations the generators fulfill the following commutator relation

$$[T^a, T^b] = i f^{abc} T^c \ . \tag{3.25}$$

The f^{abc} are the structure constants of the $SU(N)$ Lie algebra. The commutation relation hence defines the algebra, which is associated with the group. The generators are not uniquely normalized. We have

$$\mathrm{Trace}(T^a T^b) = T_R \delta^{ab} \, , \tag{3.26}$$

where T_R is the Dynkin-Index. It depends on the representation. For the fundamental representation it is mostly chosen as

$$T_R \equiv T_F = 1/2 \ . \tag{3.27}$$

From Eq. (3.25) follows

$$[T^a, T^b]T^c = i f^{abd} T^d T^c \quad \Rightarrow \quad i f^{abc} T_R = \mathrm{Trace}([T^a, T^b]T^c) \ . \tag{3.28}$$

The structure constants f^{abc} are hence totally anti-symmetric and define $(N^2-1)(N^2-1)$-dimensional matrices $T^a_{lk} \equiv -i f^a_{lk} \equiv -i f^{alk}$. For the $SU(2)$ we have

$$[J_i, J_j] = \epsilon_{ijk} J_k \ . \tag{3.29}$$

The generators of Lie groups fulfill the Jacobi identity

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0 \ . \tag{3.30}$$

Using (3.25), one obtains

$$0 = (-i f^b_{cl})(-i f^a_{lk}) + (-i f^a_{lc})(-i f^b_{lk}) + i f^{abl}(-i f^l_{ck}) \ . \tag{3.31}$$

And thereby

$$0 = (T^b T^a)_{ck} - (T^a T^b)_{ck} + i f^{abl}(T^l)_{ck} \ . \tag{3.32}$$

We thus have obtained an N^2-1-dimensional representation of the $SU(N)$ Lie algebra,

$$[T^a, T^b] = i f^{abc} T^c \ . \tag{3.33}$$

This is the *adjoint representation.* There are the following $SU(N)$ representations,

- $d = 1$: trivial representation (singlet).
- $d = N$: fundamental representation ($\lambda^a/2$), anti-fundamental representation ($-\lambda^{*a}/2$). The generators are $N \times N$ matrices.
- $d = N^2 - 1$: adjoint representation. The generators are $(N^2 - 1) \times (N^2 - 1)$-matrices. The indices of the representation run over the same range as the number of generators, which forms the dimension of the group. In the adjoint representation hence the dimension of the vector space, in which the matrices act, is equal to the dimension of the group.

If a representation r and its complex conjugate representation $\bar{r}$ with

$$T^a_{\bar{r}} = -(T^a_r)^* \; , \tag{3.34}$$

are equivalent, hence $T^a_{\bar{r}} = U T^a_r U^\dagger$, then the representation is called real. The fundamental representation of $SU(2)$ is real, but not the one of $SU(3)$. This is why the anti-quarks have an anti-colour. The adjoint representation of the $SU(3)$ is real.

Casimir operators Casimir operators allow to characterise representations independently of the chosen basis. The quadratic Casimir operator is defined by

$$\sum_a T^a T^a = C_2(R) 1 \; , \tag{3.35}$$

where $C_2(R)$ depends on the representation, but not on the basis of the generators T^a.

3.4 The Matrices of the $SU(N)$

The elements of the $SU(N)$ in general are represented through

$$U = \exp\left(i\theta^a \frac{\lambda^a}{2}\right) \qquad \text{with} \qquad \theta^a \in \mathbb{R} \; . \tag{3.36}$$

Here the $\lambda^a/2$ are the generators of the group $SU(N)$. For the $SU(2)$ the λ^a are given by the Pauli matrices σ^a ($a = 1, 2, 3$) and θ^a is a 3-component vector given by $\vec{\omega}$ for the $SU(2)$. For an element of the group $SU(2)$ we hence have

$$U = \exp\left(i\vec{\omega}\frac{\vec{\sigma}}{2}\right) \; . \tag{3.37}$$

For a general U we have

$$U^\dagger = \exp\left(-i\theta^a \left(\frac{\lambda^a}{2}\right)^\dagger\right) \stackrel{!}{=} U^{-1} = \exp\left(-i\theta^a \frac{\lambda^a}{2}\right) \; . \tag{3.38}$$

The generators hence have to be hermitian, i.e.

$$(\lambda^a)^\dagger = \lambda^a \; . \tag{3.39}$$

In addition, for the $SU(N)$ it has to hold that

$$\det(U) = 1 \; . \tag{3.40}$$

With

$$\det(\exp(A)) = \exp(\mathrm{Tr}(A)) \tag{3.41}$$

we get

$$\det\left(\exp\left(i\theta^a \frac{\lambda^a}{2}\right)\right) = \exp\left(i\theta^a \mathrm{Tr}\left(\frac{\lambda^a}{2}\right)\right) \stackrel{!}{=} 1 \ . \tag{3.42}$$

From this follows that

$$\mathrm{Tr}(\lambda^a) = 0 \ . \tag{3.43}$$

The generators of the $SU(N)$ have to be traceless. The group $SU(N)$ has N^2-1 generators λ^a with $\mathrm{Tr}(\lambda^a) = 0$. For the $SU(3)$ these are the Gell-Mann matrices

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \ .
\end{aligned} \tag{3.44}$$

The matrices $\lambda^a/2$ are normalised as

$$\mathrm{Tr}\left(\frac{\lambda^a}{2}\frac{\lambda^b}{2}\right) = \frac{1}{2}\delta^{ab} \ . \tag{3.45}$$

For the Pauli matrices ($i = 1, 2, 3$) we have

$$\mathrm{Tr}(\sigma_i^2) = 2 \qquad \text{und} \qquad \mathrm{Tr}(\sigma_1\sigma_2) = \mathrm{Tr}(i\sigma_3) = 0 \ . \tag{3.46}$$

Multiplied by $1/2$ they form the generators of the group $SU(2)$. The generator matrices fulfill the completeness relation

$$\frac{\lambda^a_{ij}}{2}\frac{\lambda^a_{kl}}{2} = \frac{1}{2}\left(\delta_{il}\delta_{kj} - \frac{1}{N}\delta_{ij}\delta_{kl}\right) \ , \tag{3.47}$$

because

$$0 \stackrel{!}{=} \frac{\lambda^a_{ii}}{2}\frac{\lambda^a_{kl}}{2} = \frac{1}{2}\delta_{il}\delta_{ki} - \frac{1}{2N}\delta_{ii}\delta_{kl} = \frac{1}{2}\delta_{kl} - \frac{1}{2}\delta_{kl} = 0 \ . \tag{3.48}$$

The gauge group underlying quantum chromo dynamics (QCD) is the $SU(3)$. The QCD describes the strong interaction between colour charged particles. The quarks are in the fundamental representation of the $SU(3)$. The Feynman rule for the interaction between one gluon and two quarks contains the $T^a_{ij} = \lambda^a_{ij}/2$, with $i, j = 1, ..., N_c$ ($N_c = 3$) and $a = 1, ..., 8$. N_c denotes the number of the quark colours. The gluons are in the adjoint representation of the $SU(3)$, which is expressed through the matrices $(F^a)_{bc} = -if^{abc}$.

3.5 Non-Abelian Gauge Theories, $SU(N)$ Symmetries

In the following we consider a Lagrangian which is invariant under transformations of the group $SU(N)$, where

$$SU(N) = \{U \in \mathbb{C}^{N\times N} | UU^\dagger = 1 \wedge \det U = 1\} \ . \tag{3.49}$$

Each $U \in SU(N)$ can be written as

$$U = \exp(i\theta_a T^a) \ , \ \theta_a \in \mathbb{R} \ . \tag{3.50}$$

From $UU^\dagger = 1$ follows that $T^a = (T^a)^\dagger$, from $\det U = 1$ follows with $\det U = e^{\mathrm{Tr}(\ln U)}$ that $\mathrm{Tr}(T^a) = 0$.

Fermion Fields Starting point is the Lagrangian for N Dirac fields $\psi_i(x)$ $(i = 1, .., N)$,

$$\mathcal{L} = \sum_{i=1...N} \bar{\psi}_i(i\gamma^\mu\partial_\mu - m)\psi_i = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi \qquad \text{with} \quad \bar{\Psi} = (\bar{\psi}_1, \bar{\psi}_2, ..., \bar{\psi}_N) \ . \tag{3.51}$$

The Lagrangian is invariant under a global $SU(N)$ gauge transformation (the index a runs over $a = 1, ..., N^2 - 1$)

$$\Psi \to \Psi' = \exp\left(i\theta^a T^a\right)\Psi = \left(1 + i\theta^a T^a + \mathcal{O}((\theta^a)^2)\right)\Psi = U\Psi = \ \text{and}\ \bar{\Psi} \to \bar{\Psi}' = \bar{\Psi}U^{-1} \tag{3.52}$$

respectively, $(i, j = 1, ..., N)$

$$\psi_i(x) \to U_{ij}\psi_j(x) \ . \tag{3.53}$$

The generators T^a are

$$\begin{array}{lll} \text{fundamental representation:} & (T^a)_{ij} = \left(\frac{\lambda^a}{2}\right)_{ij} & d = N \\ \text{adjoint representation} & (T^a)_{bc} = -if^{abc} & d = N^2 - 1 \\ \text{trivial representation} & T^a = 0 \Leftrightarrow U(\theta) = 1 \ . & \end{array} \tag{3.54}$$

Examples:

- $\Psi = \begin{pmatrix} p \\ n \end{pmatrix}$: $SU(2)$ transformations in the isospin space, proton-neutron doublet.
- $\Psi = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$: $SU(2)_L$, weak interaction on left-handed fermions.
- $\Psi = (q_1, q_2, q_3)^T$, quarks, $SU(3)_C$. Here, each q_i $(i = 1, 2, 3)$ is a four-component spinor. The QCD Lagrangian is invariant under $SU(3)_C$ transformations.

Representation of the Gauge Fields The gauge fields are in the adjoint representation of the $SU(N)$. Thereby, we have $N^2 - 1$ gauge fields $G^a_\mu(x)$ $(a = 1, ..., N^2 - 1)$. In a non-Abelian gauge theory also the gauge fields carry charge (e.g. in the QCD the colour charge), in an Abelian gauge theory, however, not (the photon does not have an electric charge). The adjoint representation of the $SU(N)$ is given by the matrices $(T^a)_{bc}$, which are obtained from the structure constants of the group,

$$(T^a)_{bc} = -if^{abc} \ , \quad a, b, c = 1, ...N^2 - 1 \ . \tag{3.55}$$

Fermion Gauge Boson Interaction In analogy to QED we can write the interaction between fermions and gauge bosons as

$$\mathcal{L}_{\text{int}} = \sum_{i,j=1}^{N} \bar{\psi}_i (i\gamma_\mu (D^\mu[G])_{ij} - m_j \delta_{ij}) \psi_j \ . \tag{3.56}$$

The covariant derivative is given by

$$(D^\mu[G])_{ij} = \delta_{ij}\partial^\mu - ig \sum_{a=1}^{N^2-1} G_a^\mu(x) T_{ij}^a \equiv \delta_{ij}\partial^\mu - ig(\mathcal{G}^\mu)_{ij} \ . \tag{3.57}$$

The T^a can be different, but G_μ^a is identical in all D_μ. For example in supersymmetry (SUSY),

$$\begin{array}{lll} \text{squark, quark} & T^a = \frac{\lambda^a}{2} & (d = N) \\ \text{gluino, gluon} & (T^a)_{bc} = -if^{abc} & (d = N^2-1) \end{array} \tag{3.58}$$

Gauge-Invariant Lagrangian Let us now look at local symmetries, hence $\theta^a = \theta^a(x)$. The transformation of Ψ is given by $\Psi' = U\Psi$. We want to achieve that the Lagrangian is invariant under these gauge transformations. This is fulfilled if the covariant derivative transforms exactly as Ψ, hence $(D_\mu\Psi)' = U(D_\mu\Psi)$. Thereby

$$(D_\mu\Psi)' = D'_\mu\Psi' = D'_\mu U\Psi \Rightarrow D'_\mu U = UD_\mu \ . \tag{3.59}$$

This if fulfilled, because

$$\partial_\mu - ig\mathcal{G}'_\mu = D'_\mu = UD_\mu U^{-1} = U(\partial_\mu - ig\mathcal{G}_\mu)U^{-1} = UU^{-1}\partial_\mu + U(\partial_\mu U^{-1}) - igU\mathcal{G}_\mu U^{-1} \Rightarrow \tag{3.60}$$

$$\mathcal{G}'_\mu = \frac{i}{g} U(\partial_\mu U^{-1}) + U\mathcal{G}_\mu U^{-1} \ . \tag{3.61}$$

Important: G'^a_μ is independent of the representation U. With infinitesimal

$$U = \exp(iT^a\theta^a) = 1 + iT^a\theta^a + \mathcal{O}(\theta^{a2}) \tag{3.62}$$

we have

$$\begin{aligned} \mathcal{G}'_\mu &= G'^b_\mu T^b = \frac{i}{g} U(-i)T^a\left(\partial_\mu\theta^a\right)U^{-1} + \underbrace{(1 + i\theta^a T^a)G_\mu^c T^c (1 - i\theta^b T^b)}_{G_\mu^c T^c + iG_\mu^c \underbrace{(T^a T^c - T^c T^a)}_{if^{acb}T^b}\theta^a + \mathcal{O}(\theta^2)} \\ &= T^b \underbrace{\left(\frac{1}{g}\partial_\mu\theta^b + G_\mu^b + i(-if^{abc})\theta^a G_\mu^c\right)}_{G'^b_\mu} \ . \end{aligned} \tag{3.63}$$

The field strength tensor is defined as $\mathcal{F}^{\mu\nu} \sim [D^\mu, D^\nu]$. Let us look at the commutator,

$$\begin{aligned} [D^\mu, D^\nu] &= [\partial_\mu - igT^a G_\mu^a, \partial_\nu - igT^b G_\nu^b] = -igT^b\partial_\mu G_\nu^b - igT^a(-\partial_\nu G_\mu^a) + (-ig)^2 G_\mu^a G_\nu^b \underbrace{[T^a, T^b]}_{if^{abc}T^c} \\ &= -igT^a \underbrace{\left(\partial_\mu G_\nu^a - \partial_\nu G_\mu^a + g \underbrace{f^{bca}}_{f^{abc}} G_\mu^b G_\nu^c\right)}_{=:F_{\mu\nu}^a} = -igT^a F_{\mu\nu}^a \equiv -ig\mathcal{F}_{\mu\nu} \ . \end{aligned} \tag{3.64}$$

The $F^a_{\mu\nu}$ are independent of the representation of the T^a. We have for the transformation behaviour

$$\mathcal{F}'_{\mu\nu} = \frac{i}{g}[D'^{\mu}, D'^{\nu}] = \frac{i}{g}[UD_\mu U^{-1}, UD_\nu U^{-1}] = U\mathcal{F}_{\mu\nu}U^{-1} \qquad \text{homogenuous transformation} \tag{3.65}$$

And with Eq. (3.63)

$$(F^a_{\mu\nu})' = F^a_{\mu\nu} + i(-if^{bac})\theta^b F^c_{\mu\nu} + ... \tag{3.66}$$

Furthermore, from this follows that

$$F^{a\mu\nu}F^a_{\mu\nu} = 2\mathrm{Tr}(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}) \left(= 2\mathrm{Tr}(F^{a\mu\nu}T^a F^b_{\mu\nu}T^b) = 2F^{a\mu\nu}F^b_{\mu\nu}\underbrace{\mathrm{Tr}(T^aT^b)}_{\frac{1}{2}\delta^{ab}} = F^{\mu\nu a}F^a_{\mu\nu}\right) \qquad \text{is gauge invariant} \tag{3.67}$$

Thereby we have for the kinetic Lagrangian

$$\mathcal{L}_{kin,A} = -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} = -\frac{1}{2}\mathrm{Tr}(\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}) \ . \tag{3.68}$$

This Lagrangian for the gauge fields is also called Yang-Mills Lagrangian. It contains cubic and quartic terms in the the gauge fields. This leads in QCD to the 3-gluon and the 4-gluon vertices. Remark that the gauge fields as in the case of the photon have to be massless. A mass term bilinear in the G^a_μ would break the $SU(N)$ gauge invariance.

3.6 The QCD Lagrangian

Example: QCD is invariant under the colour $SU(3)$. The 6 quark fields carry colour charge and are in the fundamental representation,

$$\Psi_q = \begin{pmatrix} \psi_{q1} \\ \psi_{q2} \\ \psi_{q3} \end{pmatrix} \qquad q = u, d, c, s, t, b \ . \tag{3.69}$$

They form triplets. The 8 gluons G^μ are in the adjoint representation. The QCD Lagrangian reads

$$\mathcal{L}_{QCD} = -\frac{1}{4}G^{a\mu\nu}G^a_{\mu\nu} + \sum_{q=1...6} \bar{\Psi}_q(i\gamma^\mu D_\mu - m_q)\Psi_q \ , \tag{3.70}$$

with

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + gf^{abc}G^b_\mu G^c_\nu \ . \tag{3.71}$$

The quark masses have the values

$$m_u \approx 1.7...3.1 \text{ MeV} \qquad m_d \approx 4.1...5.7 \text{ MeV} \qquad m_s \approx 100 \text{ MeV} \tag{3.72}$$

$$m_c \approx 1.29 \text{ GeV} \qquad m_b \approx 4.19 \text{ GeV} \qquad m_t \approx 173 \text{ GeV} \ . \tag{3.73}$$

3.7 Chiral Gauge Theories

Let us look at

$$\mathcal{L}_f = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi \; . \tag{3.74}$$

In the chiral representation the 4×4 γ matrices are given by

$$\gamma^\mu = \left(\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} & -\vec{\sigma} \\ \vec{\sigma} & \mathbf{0} \end{pmatrix} \right) = \begin{pmatrix} 0 & \sigma_-^\mu \\ \sigma_+^\mu & 0 \end{pmatrix} \tag{3.75}$$

$$\gamma^5 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} , \tag{3.76}$$

where σ_i $(i = 1, 2, 3)$ are the Pauli matrices. With

$$\Psi = \begin{pmatrix} \chi \\ \varphi \end{pmatrix} \quad \text{and} \quad \bar{\Psi} = \Psi^\dagger \gamma^0 = (\chi^\dagger, \varphi^\dagger) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (\varphi^\dagger, \chi^\dagger) \tag{3.77}$$

we get

$$\bar{\Psi} i\gamma^\mu D_\mu \Psi = i(\varphi^\dagger, \chi^\dagger) \underbrace{\begin{pmatrix} 0 & \sigma_-^\mu \\ \sigma_+^\mu & 0 \end{pmatrix} \begin{pmatrix} D_\mu \chi \\ D_\mu \varphi \end{pmatrix}}_{\begin{pmatrix} \sigma_-^\mu D_\mu \varphi \\ \sigma_+^\mu D_\mu \chi \end{pmatrix}} = \varphi^\dagger i \sigma_-^\mu D_\mu \varphi + \chi^\dagger i \sigma_+^\mu D_\mu \chi \; . \tag{3.78}$$

The gauge interaction holds independently bouth for

$$\Psi_L = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \frac{1}{2}(1 - \gamma_5)\Psi \quad \text{and} \quad \Psi_R = \begin{pmatrix} \chi \\ 0 \end{pmatrix} = \frac{1}{2}(1 + \gamma_5)\Psi \; . \tag{3.79}$$

The Ψ_L and Ψ_R can have different gauge representations. But

$$m\bar{\Psi}\Psi = m(\varphi^\dagger, \chi^\dagger) \begin{pmatrix} \chi \\ \varphi \end{pmatrix} = m(\varphi^\dagger \chi + \chi^\dagger \varphi) = m(\bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L) \; . \tag{3.80}$$

The mass term mixes Ψ_L and Ψ_R. This implies *symmetry breaking* if Ψ_L and Ψ_R have different representations.

What about a mass term for gauge bosons? Let us look at

$$\mathcal{L} = -\frac{1}{4} \underbrace{F^{a\mu\nu} F^a_{\mu\nu}}_{\text{gauge invariant}} + \frac{m^2}{2} \underbrace{A^{a\mu} A^a_\mu}_{\text{not gauge invariant}} \; . \tag{3.81}$$

For example for the $U(1)$

$$(A_\mu A^\mu)' = (A_\mu + \partial_\mu \theta)(A^\mu + \partial^\mu \theta) = A_\mu A^\mu + 2A_\mu \partial^\mu \theta + (\partial_\mu \theta)(\partial^\mu \theta) \; . \tag{3.82}$$

The mass term for A^μ breaks the gauge symmetry.

Chapter 4

Path Integrals

Theoretically, processes in the world of elementary particles can be formulated through two approaches. We have introduced in TTP1 the *canonical quantisation*. In this approach, fields become operators and have a Fourier representation with creation and annihilation operators. The particle interpretation as excitations of fields is obvious in this formalism. The canonical formalism is based on quantum mechanics. It is suitable to show basic properties of the fields like e.g. the spin-statistics theorem. The computation of interaction amplitudes and the quantisation of non-Abelian gauge theories is rather complicated, however.

Starting point in the *path integral formalism* is the principle of minimal action. The integration is performed over all possible field configurations. The fields are here functions and no operators. The contributions that do not cancel each other, stem for weak couplings from paths near the minima of the action. The computation of the interaction amplitudes is relatively simple in this formalism and the symmetries of the fields are obvious. The path integral formalism is closer to the wave character of the elementary particles. The convergence of the path integrals is not proven in a mathematically strict way, however.

4.1 Path Integrals in Quantum Mechanics

The Lagrangian function L is the fundamental object in classical mechanics. It is the starting point for the construction of the classical action,

$$S \equiv \int_{t_1}^{t_2} dt L(q, \dot{q}) \; , \tag{4.1}$$

where $q(t)$ is the generalised coordinate and $\dot{q}(t) \equiv dq/dt$ is the generalised velocity. The equations of motion follow from the Hamilton principle of the minimal action which means that the variation

$$\delta S = \delta \int_{t_1}^{t_2} dt L(q, \dot{q}) = 0 \tag{4.2}$$

vanishes considering the additional conditions that the variatons of the generalised coordinates at the endpoints t_1 and t_2 vanish. The physical path is hence that specific trajectory (cf. Fig. 4.1), which connects the $q_1 \equiv q(t_1)$ and $q_2 \equiv q(t_2)$ and along which the action is stationary. An important generalisation to quantum mechanics as weighted sum over the paths has been developed by Feynman. We have in quantum mechanics

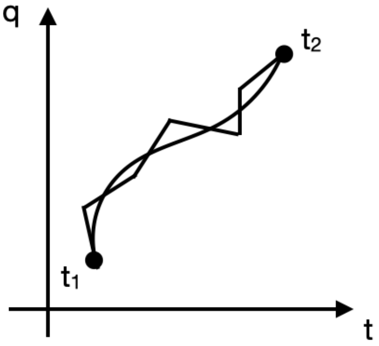


Figure 4.1: A possible trajectory between the fixed starting and end points $q(t_1)$ and $q(t_2)$.

$$< t_2|t_1 > \sim \int Dq \exp iS \tag{4.3}$$

In quantum field theory, the theory is defined by the path integral

$$\boxed{\begin{aligned} W &\sim \int D\phi\, e^{iS} && (4.4) \\ S &= \int d^4x\, \mathcal{L} && (4.5) \end{aligned}}$$

The integral is performed over all field values ϕ at each point x,

$$W \sim \lim_{\epsilon\to 0} \int d\Pi_\alpha d\phi_\alpha \exp\left\{ i \sum_\beta \epsilon^4 \mathcal{L}(\phi_\beta) \right\} \tag{4.6}$$

Quantum mechanics: Without restriction of generality, we discuss quantum mechanics in one space dimension. Be q the space coordinate. A state in the Schrödinger picture is connected to a state in the Heisenberg picture through

$$|\psi, t >_S = \exp\left(-\frac{i}{\hbar}\hat{H}t\right) |\psi >_H \quad . \tag{4.7}$$

The space operator in the Heisenberg picture is related to the one in the Schrödinger picture as

$$\hat{Q}_H(t) = e^{i\hat{H}t}\hat{Q}_S e^{-i\hat{H}t} \ . \tag{4.8}$$

We define:

$$|q, t >= \exp\left(\frac{i}{\hbar}\hat{H}t\right) |q > \quad . \tag{4.9}$$

Thereby

$$\psi(q,t) =< q|\psi, t >_S =< q| \exp^{\left(-\frac{i}{\hbar}\hat{H}t\right)} |\psi >_H =< q, t|\psi >_H \quad . \tag{4.10}$$

We are interested in the state at the position q_f at the time t_f,

$$\begin{aligned}\psi(q_f,t_f) &= <q_f,t_f|\psi>_H = \int dq_i <q_f,t_f|q_i,t_i><q_i,t_i|\psi>_H \\ &= \int dq_i K(q_f,t_f;q_i,t_i)\psi(q_i,t_i) \ .\end{aligned} \tag{4.11}$$

The whole information on the dynamics of the system is in the integrand $K(q_f,t_f;q_i,t_i)$. It is called *propagator*. We now look at the transition matrix element (in the following, we set $\hbar = 1$)

$$<q',t'|q,t> = <q'|e^{-i\hat{H}(t'-t)}|q> \ . \tag{4.12}$$

We devide it up into $(n+1)$ partial intervals $\tau = (t'-t)/(n+1)$ (cf. Fig. 4.2).

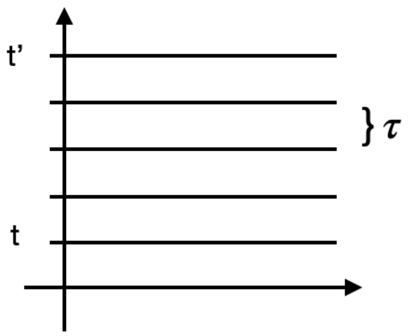


Figure 4.2: Partial intervals.

We use the completeness relation $1 = \int dq|q><q|$ and obtain for Eq. (4.12)

$$\int dq_n ... dq_1 <q',t'|q_n,t_n><q_n,t_n|q_{n-1},t_{n-1}> ... <q_1,t_1|q,t> \ . \tag{4.13}$$

We look more closely at the matrix element

$$\begin{aligned}<q_{j+1},t_{j+1}|q_j,t_j> &= <q_{j+1}|e^{-i\hat{H}\tau}|q_j> = <q_{j+1}|1-i\hat{H}\tau+\mathcal{O}(\tau^2)|q_j> \\ &= \delta(q_{j+1}-q_j) - i\tau <q_{j+1}|\hat{H}|q_j> + \mathcal{O}(\tau^2) \ .\end{aligned} \tag{4.14}$$

With the Hamilton operator $\hat{H} = \hat{P}^2/(2m) + V(\hat{Q})$ we obtain

$$\begin{aligned}<q_{j+1}|\frac{\hat{P}^2}{2m}|q_j> &= \int dp dp' <q_{j+1}|p'><p'|\frac{\hat{P}^2}{2m}|p><p|q_j> \\ &= \int dp dp' \frac{1}{\sqrt{2\pi}^2} e^{i(p'q_{j+1}-pq_j)} \frac{p^2}{2m}\delta(p-p') = \int \frac{dp}{2\pi} e^{ip(q_{j+1}-q_j)} \frac{p^2}{2m} \ .\end{aligned} \tag{4.15}$$

And

$$<q_{j+1}|V(\hat{Q})|q_j> = V(q_j) <q_{j+1}|q_j> = V(q_j)\delta(q_{j+1}-q_j) = \int \frac{dp}{2\pi} e^{ip(q_{j+1}-q_j)} V(q_j) \ . \tag{4.16}$$

Thereby, we obtain for the matrix element

$$< q_{j+1}|\hat{H}|q_j >= \int \frac{dp}{2\pi} e^{ip(q_{j+1}-q_j)} H(p, q_j) \ . \tag{4.17}$$

And finally

$$< q_{j+1}, t_{j+1}|q_j, t_j >= \int \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-i\tau H(p_j,q_j)} \ . \tag{4.18}$$

Thereby, we have for the transition matrix element (4.12)

$$< q', t'|q, t >= \lim_{n\to\infty} \int \Pi_{j=0}^n \frac{dp_j}{2\pi} \left(\Pi_{j=1}^n dq_j\right) e^{i\sum_{j=0}^n [p_j(q_{j+1}-q_j)-\tau H(p_j,q_j)]} \ . \tag{4.19}$$

With the symbolic notation

$$\lim_{n\to\infty} \int \Pi_{a=1}^n dq_a \Pi_{b=0}^n \frac{dp_b}{2\pi} = \int \mathcal{D}q(t)\,\mathcal{D}p(t) \ , \tag{4.20}$$

which is the definition of the functional integral (=”integration over functions”), we have

$$< q', t_f|q, t_i >= \int \mathcal{D}q(t)\,\mathcal{D}p(t)\, e^{i\int_{t_i}^{t_f} dt[p\dot{q}-H(p,q)]} \ , \tag{4.21}$$

where $q(t_i) = q$ and $q(t_f) = q'$. The quantum mechanical transition matrix element is given by the ∞-dimensional integral over the classical “paths”. For $H = \frac{p^2}{2m} + V(q)$, the integration over p can be performed. With the formula

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2+bx+c} = e^{\frac{b^2}{4a}+c}\sqrt{\frac{\pi}{a}} \tag{4.22}$$

we obtain from Eq. (4.19)

$$< q', t_f|q, t_i >= \lim_{n\to\infty} \left(\frac{1}{2\pi}\right)^{n+1} \left(\frac{2\pi m}{i\tau}\right)^{\frac{n+1}{2}} \int \Pi_{j=1}^n dq_j\, e^{i\sum_j[\tau\frac{m}{2}\left(\frac{q_{j+1}-q_j}{\tau}\right)^2 - V\tau]} \ . \tag{4.23}$$

In the continuum limit we obtain

$$< q', t_f|q, t_i >= \mathcal{N} \int \mathcal{D}q\, e^{i\int_{t_i}^{t_f} dt\, L(q,\dot{q})} \ , \tag{4.24}$$

where $\mathcal{N}$ is a normalization factor. In an analogous calculation we obain for

$$< q', t'|\hat{Q}(t_0)|q, t >= \int \mathcal{D}q\,\mathcal{D}p\, q(t_0) e^{i\int_t^{t'} d\tau[p\dot{q}-H]} \ . \tag{4.25}$$

Because

$$\begin{aligned}
< q_j, t_j|q_j|q_{j-1}, t_{j-1} > &= < q_j|q_j \exp(-i\hat{H}(t_j - t_{j-1}))|q_{j-1} > \\
&= < q_j|\hat{Q} \exp(-i\hat{H}(t_j - t_{j-1}))|q_{j-1} > \\
&= < q_j| \exp(-i\hat{H}t_j) \exp(i\hat{H}t_j)\hat{Q} \exp(-i\hat{H}(t_j - t_{j-1}))|q_{j-1} > \\
&= < q_j, t_j|\hat{Q}(t_j)|q_{j-1}, t_{j-1} > \ .
\end{aligned} \tag{4.26}$$

Let us look at $A = < q', t' | \hat{Q}(t_a) \hat{Q}(t_b) | q, t >$. <u>If $t_a > t_b$</u>:

$$\begin{aligned} A &= \int (\Pi dq_j) < q', t' | q_n, t_n >< q_n, t_n | q_{n-1}, t_{n-1} > ... < q_{j_1}, t_{j_1} | \hat{Q}(t_a) | q_{j_1 - 1}, t_{j_1 - 1} > \\ & ... < q_{j_2}, t_{j_2} | \hat{Q}(t_b) | q_{j_2 - 1}, t_{j_2 - 1} > ... < q_1, t_1 | q, t > \quad . \end{aligned} \tag{4.27}$$

And we obtain (calculation as above)

$$A = \underbrace{\int \mathcal{D}q \, \mathcal{D}p \, q(t_a) q(t_b) \, e^{i \int_t^{t'} d\tau [p\dot{q} - H]}}_{P} \quad . \tag{4.28}$$

<u>If $t_b > t_a$</u>, P now is

$$P = < q', t' | \hat{Q}(t_b) \hat{Q}(t_a) | q, t > \quad . \tag{4.29}$$

The path integral formula hence corresponds to the matrix element of the time-ordered product ($a \equiv 1$, $b \equiv 2$) $T[\hat{Q}(t_1)\hat{Q}(t_2)]$. We then have in general

$$< q', t' | T[\hat{Q}(t_1) ... \hat{Q}(t_N)] | q, t > = \int \mathcal{D}q \, \mathcal{D}p \; q(t_1) \, q(t_2) ... q(t_N) \, e^{i \int_t^{t'} d\tau [p\dot{q} - H]} \quad . \tag{4.30}$$

We now want to have the vacuum-to-vacuum amplitude in the presence of an external "source" J (cf. Fig. 4.3), which describes the creation and annihilation of particles. We replace the Lagrangian density by

$$L \to L + \hbar J(t) \, q(t) \qquad (\hbar = 1) \; , \tag{4.31}$$

where J denotes the source. Our goal is to obtain $< 0, t = \infty | 0, t = -\infty >_J$.

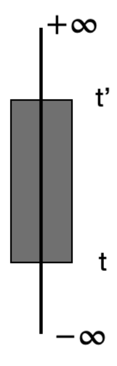


Figure 4.3: External source J turned on between t and t'.

We have the path integral formula

$$\begin{aligned} < Q', T' | Q, T >_J &= \mathcal{N} \int \mathcal{D}q \, e^{i \int_T^{T'} dt [L + Jq]} \\ &= \int dq \, dq' \, < Q', T' | q', t' >_{J=0} < q', t' | q, t >_{J \neq 0} < q, t | Q, T >_{J=0} \end{aligned} \tag{4.32}$$

$\mathcal{N}$ is a normalisation factor, which comes from the p integration. We have

$$< Q', T'|q', t' >_{J=0} = < Q'|e^{-i\hat{H}T'} e^{+i\hat{H}t'}|q' > = \sum_m \phi_m(Q')\phi_m^*(q')e^{-iE_m(T'-t')} , \tag{4.33}$$

where we have inserted $1 = \sum_E |\text{Energie} >< \text{Energie}|$. And analogously

$$< q, t|Q, T >_{J=0} = \sum_n \phi_n(q)\phi_n^*(Q)e^{iE_n(T-t)} . \tag{4.34}$$

We use a trick and rotate the time axis a little bit, $t \to te^{-i\delta}$, and take the limit $T' \to \infty e^{-i\delta}$ and $T \to -\infty e^{-i\delta}$. This means that in the sum only the contributions of the ground state $E_0 < E_i$ are left over so that we obtain

$$\begin{aligned} \lim_{\substack{T' \to +\infty e^{-i\delta} \\ T \to -\infty e^{-i\delta}}} < Q', T'|Q, T >_J &= \phi_0^*(Q)\phi_0(Q')e^{-iE_0(T'-T)} \\ &\int dq\, dq' \phi_0^*(q', t')\phi_0(q, t) < q', t'|q, t >_J . \end{aligned} \tag{4.35}$$

This integral is equivalent to $< 0, t'|0, t >_J$. We now also take the limits $t' \to \infty$, $t \to -\infty$ and obtain

$$\boxed{\begin{aligned} < 0, \infty|0, -\infty >_J &= \lim_{\substack{T' \to +\infty e^{-i\delta} \\ T \to -\infty e^{-i\delta}}} \frac{< Q', T'|Q, T >_J}{\phi_0^*(Q)\phi_0(Q')e^{-iE_0(T'-T)}} \\ \text{with} \quad & < Q', T'|Q, T >_J = \mathcal{N} \int \mathcal{D}q\, e^{i\int_T^{T'} dt\, [L(q,\dot{q}) + J(t)q(t)]} \qquad (4.36) \end{aligned}}$$

Remark: Instead of rotating the t-axis the contribution of the ground state can also be isolated through $H \to H - i\frac{\epsilon}{2}q^2$, $\epsilon > 0, \epsilon \to 0$, *i.e.* $L \to L + \frac{i}{2}\epsilon q^2$. Thereby, we finally have

$$\boxed{\begin{aligned} < 0, \infty|0, -\infty >_J \;&\sim\; \mathcal{Z}[J] \qquad \text{generating functional} \\ \text{where} \qquad & \mathcal{Z}[J] \equiv \int \mathcal{D}q\, e^{i\int_{-\infty}^{\infty} dt\, [L + J\cdot q + \frac{i}{2}\epsilon q^2]} \qquad (4.37) \end{aligned}}$$

We define the functional derivative for a functional $F[f]$, which maps $C^n(M) \to \mathbb{C}$, where the space of the functions is given by $M = \mathbb{R}, \mathbb{C}$:

$$\frac{\delta F[f(x)]}{\delta f(y)} \equiv \lim_{\epsilon \to 0} \frac{F[f(x) + \epsilon\delta(x - y)] - F[f(x)]}{\epsilon} . \tag{4.38}$$

We then have e.g.

$$\frac{\delta f(x)}{\delta f(y)} = \delta(x - y) . \tag{4.39}$$

For the nth derivative of $\mathcal{Z}[J]$ with respect to J we obtain

$$\boxed{\begin{aligned} \left.\frac{\delta^n \mathcal{Z}[J]}{\delta J(t_1)...\delta J(t_n)}\right|_{J=0} &= (i)^n \int \mathcal{D}q\, q(t_1)...q(t_n)e^{i\int_{-\infty}^{\infty} dt\, [L + \frac{i}{2}\epsilon q^2]} \\ &\sim (i)^n < 0|T[\hat{Q}(t_1)...\hat{Q}(t_n)]|0 > \qquad (4.40) \end{aligned}}$$

4.2 Scalar Fields

A 1-dimensional quantum-mechanical system corresponds to a field theory in 0 space dimensions. (In quantum mechanics we have the operators $\hat{Q}(t), \hat{P}(t), \hat{H}(t)$.)

In scalar field theory we have the scalar fields $\hat{\phi}(\vec{x},t)$ and the momentum $\hat{\Pi}(\vec{x},t)$ and the Hamilton and Lagrangian densities $\hat{\mathcal{H}}(\hat{\phi},\hat{\Pi}), \hat{\mathcal{L}}(\hat{\phi},\partial_\mu\hat{\phi})$. We take as an example the classical Lagrangian of the ϕ^4 theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \; . \tag{4.41}$$

The vacuum-to-vacuum amplitude of the quantum field theory with presence of a "source" $J(x)$ is given by

$$< 0,\infty|0,-\infty >_J \sim \mathcal{Z}[J] \tag{4.42}$$

with the generating functional

$$\mathcal{Z}[J] = \int \mathcal{D}\phi \, e^{i\int_{-\infty}^{\infty} d^4x \, [\mathcal{L}(x) + \frac{1}{2}i\epsilon\phi^2(x) + J(x)\phi(x)]} \; . \tag{4.43}$$

The fields ϕ are classical fields. And

$$\mathcal{D}\phi = \Pi_{\vec{x},t} d\phi(\vec{x},t) \; . \tag{4.44}$$

In order to define the functional integrals, we discretise space and time.
We have the partial derivative

$$\left.\frac{\partial\phi}{\partial x}\right|_{(\vec{x}_n,t_n)} = \lim_{a\to 0} \frac{\phi(x_n+a,y_n,z_n,t_n) - \phi(x_n,y_n,z_n,t_n)}{a} \; . \tag{4.45}$$

For the integral over the Lagrangian density we obtain

$$\int d^4x \, \mathcal{L}(\phi) \approx \sum_{\text{lattice points}} a^4 \, \mathcal{L}(\phi(\vec{x}_n,t_n)) \; . \tag{4.46}$$

The measure $\Pi_x d\phi(x)$ becomes

$$\mathcal{D}\phi = \Pi_{j=1}^{N^4} d\phi(x_j) \; . \tag{4.47}$$

The functional integral $\mathcal{Z}[J]$ then becomes an N^4-dimensional, hence a usual finite-dimensional integral, which can be determined (e.g. on a computer) $\rightarrow$ numerical simulations of lattice field theory. Finally, we perform the limit

$$\begin{array}{lll} \text{a)} & L \to \infty, \quad i.e.\ N = \dfrac{L}{a} \to \infty & \text{infinite-volume limit} \\ \text{b)} & a \to 0 & \text{continuum limit} \; . \end{array} \tag{4.48}$$

We obtain the Green function of the scalar quantum field theory through the functional derivative:

$$\boxed{\begin{aligned} \frac{1}{i^n} \left.\frac{\delta^n \mathcal{Z}[J]}{\delta J(x_1)...\delta J(x_n)}\right|_{J=0} &= \frac{1}{\mathcal{N}} \int \mathcal{D}\phi \, \phi(x_1)...\phi(x_n) e^{i\int d^4x \, [\mathcal{L} + \frac{i\epsilon}{2}\phi^2]} \\ &= < 0,\infty|T[\hat{\phi}(x_1)...\hat{\phi}(x_n)]|0,-\infty >_{J=0} \quad (4.49) \end{aligned}}$$

with

$$\mathcal{N} = \int \mathcal{D}\phi \, e^{i\int d^4x\,[\mathcal{L}+\frac{i\epsilon}{2}\phi^2]} \; . \tag{4.50}$$

In order to demonstrate (4.49), we first need some formulae for the functional integration over c-number functions. We consider first integrals over finite dimensions:
1) 1-dimensional Gauß formula, $x\epsilon\mathbb{R}$:

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2}\, dx = \left(\frac{2\pi}{a}\right)^{1/2} \; . \tag{4.51}$$

2) Real integral over n dimensions:

$$\int \Pi_{j=1}^n dx_j \, e^{-\frac{1}{2}\sum_{k=1}^n a_k\, x_k^2} = \frac{(2\pi)^{n/2}}{\sqrt{\Pi_k a_k}} \; . \tag{4.52}$$

Be

$$A = \begin{pmatrix} a_1 & & & & \\ & a_2 & & & \\ & & . & & \\ & & & . & \\ & & & & a_n \end{pmatrix} , \tag{4.53}$$

and we use for the scalar product the notation

$$\sum_k a_k x_k^2 = (\vec{x}, A\vec{x}) \qquad \vec{x} = \begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix} \; . \tag{4.54}$$

We define the measure

$$(dx) \equiv \frac{d^n x}{(2\pi)^{n/2}} \; . \tag{4.55}$$

Then we have

$$\boxed{\int e^{-\frac{1}{2}(\vec{x},A\vec{x})}(dx) = \frac{1}{\sqrt{\det A}} \; . \qquad (4.56)}$$

The formula holds for any real, symmetric and positive matrix A.

3) Generalisation to arbitrary quadratic forms of the type

$$Q(\vec{x}) = \frac{1}{2}(\vec{x}, A\vec{x}) + (\vec{b}, \vec{x}) + c \; , \tag{4.57}$$

where A is a positive matrix. Q can be written as

$$Q = Q(\vec{x}_0) + \frac{1}{2}(\vec{x} - \vec{x}_0, A(\vec{x} - \vec{x}_0)) \tag{4.58}$$

with $\vec{x}_0 = -A^{-1}\vec{b}$. Thereby we have

$$\boxed{\int_{-\infty}^{\infty} e^{-\left[\frac{1}{2}(\vec{x},A\vec{x})+(\vec{b},\vec{x})+c\right]}(dx) = \frac{e^{\frac{1}{2}(\vec{b},A^{-1}\vec{b})-c}}{\sqrt{\det A}} \ .} \tag{4.59}$$

4) Complex variables. Be $z = x + iy \,\epsilon\, \mathbb{C}$ and $z^* = x - iy$. Thereby $dx\,dy = -\frac{1}{2}i\,dz\,dz^*$. Using

$$\int e^{-a(x^2+y^2)}dx\,dy = \frac{2\pi}{2a} \tag{4.60}$$

we obtain

$$\int e^{-az^*z}\underbrace{\frac{dz^*}{\sqrt{2\pi\,i}}\frac{dz}{\sqrt{2\pi\,i}}}_{\equiv(dz^*)(dz)} = \frac{1}{a} \ . \tag{4.61}$$

With n complex variables $\vec{z}$, A as positive definite Hermitian matrix and the definition of the measure $(dz) \equiv \frac{d^n z}{(2\pi\,i)^{n/2}}$, analogously (dz^*), we have

$$\boxed{\int (dz^*)\,(dz)\,e^{-(\vec{z}^*,A\vec{z})} = \frac{1}{\det A} \ .} \tag{4.62}$$

We generalise the equations (4.56),(4.59),(4.62) to infinite-dimensional functional integrals

$$\vec{x} = (x_i)\,\epsilon\,\mathbb{R}^n \to \phi(x)\,\epsilon\,\mathcal{F}(M_4) \ , \tag{4.63}$$

x is a continuous index, ϕ a real function. The scalar product is defined as

$$(\phi_1,\phi_2) = \int d^4x\,\phi_1(x)\,\phi_2(x). \tag{4.64}$$

The generalisation of Eq. (4.56) is

$$\boxed{\int \Pi_x\left(\frac{d\phi(x)}{\sqrt{2\pi}}\right)e^{-\frac{1}{2}\int d^4y\,\phi(y)\,A\,\phi(y)} = \frac{1}{\sqrt{\det A}} \ ,} \tag{4.65}$$

where A is a positive operator and ϕ a real function. If $\phi(x)$ is a complex function, then

$$\boxed{\int \left(\Pi_x\frac{d\phi^*(x)}{\sqrt{2\pi i}}\frac{d\phi(x)}{\sqrt{2\pi i}}\right)e^{-\int d^4y\,\phi^*(y)\,A\,\phi(y)} = \frac{1}{\det A} \ ,} \tag{4.66}$$

The generalisation of Eqs. (4.56),(4.59),(4.62) is, written up more precisely, in case of complex fields (analogously for real fields)

$$\boxed{\int \left(\Pi_x\frac{d\phi^*(x)}{\sqrt{2\pi i}}\frac{d\phi(x)}{\sqrt{2\pi i}}\right)e^{-\int d^4x_1\,d^4x_2\,\phi^*(x_1)\,A(x_1,x_2)\,\phi(x_2)} = \frac{1}{\det A} \ ,} \tag{4.67}$$

where $A(x_1, x_2)$ is a positive operator, which is independent of ϕ.

We now apply this to the real scalar field theory. Be the classical Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2)\ . \tag{4.68}$$

The normalised generating functional is

$$\mathcal{Z}_0[J] = \frac{1}{\mathcal{N}}\int \mathcal{D}\phi\, e^{i\int d^4x\,[\mathcal{L}_0 + \frac{i\epsilon}{2}\phi^2 + J\phi]}\ . \tag{4.69}$$

The exponent is

$$\int d^4x\,[-\frac{i}{2}\phi(\Box + m^2 - i\epsilon)\phi + iJ\phi] + \underbrace{\frac{i}{2}\int_{\text{Border of }M_4} dn_\mu\phi\,\partial^\mu\phi}_{=0\,\text{if}\,\phi(x)\text{ decreases fast enough.}}\ . \tag{4.70}$$

We use Eq. (4.59), generalised to functional integrals: Set $A = i(\Box + m^2 - i\epsilon)$, $b = -iJ$, $c = 0 \Rightarrow$

$$\mathcal{Z}_0[J] = \frac{1}{\mathcal{N}} e^{\frac{i}{2}\int J(x)(\Box+m^2-i\epsilon)^{-1}J(y)d^4x\,d^4y}\cdot\underbrace{[\det\ i(\Box + m^2 - i\epsilon)]^{-1/2}}_{\int \Pi_x \frac{d\phi(x)}{\sqrt{2\pi}} e^{-\frac{i}{2}\int d^4x\,\phi(\Box+m^2-i\epsilon)\phi}} \tag{4.71}$$

Remarks:
1) The factors $1/\sqrt{2\pi}$ in the nominator and $\mathcal{N}$ cancel each other.
2) Since $\mathcal{Z}_0[0] = 1$, we have $\mathcal{N} = [\det\ i(\Box + m^2 - i\epsilon)]^{-1/2}$.
3) The inverse of the differential operator $(\Box + m^2 - i\epsilon)$ is

$$(\Box + m^2 - i\epsilon)^{-1} = -\Delta_F(x-y)\ , \tag{4.72}$$

where Δ_F is the Feynman propagator (=causal 2-point Green function), which is defined as

$$(\Box_x + m^2 - i\epsilon)\Delta_F(x-y) = -\delta^4(x-y)\ . \tag{4.73}$$

Hence

$$\Delta_F(x-y) = \lim_{\epsilon\to 0}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2 - m^2 + i\epsilon}e^{-ik\cdot(x-y)}\ . \tag{4.74}$$

Thereby we finally have

$$\boxed{\mathcal{Z}_0[J] = e^{-\frac{i}{2}\int J(x)\,\Delta_F(x-y)\,J(y)d^4x d^4y}\ .} \tag{4.75}$$

Thus, for example the 2-point Green function is

$$\begin{aligned}
< 0|T[\hat{\phi}(x)\,\hat{\phi}(y)]|0 > &= \frac{1}{i^2}\left.\frac{\delta^2\mathcal{Z}_0[J]}{\delta J(x)\,\delta J(y)}\right|_{J=0} \\
= \left\{\frac{1}{i^2}\frac{\delta}{\delta J(x)}\left[\left(\frac{-i}{2}\int d^4x_2\,\Delta_F(y-x_2)\,J(x_2) - \frac{i}{2}\int d^4x_1\,J(x_1)\,\Delta_F(x_1-y)\right)e^{-\frac{i}{2}\int\cdots}\right]\right\}_{J=0} \\
&= \frac{-1}{i}\Delta_F(x-y) = i\Delta_F(x-y)\ ,
\end{aligned} \tag{4.76}$$

where we used $\Delta_F(x-y) = \Delta_F(y-x)$.

4.3 Grassmann Variables

In the following, we will treat anti-commuting fields in the path integral formalism. For this we need "anti-commuting numbers". These are called Grassmann variables. We start by looking at their properties, before we use them.

Usual numbers:	x_i	with	$[x_i, x_j] = 0$ commuting
Grassmann numbers:	η_i	with	$\{\eta_i, \eta_j\} = 0$ anti-commuting

The Grassmann numbers are hence defined through the algebra $\{\eta_i, \eta_j\} = \eta_i\eta_j + \eta_j\eta_i = 0$ for all i, j. This leads to the nil-potence of the Grassmann variables,

$\boxed{\eta_i^2 = 0}$

Properties:

1) Functions $f(\eta_i)$ of the Grassmann variables

Be f an analytic function, then the Taylor expansion of $f(\eta_i)$ only contains a finite number of terms. For example

$$f(\eta) \;=\; f_0 + f_1\eta \qquad \text{as } \eta^2 = 0. \qquad f(\eta_1, \eta_2) = f_0 + f_1\eta_1 + f_2\eta_2 + f_{12}\eta_1\eta_2 \tag{4.77}$$

2) Derivatives

The derivative (=left-derivative) of a Grassmann variable is defined through

$$\boxed{\frac{\partial}{\partial\eta_i}\eta_j = \delta_{ij}\;, \quad \frac{\partial}{\partial\eta_i}a = 0 \qquad \text{where } a \text{ is a } c \text{ number.}} \tag{4.78}$$

Remark: The derivative operators are anti-commuting among each other and with Grassman variables $(\partial/\partial\eta_j,\, \eta_j)$. For example

$$\frac{\partial}{\partial\eta_i}(\eta_1\eta_2) = \delta_{i1}\eta_2 - \delta_{i2}\eta_1\;. \tag{4.79}$$

Remark: Sometimes it is also useful to define a right-derivative.

$$\frac{\partial^R}{\partial\eta_i}(\eta_1\eta_2) = (\eta_1\eta_2)\frac{\overleftarrow{\partial}}{\partial\eta_i} = \eta_1\delta_{i2} - \eta_2\delta_{i1}\;. \tag{4.80}$$

Since the derivative operator itself is anti-commuting, we have

$$\left\{\frac{\partial}{\partial\eta_i}, \frac{\partial}{\partial\eta_j}\right\} = 0 \Rightarrow \frac{\partial^2}{\partial\eta_i^2} = \left(\frac{\partial}{\partial\eta_i}\right)^2 = 0\;, \tag{4.81}$$

which means that the derivatives are nil-potent, just like the η_i. This implies that the integral over Grassmann variables cannot be defined as the inverse of the derivative, as the derivative does not have an inverse.

3) Integration

The integral is defined such that it delivers the same as the derivative.

a) 1 Grassmann variable η

Be f an analytic function of η, $f(\eta) = a_0 + a_1\eta$. Then we have $\frac{d}{d\eta}f = a_1$ and $\frac{d^2}{d\eta^2}f(\eta) = 0$. The integration rules are hence given by

$$\boxed{\begin{aligned} \int d\eta\ a &= 0, \qquad \text{for a } c\text{-number } a \\ \int d\eta\ a\eta &= a \\ \Rightarrow \int d\eta f(\eta) &= \int d\eta(a_0 + a_1\eta) = a_1 \end{aligned} \qquad (4.82)}$$

b) n Variables η_i

$$\boxed{\int d\eta_j = 0\ , \qquad \int d\eta_j\ \eta_i = \delta_{ij} \qquad (4.83)}$$

c) Be $\eta, \bar{\eta}$ independent Grassmann variables, i.e.

$$\begin{aligned} \int d\eta &= \int d\bar{\eta} = 0 \\ \int d\eta\eta &= \int d\bar{\eta}\bar{\eta} = 1\ . \end{aligned} \tag{4.84}$$

We have

$$\begin{aligned} e^{-\bar{\eta}\eta} &= 1 - \bar{\eta}\eta + \underbrace{\frac{(\bar{\eta}\eta)^2}{2} + \ldots}_{0,\text{ as } \eta^2 = \bar{\eta}^2 = 0} = 1 - \bar{\eta}\eta \\ \Rightarrow &\quad \int d\bar{\eta}\ d\eta\ e^{-\bar{\eta}\eta} = \int d\bar{\eta}\ d\eta\ - \int d\bar{\eta}\ d\eta\ \bar{\eta}\eta = 0 + \int d\bar{\eta}\ \bar{\eta} \int d\eta\ \eta = +1\ . \end{aligned} \tag{4.85}$$

d) Generalisation to several variables
Be

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \qquad \bar{\eta} = \begin{pmatrix} \bar{\eta}_1 \\ \bar{\eta}_2 \end{pmatrix} \qquad \text{and} \quad \bar{\eta}\eta = \bar{\eta}_1\eta_1 + \bar{\eta}_2\eta_2 \tag{4.86}$$

then

$$(\bar{\eta}\eta)^2 = 2\bar{\eta}_1\eta_1\bar{\eta}_2\eta_2 \tag{4.87}$$

and higher powers

$$(\bar{\eta}\eta)^P = 0 \qquad \text{for } p \geq 3 \tag{4.88}$$

Hence

$$e^{-\bar{\eta}\eta} = 1 - (\bar{\eta}_1\eta_1 + \bar{\eta}_2\eta_2) + \bar{\eta}_1\eta_1\bar{\eta}_2\eta_2\ . \tag{4.89}$$

With the definition

$$d\bar{\eta}d\eta \equiv d\bar{\eta}_1 d\eta_1 d\bar{\eta}_2 d\eta_2 \tag{4.90}$$

we find

$$\int d\bar{\eta}\ d\eta\ e^{-\bar{\eta}\eta} = 0 + \int d\bar{\eta}_1\ d\eta_1\ d\bar{\eta}_2\ d\eta_2\ \bar{\eta}_1\eta_1\bar{\eta}_2\eta_2 = +1\ . \tag{4.91}$$

Change of variables
Sei

$$\eta = Bc \qquad \text{and} \qquad \bar{\eta} = \bar{c}H \tag{4.92}$$

where B, H are 2×2 c-number matrices, with $\det H \neq 0$, $\det B \neq 0$. (η, c are Grassmann numbers.) We have

$$\eta_1\eta_2 = (B_{11}c_1 + B_{12}c_2)(B_{21}c_1 + B_{22}c_2) = (B_{11}B_{22} - B_{21}B_{12})c_1c_2 = \det B \; c_1c_2 \tag{4.93}$$

We have to demand that

$$d\eta_1 d\eta_2 = (\det B)^{-1} dc_1 dc_2 \tag{4.94}$$

so that the integration rule

$$\int d\eta_1 d\eta_2 \; \eta_1\eta_2 = \int dc_1 dc_2 \; c_1c_2 \tag{4.95}$$

is preserved. Thereby we find

$$(\det(BH))^{-1} \int d\bar{c}dc \; e^{-\bar{c}HBc} = 1 \tag{4.96}$$

Be $A = HB$. Then $\det A = \det HB = \det BH$ and thereby

$$\boxed{\int d\bar{c} \; dc \; e^{-\bar{c}Ac} = \det A \qquad (4.97)}$$

This can immediately be generalised to $2n$ variables $c_j, \bar{c}_j$.

$$\boxed{\begin{array}{rcl} \int d\bar{c} \; dc & & e^{-\sum_{ij}\bar{c}_i A_{ij} c_j} = \det A \\ d\bar{c} \; dc & \equiv & d\bar{c}_1 \; dc_1 \; ... \; d\bar{c}_n \; dc_n \qquad (4.98) \end{array}}$$

4.4 Gauge Fixing

In the following we consider the gauge group $SU(N)$ with the gauge fields $A^a_\mu(x)$, $a = 1, ..., N^2 - 1$. With respect to the gauge invariance there is a problem in the path integral formulation: We look at the gauge field $\bar{A}_\mu(x) = \bar{A}^a_\mu(x)T^a$. The so-called orbit $\bar{A}_\mu$ is defined as the set of functions $\{\bar{A}^U_\mu\}$ with

$$\bar{A}^U_\mu = U\bar{A}_\mu U^{-1} - \frac{i}{g} U\partial_\mu U^{-1} \; , \tag{4.99}$$

where

$$U(x) = \exp\{i\omega_a(x)T^a\} \quad \epsilon \; SU(N) \; . \tag{4.100}$$

This means that the whole gauge field space can be decomposed into equivalence classes $\{\bar{A}^U_\mu\}$, with $\bar{A}_\mu$ as representative. We look at the path integral

$$\int \mathcal{D}A_\mu e^{iS} \tag{4.101}$$

with

$$\mathcal{D}A = \Pi_{a=1}^{N^2-1}\Pi_{\mu=0}^{3}\Pi_x dA_\mu^a(x)\ . \tag{4.102}$$

The action S is invariant under local gauge transformations, i.e.

$$S[\bar{A}_\mu^U] = S[\bar{A}_\mu] \tag{4.103}$$

And for the integration measure we have (schematically)

$$\int \mathcal{D}A = \int \mathcal{D}\bar{A} \int \mathcal{D}U\ , \tag{4.104}$$

so that we obtain for the path integral

$$\int \mathcal{D}A\ e^{iS[A]} = \int \mathcal{D}\bar{A}\ e^{iS[\bar{A}]} \int \mathcal{D}U\ . \tag{4.105}$$

The latter results in ∞. In order to avoid the infinite factor, we restrict the gauge freedom. I.e. we introduce the gauge fixing condition $F(A) = 0$.

Gauge Fixing, Faddeev-Popov Trick

We want to implement the gauge fixing condition $F^a(A_\mu^b) = 0$ into the functional integral, in a gauge-invariant way.
Remark: In non-Abelian gauge theories the Coulomb gauge $\vec{\nabla}\vec{A}_\mu^a = 0$ or the Euclidian Lorentz condition $\partial_\mu^E A_E^{a\mu} = 0$ are not unique for "large" gauge fields (i.e. gauge field configurations beyond perturbation theory). This means that $\vec{\nabla}\vec{A}^U = 0$ beyond perturbation theory has several solutions $U(x)$ for one given $\vec{A}^a$. This phenomenon is called Gribov ambiguity.
Mathematical Remarks:
a) The integration measure of the functional integral $\mathcal{D}A \equiv \Pi_{x,a,\mu} dA_\mu^a(x)$ is gauge invariant.
Proof:
We consider the gauge transformation $A_\mu \to A'_\mu = UA_\mu U^{-1} - \frac{i}{g}U\partial_\mu U^{-1}$. The gauge fields are $A_\mu^a = \mathrm{Tr}(2T^aA_\mu)$ and thereby $A_\mu^a \to A_\mu'^b = \mathrm{Tr}(2T^bA'_\mu)$. The integration measure becomes

$$\mathcal{D}A' = \mathcal{D}A \det_{\substack{x,x'\\ \mu,\nu\\ a,b}} \left(\frac{\partial A_\mu'^b(x)}{\partial A_\nu^a(x')}\right) \tag{4.106}$$

and ($A_\mu = A_\mu^a T^a$)

$$\frac{\partial A_\mu'^b(x)}{\partial A_\nu^a(x')} = \mathrm{Tr}[2T^bU(x)T^aU^{-1}(x)]\delta^{(4)}(x-x')g_\mu{}^\nu \tag{4.107}$$

where $U(x) = e^{i\omega_s(x)T^s}$. We use the formula

$$e^{iB}T^ae^{-iB} = T^a + i[B,T^a] + \frac{i^2}{2}[B,[B,T^a]] + ... \tag{4.108}$$

With $e^{iB} = U(x)$ we have

$$[B,T^a] = \sum_s \omega_s[T^s,T^a] = i\sum_s f_{sac}T^c\omega_s \tag{4.109}$$

And thereby we obtain for Eq. (4.107) with Eq. (4.108)

$$\delta^{(4)}(x-x')g_\mu{}^\nu \underbrace{(\delta_{ab}+i^2\sum_s f_{sab}\omega_s + ...)}_{\text{Matrix } \delta_{ab}+C_{ab}} \tag{4.110}$$

We use the formula

$$\boxed{\det(I+C) = \exp \operatorname{Tr} \ln(I+C) = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \operatorname{Tr}(C^n)\right) \qquad (4.111)}$$

We apply this on Eq. (4.110) and obtain

$$\det\left(\frac{\partial A'^b_\mu}{\partial A^a_\nu}\right) = 1 \tag{4.112}$$

in the lattice regularisation. (The lattice regularisation changes $\delta^{(4)}(x-x')$ into the Kronecker $\delta_{x,x'}$.) This means that det(...) is independent of $\omega_a(x)$ and thereby leads to the invariance of $\mathcal{D}A$.

(Remark: It would also be sufficient to show the invariance of $\mathcal{D}A$ under the infinitesimal gauge transformation $U(x) = I + i\omega_s(x)T^s + \mathcal{O}(\omega^2)$.)

b) Invariant group integration for compact groups G (= Haar measure)

Be $g\epsilon G$ a compact Lie group and $f(g)$ a function of g. For compact Lie groups there exists an invariant measure dg, for which holds

$$\begin{aligned} \int_G dg\; f(\underbrace{gg_0}_{g'}) &= \int_G dg'\; f(g') \qquad \text{right invariance} \\ \int_G dg\; f(\underbrace{g_0 g}_{g''}) &= \int_G dg''\; f(g'') \qquad \text{left invariance} \end{aligned} \tag{4.113}$$

for arbitrary $g_0 \,\epsilon\, G$.

An integration over G corresponds to an integration over group parameters. With

$$G = \{g(\omega)|\omega = (\omega_1, ..., \omega_{d(G)}) \,\epsilon\, D \subset \mathbf{R}^{d(G)}\} \tag{4.114}$$

and the metric tensor on the group

$$M_{ij} = \operatorname{Tr}[g^{-1}(\partial_i g)g^{-1}(\partial_j g)] \;, \tag{4.115}$$

where

$$\partial_i g = \frac{\partial}{\partial\omega_i} g(\omega) \tag{4.116}$$

we have the explicit formula

$$\boxed{\int_G dg\; f(g) = K \int_D \Pi_{j=1}^{d(G)} d\omega_j \;|\det M|^{1/2} f(g(\omega)) \qquad (4.117)}$$

in which the normalisation constant K is fixed by the requirement

$$1 \stackrel{!}{=} \int_G dg = K \int_D d\omega \, |\det M|^{1/2} \, . \tag{4.118}$$

Example:

$$\begin{aligned} U(1) &= \{e^{i\theta} | -\pi \leq \theta \leq \pi\} \\ |\det M|^{1/2} &= |i^2|^{1/2} = 1 \end{aligned}$$
$$\int dg f(g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \; f(e^{i\theta}) \tag{4.119}$$

Faddeev-Popov-Trick

We consider the functional

$$\Delta^{-1}[A] \equiv \int DU \; \delta(F[A^U]) \tag{4.120}$$

where $DU = \Pi_x dU(x)$ is the group measure (left- and right-invariant), since U is a compact group. The δ functional is explicitly $\Pi_{x,a}\delta(F^a[A_\mu^{bU}(x)])$ und $A_\mu^U = UA_\mu U^{-1} - \frac{i}{g} U\partial_\mu U^{-1}$.

Δ^{-1} is gauge invariant

$$\begin{aligned} \Delta^{-1}[A^U] &= \int DU' \; \delta(F[A^{UU'}]) = \int D(UU')\delta(F[A^{UU'}]) \\ &= \int DU''\delta(F[A^{U''}]) = \Delta^{-1}[A] \end{aligned} \tag{4.121}$$

Trick: Implement this into the path integral. We have

$$1 = \Delta[A] \int DU \; \delta(F[A^U]) \tag{4.122}$$

and thereby

$$\int DA \; e^{iS} = \int D\bar{A} \; \Delta[\bar{A}] \int DU \; \delta(F[\bar{A}^U])e^{iS} \, . \tag{4.123}$$

We perform a gauge transformation $\bar{A}_\mu^U \to \bar{A}_\mu$ and use that $DA, S[A], \Delta[A]$ are gauge invariant so that we obtain

$$\int DA \; e^{iS} \;\; = \;\; \int D\bar{A} \; \Delta[\bar{A}] \int DU \; \delta(F[\bar{A}])e^{iS} = \int D\bar{A} \; \Delta[\bar{A}] \, \delta(F[\bar{A}]) \, e^{iS} \int DU \tag{4.124}$$

and thereby now define the functional $\mathcal{Z} = \int DAe^{iS}$ as $(\bar{A} \to A)$

$$\boxed{\mathcal{Z} \to \mathcal{Z} = \frac{1}{\mathcal{N}} \int DA \; \Delta[A] \, \delta(F[A])e^{iS} \tag{4.125}}$$

We now calculate $\Delta^{-1}[A]$. By rescaling the minimal gauge transformation parameter with the gauge coupling g, $\omega^a \to g\omega^a$, und $U(x) = 1 + ig\omega_a T^a + \mathcal{O}(\omega^2)$ und $U^{-1} = 1 - ig\omega_a(x)T^a + \mathcal{O}(\omega^2)$, we obtain

$$\boxed{A^a_\mu(x) \to A'^a_\mu(x) = \mathrm{Tr}(2T^a A'_\mu) = A^a_\mu + g f_{abc} A^b_\mu \omega^c - \partial_\mu \omega^a + \mathcal{O}(\omega^2) \qquad (4.126)}$$

We have

$$DU = \Pi_x dU(x) = \Pi_x \Pi_{a=1}^{d(G)} d\omega_a(x) \sqrt{|\det M|} K \equiv D\omega \tag{4.127}$$

and thereby

$$\Delta^{-1}[A] = \int D\omega \; \Pi_{a,x} \delta(F^a[A^{U(\omega)}(x)]) \tag{4.128}$$

which leads after a variable transformation $\omega^a(x) \to F^a[A^{U(\omega)}]$ to

$$= \int DF \, \frac{\delta(F[A^U])}{|\det \frac{\delta F^a}{\delta \omega^b}|} = \sum_{\bar{A}} \frac{1}{|\det \frac{\delta F^a[A^U]}{\delta \omega^b}|_{A^U = \bar{A}}} \, , \tag{4.129}$$

where $\bar{A}$ is the solution of $F[A^U] = 0$ for a given A: (In general, there are several solutions for $F[A^U] = 0$, Gribov ambiguity. But we here do perturbation theory.) We want to do perturbation theory and only look at fluctuations around (the field configuration) $A^a_\mu = 0$. Thereby, $F^a[A^U] = 0$ has a unique solution, and we obtain

$$= \frac{1}{|\det \frac{\delta F}{\delta \omega}|_{\vec{\omega}=0}} \tag{4.130}$$

so that

$$\Delta[A] = |\det \frac{\delta F^a}{\delta \omega^b}[A^U]|_{\vec{\omega}=0} \tag{4.131}$$

This is called:

Faddeev-Popov Determinant

$$\Delta[A] = |\det_{x,y} M_{ab}(x,y)| \tag{4.132}$$

where

$$M_{ab}(x,y) = \frac{\delta F^a[A_\omega(x)]}{\delta \omega^b(y)}|_{\vec{\omega}=0} \tag{4.133}$$

It is sufficient to derive $F^a[A^U]$ for infinitesimal gauge transformations. For the calculation of $M_{ab}(x,y)$ we use that

$$\frac{\delta A^a_{\omega\mu}(x)}{\delta \omega^b(y)} = -\underbrace{(\partial_\mu \delta_{ab} + g f_{eab} A^e_\mu(x))}_{\equiv \Delta^{ab}_\mu} \delta^{(4)}(x-y) \tag{4.134}$$

Δ_μ is the covariant derivative in the adjoint representation. For the <u>covariant</u> gauge fixing condition $F^a[A] = \partial^\mu A^a_\mu = 0$ we find

$$F^a[A_\omega] = \partial^\mu A^a_{\mu\omega}(x) = \partial^\mu A^a_\mu(x) - \partial^\mu(\Delta^{ac}_\mu \omega^c(x)) \tag{4.135}$$

The first summand is 0 because of the gauge fixing condition. And thereby

$$\boxed{\begin{aligned} M_{ab}(x,y) &= \frac{\delta F^a[A_\omega(x)]}{\delta\omega^b(y)}|_{\vec{\omega}=0} = -\partial^\mu \Delta_\mu^{ac}\delta_{cb}\delta^{(4)}(x-y) \\ &= -\partial^\mu(\partial_\mu\delta_{ab} + g f_{cab}A_\mu^c(x))\delta^{(4)}(x-y) \qquad (4.136) \end{aligned}}$$

For the gauge fixing condition $F^a[A] = 0 \to F^a[A] = B^a(x)$ (B^a are functions independent of A) we have the same Faddeev-Popov determinant as for the covariant gauge fixing case and thereby the functional $\mathcal{Z}$

$$\mathcal{Z} \sim \int DA\ \Delta[A]\,\delta(F[A]-B)e^{iS} \qquad (4.137)$$

Gauge-invariant quantities are independent of a change of the gauge fixing condition. We therefore average over $B^a(x)$ with the weighting factor

$$\rho = \int DB\ \exp\left(-\frac{i}{2\xi}\int d^4x\ \sum_a B_a^2(x)\right) \qquad \xi\epsilon\mathbf{R} \qquad (4.138)$$

This solely changes the normalisation factor. We use that

$$\det(A_{ij}) = \int d\bar{c}\,dc\ e^{-\sum_{ij}\bar{c}_i A_{ij} c_j} \qquad (4.139)$$

where $\{c_i, c_j\} = 0$, c_i are Grassmann variables. For the Faddeev-Popov determinant

$$\Delta[A] = |\det_{\substack{x,y\\a,b}}(-iM)| \qquad (4.140)$$

we then have (the factor $(-i)$ is convention)

$$|\det(-iM)| = \text{const.}\int D\bar{c}\,Dc\ \exp\left(i\int d^4x_1\,d^4x_2\ \bar{c}_a(x_1)M_{ab}(x_1,x_2)c_b(x_2)\right) \qquad (4.141)$$

where

$$D\bar{c}\,Dc = \Pi_{x,a}d\bar{c}_a(x)\,dc_a(x) \qquad (4.142)$$

and $c_a(x), \bar{c}_a(x)$ are Grassmann fields. I.e.

$$\{c_a(x), c_b(y)\} = 0 \qquad \{\bar{c}_a(x), \bar{c}_b(y)\} = 0 \qquad \{c_a(x), \bar{c}_b(y)\} = 0\ . \qquad (4.143)$$

The fields $c_a, \bar{c}_a$ transform as scalar fields under Lorentz transformations, i.e. they are anti-commuting spin-0 fields. They have the wrong statistics. They are called Faddeev-Popov ghosts and are pure help fields. In the covariant gauge $\partial_\mu A^{a\mu} = 0$ we have Eq. (4.136) and after partial integration

$$i\int d^4x\,d^4y\ \bar{c}_a(x)M_{ab}c_b(y) = i\int d^4x\ \partial^\mu\bar{c}_a(x)(\partial_\mu\delta_{ab} + g f_{cab}A_\mu^c(x))c_b(x) \qquad (4.144)$$

so that the functional

$$\mathcal{Z} \sim \int DA\,D\bar{c}\,Dc\ e^{i\int d^4x\ \mathcal{L}_{eff}(x)} \qquad (4.145)$$

with

$$\begin{aligned} \mathcal{L}_{eff} &= \mathcal{L}_{class}(x) + \mathcal{L}_{gauge\,fix}(x) + \mathcal{L}_{ghost}(x) && (4.146)\\ &= \mathcal{L}_{class} - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 + \partial^\mu \bar{c}_a(x)(\partial_\mu \delta_{ab} + g f_{cab} A^c_\mu) c_b(x) && (4.147) \end{aligned}$$

Let us summarise: We have for the total action functional with fermions

$$\boxed{\begin{aligned} \mathcal{Z} &\sim \int D\bar{\psi}\, D\psi\, DA\, D\bar{c}\, Dc\ \exp i \int d^4x\, [\mathcal{L} + \mathcal{L}_{GF} + \mathcal{L}_{FP}] && (4.148)\\ \mathcal{L} &= \text{usual Lagrangian}\\ \mathcal{L}_{GF} &= \text{Gauge fixing, } \mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial A)^2 \text{ etc.}\\ \mathcal{L}_{FP} &= \partial\bar{c}\Delta c \text{ for gauge theories, non-Abelian and non-linear gauge fixing} \end{aligned}}$$

Propagators: The matrices between the bilinear forms of the fields in the total Lagrangian depend on the gauge fixing.
Example:

$$\begin{aligned} \mathcal{L}_{GF} &= \frac{-1}{2\xi}\partial A \cdot \partial A\\ \mathcal{Z} &\sim \int DA\ \exp i \int d^4x\, d^4y\ \frac{1}{2} A^a_\mu [\Delta_F^{-1}(x-y)]^{ab}_{\mu\nu} A^b_\nu\\ (\Delta_F^{-1})^{ab}_{\mu\nu} &\equiv [\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu + \frac{1}{\xi}\partial_\mu\partial_\nu]\delta_4(x-y)\delta^{ab}\\ (\Delta_F)^{ab}_{\mu\nu}(q) &= \frac{d_{\mu\nu}\delta_{ab}}{q^2 + i\epsilon} \qquad d_{\mu\nu} = -g_{\mu\nu} + (1-\xi)\frac{q_\mu q_\nu}{q^2} && (4.149) \end{aligned}$$

't Hooft-Feynman gauge:
Higgs phenomenon in $SO(2)$:

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F^2_{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) - V(\phi)\\ V(\phi) &= \frac{\lambda}{4}\left(\phi^2 - \frac{\mu^2}{\lambda}\right)^2 \qquad iD_\mu = i\partial_\mu - gA_\mu\ . && (4.150) \end{aligned}$$

With $\phi = 1/\sqrt{2}(\phi_1 + v + i\phi_2)$ we have

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F^2_{\mu\nu} + \frac{g^2v^2}{2}A^2_\mu + \frac{1}{2}(\partial_\mu\phi_2)^2 - \frac{1}{2}(2\lambda v^2)\phi_1^2 + \frac{1}{2}(\partial_\mu\phi_1)^2\\ &\quad + gvA_\mu(\partial^\mu\phi_2) + \text{ 3- and 4-point couplings} && (4.151) \end{aligned}$$

Gauge fixing The gauge fixing is chosen such that $\mathcal{L}_{tot}$ is diagonal in the bilinear expressions (gauge fields, Goldstone fields). There are hence no transitions between gauge and Goldstone field.

$$\mathcal{L}_{fix} = -\frac{1}{2\xi}[\partial_\mu A^\mu - \xi m_A \phi_2]^2 \qquad m_A = gv \tag{4.152}$$

The diagonal progagators are:

$$\text{Goldstone field:} \quad \frac{i}{q^2 - \xi m_A^2} \tag{4.153}$$

$$\text{Gauge field:} \quad i\frac{-g_{\mu\nu} + (1-\xi)\frac{q_\mu q_\nu}{q^2 - \xi m_A^2}}{q^2 - m_A^2} \tag{4.154}$$

and

$$\begin{aligned} \xi \to \infty \quad &: \quad \text{unitary theory (no Goldstone contribution)} \\ \xi \to 1 \quad &: \quad \text{Feynman gauge (Goldstone propagator} \sim m_A \text{ mass)} \\ & \quad\;\; \text{Renormalisation maximally simplified} \end{aligned} \tag{4.155}$$

Ghosts: Cancel the unphysical longitudinal contributions in the propagators of the gauge fields

$$\begin{aligned} \Delta_F^{\mu\nu} &= \frac{i}{q^2 - m_A^2}[-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}] - \frac{i\xi}{q^2 - \xi m_A^2}\frac{q_\mu q_\nu}{q^2} \\ &= \frac{i}{q^2 - m_A^2}[-g_{\mu\nu} + \frac{q_\mu q_\nu}{m_A^2}] - \frac{i}{q^2 - \xi m_A^2}\frac{q_\mu q_\nu}{m_A^2} \end{aligned} \tag{4.156}$$

The gauge fixing Lagrangian for the GSW theory in the R_ξ gauge reads:

$$\mathcal{L}_{GF} = -\frac{1}{2}[F_\gamma^2 + F_Z^2 + 2F_+F_-] \tag{4.157}$$

$$F_\gamma = \frac{1}{\xi_\gamma^{1/2}}\partial_\mu A^\mu \tag{4.158}$$

$$F_Z = \frac{1}{\xi_Z^{1/2}}[\partial_\mu Z^\mu - \xi_Z m_Z \chi] \tag{4.159}$$

$$F_\pm = \frac{1}{\xi_W^{1/2}}[\partial_\mu W^{\pm\mu} \mp i\xi_W m_W \phi^\pm] \tag{4.160}$$

The propagators for the gauge bosons in the R_ξ gauge are

$$\begin{aligned} &\frac{i}{k^2 - m_V^2 + i\epsilon}[-g_{\mu\nu} + (1-\xi_V)\frac{k_\mu k_\nu}{k^2 - \xi_V m_V^2 + i\epsilon}] \qquad \text{for } V = W^\pm, Z \\ &\frac{-ig_{\mu\nu}}{k^2 + i\epsilon} \qquad \text{for } V = A\,. \end{aligned} \tag{4.161}$$

The Goldstone propagators are

$$\begin{aligned} &\frac{i}{k^2 - \xi_V m_V^2 + i\epsilon} \qquad \text{for } V = W, Z \\ &\frac{i}{k^2 + i\epsilon} \qquad \text{for } V = A\,. \end{aligned} \tag{4.162}$$

4.5 Interactions

We start from the following Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}(m^2 - i\epsilon)\phi^2 - \frac{g}{n!}\phi^n(x) = \mathcal{L}_0 + \mathcal{L}_{WW} \text{ with } \mathcal{L}_{WW} = -\frac{g}{n!}\phi^n(x) \tag{4.163}$$

The vacuum functional can be written as

$$\mathcal{Z}[J] = \mathcal{N} \int D\phi \exp\left(\frac{i}{\hbar}\int d^4z \mathcal{L}_{WW}(\phi(z))\right) \exp\left(\frac{i}{\hbar}\int d^4x [\mathcal{L}_0 + J\phi]\right) \ . \tag{4.164}$$

With

$$\frac{\hbar}{i}\frac{\delta}{\delta J(x_1)} ... \frac{\hbar}{i}\frac{\delta}{\delta J(x_n)} \mathcal{Z}_0[J] = \mathcal{N} \int D\phi \ \phi(x_1)...\phi(x_n) \exp\left(\frac{i}{\hbar}\int d^4x [\mathcal{L}_0 + J\phi]\right) \tag{4.165}$$

we obtain:

$$\begin{aligned}\mathcal{Z}[J] &= \mathcal{N} \int D\phi \exp\left(\frac{i}{\hbar}\int d^4z \mathcal{L}_{WW}\left(\frac{\hbar}{i}\frac{\delta}{\delta J(z)}\right)\right) \exp\left(\frac{i}{\hbar}\int d^4x [\mathcal{L}_0 + J\phi]\right) \\ &= \mathcal{N}' \exp\left(\frac{i}{\hbar}\int d^4z \mathcal{L}_{WW}\left(\frac{\hbar}{i}\frac{\delta}{\delta J(z)}\right)\right) \mathcal{Z}_0[J] \ .\end{aligned} \tag{4.166}$$

$\mathcal{N}'$ is given by $\mathcal{Z}[0] = 1$. Thereby the central formula of perturbation theory is

$$\boxed{\mathcal{Z}[J] = \frac{\exp\left(\frac{i}{\hbar}\int d^4z \ \mathcal{L}_{WW}\left(\frac{\hbar}{i}\frac{\delta}{\delta J(z)}\right)\right) \mathcal{Z}_0[J]}{\exp\left(\frac{i}{\hbar}\int d^4z \ \mathcal{L}_{WW}\left(\frac{\hbar}{i}\frac{\delta}{\delta J(z)}\right)\right) \mathcal{Z}_0[J]\Big|_{J=0}}} \tag{4.167}$$

For interacting n-point functions we then have

$$\begin{aligned}\langle 0|T[\hat{\phi}(x_1)...\hat{\phi}(x_n)]|0\rangle_{WW} &= \\ &\frac{\frac{\hbar}{i}\frac{\delta}{\delta J(x_1)} ... \frac{\hbar}{i}\frac{\delta}{\delta J(x_n)} \exp\left(\frac{i}{\hbar}\int d^4z \ \mathcal{L}_{WW}\left(\frac{\hbar}{i}\frac{\delta}{\delta J(z)}\right)\right) \mathcal{Z}_0[J]\Big|_{J=0}}{\exp\left(\frac{i}{\hbar}\int d^4z \ \mathcal{L}_{WW}\left(\frac{\hbar}{i}\frac{\delta}{\delta J(z)}\right)\right) \mathcal{Z}_0[J]\Big|_{J=0}}\end{aligned} \tag{4.168}$$

The denominator describes the vacuum graphs, which are divided out.

4.5.1 ϕ^4 Theory

We evaluate Eq. (4.167) until order λ for (from now on again $\hbar \equiv 1$)

$$\mathcal{L}_{WW} = -\frac{\lambda}{4!}\phi^4 \ . \tag{4.169}$$

For the nominator Z we have

$$\mathcal{Z} = \left[1 - \frac{i\lambda}{4!}\int d^4z \left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4 + \mathcal{O}(\lambda^2)\right] \underbrace{\exp\left(-\frac{i}{2}\int d^4x d^4y J(x)\Delta_F(x-y)J(y)\right)}_{\mathcal{Z}_0[J]} \tag{4.170}$$

Side calculation:

$$\begin{aligned}
&(1)\quad \frac{1}{i}\frac{\delta}{\delta J(z)}\mathcal{Z}_0[J] &&= -\int d^4x J(x)\Delta_F(z-x)\mathcal{Z}_0[J] \\
&(2)\quad \frac{1}{i^2}\frac{\delta^2}{\delta J(z)\delta J(z)}\mathcal{Z}_0[J] &&= \left\{i\Delta_F(0) + \left[\int d^4x\Delta_F(z-x)J(x)\right]^2\right\}\mathcal{Z}_0[J] \\
&(3)\quad \left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^3\mathcal{Z}_0[J] &&= \left\{-3i\Delta_F(0)\int d^4x\Delta_F(z-x)J(x)\right. \\
& &&\quad \left. - \left[\int d^4x J(x)\Delta_F(z-x)\right]^3\right\}\mathcal{Z}_0[J] \\
&(4)\quad \left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4\mathcal{Z}_0[J] &&= \left\{-3[\Delta_F(0)]^2 + 6i\Delta_F(0)\left[\int d^4x\Delta_F(z-x)J(x)\right]^2\right. \\
& &&\quad \left. + \left[\int d^4x\Delta_F(z-x)J(x)\right]^4\right\}\mathcal{Z}_0[J]\,.
\end{aligned} \tag{4.171}$$

Representation as diagram: The $\Delta_F(x-y)$ is a propagator propagating from x to y. The $\Delta_F(0)$ is a closed loop. Furthermore, we have for (4) (cf. Fig. 4.4):

$$\begin{aligned}
\left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4\mathcal{Z}_0[J] &= \left\{-3 \text{ Picture1 }) + 6i\int d^4x_1 d^4x_2 \text{ Picture2 } + \right. \\
&\left\{\int \Pi_{j=1}^4 d^4x_j \text{ Picture3 } + \mathcal{O}(\lambda^2)\right\}\mathcal{Z}_0[J]\,.
\end{aligned} \tag{4.172}$$

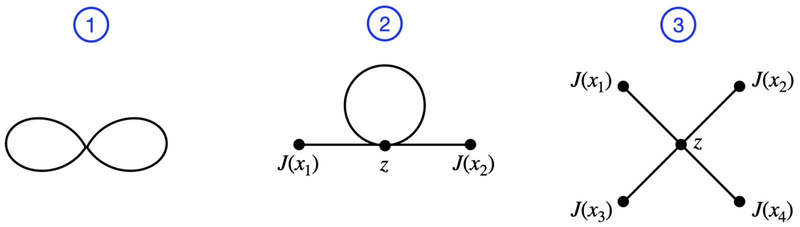


Figure 4.4: The three pictures "Picture1/2/3" appearing in Eq. (4.172)

The first "Picture1" is a vacuum diagram, which is just bubbles without external lines. The factors 3 and 6 are symmetry factors. The lines with an attached $J(x)$ are external lines. In "Picture2" we have two external lines, in "Picture3" we have four external lines.

The denominator N:

$$N = \mathcal{Z}|_{J=0} = 1 - \frac{i\lambda}{4!}\int d^4z(-3 \text{ Picture1 } + \mathcal{O}(\lambda^2)\,. \tag{4.173}$$

Thereby we have

$$\begin{aligned}
\mathcal{Z}[J] &= \frac{\text{nominator}}{\text{denominator}} = \text{nominator}\left(1 + \frac{i\lambda}{4!}\int d^4z(-3 \text{ Picture1 }) + \mathcal{O}(\lambda^2)\right) \\
&= \left\{1 - \frac{i\lambda}{4!}\int d^4z\left(6i\int d^4x_1 d^4x_2 \text{ Picture2 } + \int \Pi_{j=1}^4 d^4x_j \text{ Picture3 }\right) + \mathcal{O}(\lambda^2)\right\}\mathcal{Z}_0[J]\,.
\end{aligned} \tag{4.174}$$

This means that vacuum diagrams do not appear in normalised $\mathcal{Z}[J]$. This holds for all orders in perturbation theory. For the 4-point function

$$\langle 0|T[\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4)]|0\rangle|_{WW} \tag{4.175}$$

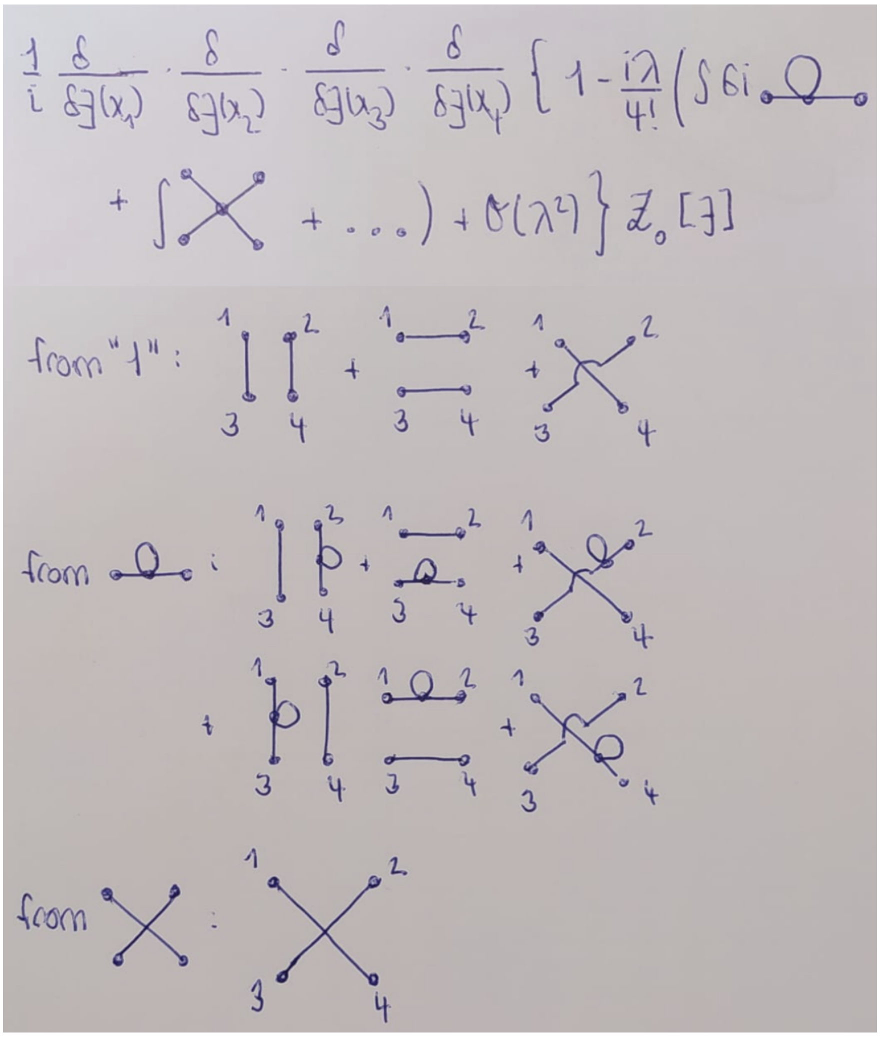


Figure 4.5: Visualisation of Eq. (4.175), the fourth derivative of $\mathcal{Z}$ w.r.t. the sources.

we obtain with Eq. (4.168) and (4.75) then (cf. Fig. **??**)

$$= \tau(x_1x_2)\tau(x_3x_4) + \tau(x_1x_3)\tau(x_2x_4) + \tau(x_1x_4)\tau(x_2x_3) + \tau_c(x_1, x_2, x_3, x_4) \ . \tag{4.176}$$

Here τ_c the denotes connected Green function. We are looking for a generating functional $W[J]$ for connected Green functions $\tau_c(x_1, ..., x_n)$. It is given by

$$\mathcal{Z}[J] = \exp(iW[J]) \ . \tag{4.177}$$

This can be seen as follows. Because of $\mathcal{Z}[0] = 1$ we have $W[0] = 0$. And for the derivatives we have

$$\frac{1}{i}\frac{\delta}{\delta J(x_1)}\mathcal{Z} = \frac{\delta W}{\delta J(x_1)}\exp(iW) \Rightarrow \left.\frac{\delta W}{\delta J(x)}\right|_{J=0} = 0 \tag{4.178}$$

$$\frac{1}{i^2}\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\mathcal{Z} = \frac{\delta W}{\delta J(x_1)}\frac{\delta W}{\delta J(x_2)}\exp(iW) - i\frac{\delta^2 W}{\delta J(x_1)\delta J(x_2)}\exp(iW) \ . \tag{4.179}$$

For $J=0$ it holds:

$$\begin{aligned}
\left.\frac{\delta^2 W}{\delta J(x_1)\delta J(x_2)}\right|_{J=0} &= i\tau(x_1,x_2) && (4.180)\\
\frac{1}{i^4}\left.\frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\mathcal{Z}\right|_{J=0} &= \tau(x_1,x_3)\tau(x_2,x_4)+\tau(x_1,x_4)\tau(x_2,x_3)\\
&\quad +\tau(x_1,x_2)\tau(x_3,x_4)\\
&\quad +(-i)\frac{1}{i^2}\frac{\delta^4 W}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} && (4.181)
\end{aligned}$$

And thereby

$$i\left.\frac{\delta^4 W}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\right|_{J=0} = \tau_c(x_1,x_2,x_3,x_4)\ . \tag{4.182}$$

Thereby $W[J]=-i\ln\mathcal{Z}[J]$ generates the connected Green functions.

4.6 Fermi Fields

With the canonical field quantisation of the Dirac field operators

$$\hat{\psi}(x)\ ,\ \hat{\bar{\psi}}(x)=\hat{\psi}^\dagger\gamma^0\ , \tag{4.183}$$

we have to postulate anti-commutation rules in order to obtain the Fermi statistics,

$$\begin{aligned}
\{\hat{\psi}_r(x),\hat{\psi}_s(y)\} &= \{\hat{\bar{\psi}}_r(x),\hat{\bar{\psi}}_s(y)\}=0\\
\{\hat{\psi}_r(x),\hat{\bar{\psi}}_s(y)\}_{x^0=y^0} &= \delta_{rs}\delta^{(3)}(\vec{x}-\vec{y})\ . && (4.184)
\end{aligned}$$

In the functional integral quantisation we need, since there is no c-number equivalent to $\hat{\psi},\hat{\bar{\psi}}$, Grassmann variables.

We have seen for the Grassmann variables $c_i,\bar{c}_i$ $(i=1,...,n)$ that

$$\begin{aligned}
\int d\bar{c}\,dc \qquad & e^{-\sum_{ij}\bar{c}_i A_{ij} c_j} = \det A\\
d\bar{c}\,dc \quad &\equiv\quad d\bar{c}_1\,dc_1\,...\,d\bar{c}_n\,dc_n\ . && (4.185)
\end{aligned}$$

In order to describe Fermi fields, we make the transition to the infinite-dimensional Grassmann algebra,

$$c_i\to\psi(x)=\psi_r(x)\quad ,\quad r=\ \text{Spinor-Index}\ . \tag{4.186}$$

Here $\psi(x)$ is a Grassmann "field", i.e. a Grassmann variable with a continuous index $x\in M_4$. The Grassmann algebra

$$\{\psi_r(x),\psi_s(y)\}=0 \tag{4.187}$$

holds and

$$\frac{\delta}{\delta\psi_s(y)}\psi_r(x)=\delta_{rs}\delta^{(4)}(x-y)\ ,\ \int d\psi_r(x)=0\ ,\ \int d\psi_r(x)\psi_s(x)=\delta_{rs}\ . \tag{4.188}$$

We furthermore need the independent Grassmann field $\bar{\psi}(x) = \bar{\psi}_r(x)$. Here, the algebra

$$\{\bar{\psi}_r(x), \bar{\psi}_s(y)\} = \{\psi_r(x), \bar{\psi}_s(y)\} = 0 \tag{4.189}$$

holds. The differentiation and integration are analogous to the case $\psi(x)$.

The Lagrangian for the free Dirac theory reads

$$\begin{aligned} \mathcal{L}_0 &= i\bar{\psi}(x)\gamma^\mu \partial_\mu \psi(x) - m\bar{\psi}(x)\psi(x) \\ &= \bar{\psi}[i\gamma^\mu \partial_\mu - m]\psi \; . \end{aligned} \tag{4.190}$$

We write up a normalised generating functional analogous to the scalar field. For this we introduce the Grassmann fields $\eta = \eta_r(x)$ und $\bar{\eta} = \bar{\eta}_s(x)$, which serve as sources for the fields $\bar{\psi}_r(x)$, respectively, $\psi_s(x)$. Thereby we have for the normalised generating functional

$$\mathcal{Z}_0[\bar{\eta}, \eta] = \frac{1}{\mathcal{N}} \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(i \int d^4x [\mathcal{L}_0 + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)] \right) \; , \tag{4.191}$$

where

$$\mathcal{N} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(i \int d^4x \mathcal{L}_0(x) \right) \tag{4.192}$$

and

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi = \Pi_x \Pi_{r=1}^4 d\bar{\psi}_r(x) d\psi_r(x) \; . \tag{4.193}$$

The Green functions are given by

$$\langle 0|T[\hat{\psi}_{r_1}(x_1)...\hat{\psi}_{r_n}(x)\hat{\bar{\psi}}_{s_1}(y_1)...\hat{\bar{\psi}}_{s_n}(y_n)]|0\rangle = \left| \frac{1}{i^n} \left(\frac{1}{-i} \right)^n \frac{\delta^{2n} \mathcal{Z}_0[\bar{\eta}, \eta]}{\delta \bar{\eta}_{r_1}(x_1)...\delta \eta_{s_n}(y_n)} \right|_{\eta=\bar{\eta}=0} \; . \tag{4.194}$$

The factors $1/(-i)$ appear because

$$\frac{\delta}{\delta \eta_{s_i}(y_i)} \bar{\psi}_r(x)\eta_r(x) = -\bar{\psi}_r(x) \frac{\delta}{\delta \eta_{s_i}(y_i)} \eta_r(x) = -\bar{\psi}_{s_i}(y_i) \; . \tag{4.195}$$

Note furthermore that

$$\frac{\delta^2}{\delta \eta(x) \delta \bar{\eta}(y)} = -\frac{\delta^2}{\delta \bar{\eta}(y) \delta \eta(x)} \; . \tag{4.196}$$

We look for a formula $\mathcal{Z}_0$ analogous to the scalar case. We introduce the following notation

$$S_F^{-1} \equiv i\gamma_\mu \partial^\mu - m \; . \tag{4.197}$$

Then

$$(i\gamma_\mu \partial_x^\mu - m)_{rr_1} S_{F\, r_1 s}(x - y) = \delta_{rs} \delta^{(4)}(x - y) \; , \tag{4.198}$$

where $S_F(x - y)$ is the Feynman propagator for a free Dirac fermion. We have

$$S_F(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{\not{k} + m}{k^2 - m^2 + i\epsilon} e^{-ik(x-y)} \; . \tag{4.199}$$

Furthermore, we have the respresentation

$$S_F(x-y) = (i\gamma\cdot\partial_x + m)\Delta_F(x-y)\ . \tag{4.200}$$

We hence write Eq. (4.191) as

$$\mathcal{Z}_0[\bar\eta,\eta] = \frac{1}{\mathcal{N}}\int \mathcal{D}\bar\psi\mathcal{D}\psi e^{i\int d^4x Q}\ , \tag{4.201}$$

where

$$Q = \underbrace{\bar\psi S_F^{-1}\psi}_{\mathcal{L}_0} + \bar\eta\psi + \bar\psi\eta\ . \tag{4.202}$$

It is

$$Q = Q_0 + (\bar\psi - \bar\psi_0)S_F^{-1}(\psi - \psi_0)\ , \tag{4.203}$$

where

$$Q_0 = -\bar\eta S_F\eta = -\bar\eta(x)\int d^4z\, S_F(x-z)\eta(z) \tag{4.204}$$

$$\psi_0 = -S_F\eta = -\int d^4z\, S_F(x-z)\eta(z) \tag{4.205}$$

$$\bar\psi_0 = -\bar\eta S_F = -\int d^4z\, \bar\eta(z)S_F(z-x)\ . \tag{4.206}$$

Thereby hence

$$\begin{aligned}\mathcal{Z}_0[\bar\eta,\eta] &= \frac{1}{\mathcal{N}}\exp\left(-i\int d^4x\, d^4z\, \bar\eta(x)S_F(x-z)\eta(z)\right)\\ &\quad \int \mathcal{D}\bar\psi\mathcal{D}\psi \exp\left(i\int d^4x\, d^4z\, (\bar\psi-\bar\psi_0)S_F^{-1}(\psi-\psi_0)\right)\ .\end{aligned} \tag{4.207}$$

We perform a field transformation:

$$\psi'(x) = \psi(x) - \psi_0(x) \qquad \text{and} \qquad \bar\psi'(x) = \bar\psi(x) - \bar\psi_0(x)\ . \tag{4.208}$$

It is

$$\mathcal{D}\bar\psi'\mathcal{D}\psi' = \mathcal{D}\bar\psi\mathcal{D}\psi\ . \tag{4.209}$$

We apply Eq. (4.185) to the second integral and obtain

$$\det(-iS_F^{-1}) = \det[-i(i\gamma\cdot\partial - m)]\ . \tag{4.210}$$

Thereby we have

$$\mathcal{N} = \det(-iS_F^{-1}) \tag{4.211}$$

and hence for the normalised generating functional

$$\mathcal{Z}_0[\bar\eta,\eta] = e^{-i\int d^4x d^4y \bar\eta(x)S_F(x-y)\eta(y)}\ . \tag{4.212}$$

<u>Check:</u>

$$\begin{aligned}\langle 0|T[\hat\psi(x)\hat{\bar\psi}(y)]|0\rangle &= \frac{-1}{i^2}\frac{\delta^2\mathcal{Z}_0}{\delta\bar\eta(x)\delta\eta(y)}\bigg|_{\eta=\bar\eta=0}\\ &= \frac{\delta^2}{\delta\bar\eta(x)\delta\eta(y)}\left\{1 - i\int d^4z_1\, d^4z_2\, \bar\eta(z_1)S_F(z_1-z_2)\eta(z_2) + ...\right\}\bigg|_{\eta=\bar\eta=0}\\ &= iS_F(x-y)\ .\end{aligned} \tag{4.213}$$

4.7 Generating Functional for Interacting Field Theories

We consider an interacting field theory for a hermitian scalar field and a Dirac field. The classical Lagrangian be of the form

$$\mathcal{L} = \mathcal{L}_0(\Phi) + \mathcal{L}_0(\bar{\psi}, \psi) + \mathcal{L}_I(\bar{\psi}, \psi, \Phi) \; , \tag{4.214}$$

where the index 0 denotes the free Lagrangians and the index I stands for the interaction Langrangian. An example for an interaction is the Yukawa interaction

$$\mathcal{L}_I = -y\bar{\psi}\psi\Phi \; , \tag{4.215}$$

where y denotes the Yukawa coupling. Note: In the case of Dirac fermions the $\mathcal{L}_I$ contains the same number of ψ and $\bar{\psi}$, as otherwise the Lagrangian would violate charge conservation. We can derive the generating functional of the above theory by making in $\mathcal{L}_I$ the following replacements:

$$\bar{\psi}(z) \to \frac{-1}{i}\frac{\delta}{\delta\eta(z)} \quad , \quad \psi(z) \to \frac{1}{i}\frac{\delta}{\delta\bar{\eta}(z)} \quad , \quad \Phi(z) \to \frac{1}{i}\frac{\delta}{\delta J(z)} \; . \tag{4.216}$$

Thereby we obtain the central formula of functional perturbation theory,

$$\mathcal{Z}[J,\bar{\eta},\eta] = \frac{\exp\left(i\int d^4z\, \mathcal{L}_I\left(-\frac{1}{i}\frac{\delta}{\delta\eta(z)}, \frac{1}{i}\frac{\delta}{\delta\bar{\eta}(z)}, \frac{1}{i}\frac{\delta}{\delta J(z)}\right)\right)\mathcal{Z}_0[J]\mathcal{Z}_0[\bar{\eta},\eta]}{\exp\left(i\int d^4z\, \mathcal{L}_I\left(\frac{1}{i}\frac{\delta}{\delta\eta(z)}, \frac{1}{i}\frac{\delta}{\delta\bar{\eta}(z)}, \frac{1}{i}\frac{\delta}{\delta J(z)}\right)\right)\mathcal{Z}_0[J]\mathcal{Z}_0[\bar{\eta},\eta]\Big|_{J=\bar{\eta}=\eta=0}} \; . \tag{4.217}$$

4.8 Non-Abelian Gauge Theories

In the following we consider the gauge group $SU(N)$, i.e. we have the gauge fields $A^a_\mu(x)$ ($a = 1, ..., N^2-1$) and a Dirac field $\psi_j(x)$ ($j = 1, ..., N$) in the fundamental representation F, respectively $\bar{\psi}_j(x)$ in the representation $\bar{F}$. The fields transform according to

$$\begin{aligned} A_\mu \quad &\to \quad A'_\mu = UA_\mu U^{-1} - \frac{i}{g}U\partial_\mu U^{-1} && (4.218)\\ \psi \quad &\to \quad \psi'(x) = U(x)\psi(x) && (4.219)\\ \bar{\psi} \quad &\to \quad \bar{\psi}'(x) = \bar{\psi}(x)U^\dagger(x) \; , && (4.220)\end{aligned}$$

where

$$U(x) = \exp i\omega_a(x)T^a \qquad \in SU(N) \; . \tag{4.221}$$

The generators of the group $SU(N)$ are denoted by T^a. The classical Lagrangian is given by

$$\mathcal{L}(x) = -\frac{1}{4}F^a_{\mu\nu}(x)F^{\mu\nu a}(x) + \bar{\psi}(x)(i\gamma^\mu D_\mu - m)\psi(x) \; . \tag{4.222}$$

Here $F^{\mu\nu}$ is the field strength tensor, D_μ the covariant derivative and m the mass of the fermion. We have gotten to know the Faddeev-Popov trick, with which we can solve the problem of gauge invariance in the path integral quantisation. Thereby the gauge fixing condition is implemented in the Lagrangian. Furthermore, the Faddeev-Popov ghosts appear, which are pure help fields. They are scalar fields, which obey the Fermi statistics, however. Furthermore, we have seen how the path integral formalism looks for Fermi fields. With all this, we can finally write up the generating functional for non-Abelian gauge theories. It is proportional to

$$\mathcal{Z} \sim \int \mathcal{D}A\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\mathcal{D}\bar{c}\,\mathcal{D}c\; e^{i\int d^4x \mathcal{L}_{eff}(x)} \; . \tag{4.223}$$

Here $\mathcal{L}_{eff}$ is given in the covariant gauge through

$$\begin{aligned}\mathcal{L}_{eff} &= \mathcal{L}_{class}(x) + \mathcal{L}_{gaugefix}(x) + \mathcal{L}_{ghost}(x) \\ &= -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{\psi}_l(i\gamma^\mu D_{\mu lj} - m\delta_{lj})\psi_j - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 \\ &\quad +\partial^\mu \bar{c}_a(x)(\partial_\mu \delta_{ab} + g f_{cab} A^c_\mu) c_b(x) \; .\end{aligned} \tag{4.224}$$

For the computation of the Green functions, we introduce sources:

$$A^a_\mu(x) \;\leftrightarrow\; J^{a\mu}(x) \quad \text{real function} \tag{4.225}$$
$$\psi_{s_j}(x) \;\leftrightarrow\; \bar{\eta}_{s_j}(x) \quad \text{Grassmann variable} \tag{4.226}$$
$$\bar{\psi}_{s_j}(x) \;\leftrightarrow\; \eta_{s_j}(x) \quad \text{Grassmann variable} \; . \tag{4.227}$$

For mathematical reasons, we also introduce sources for the ghost fields:

$$c_a(x) \;\leftrightarrow\; \bar{\zeta}_a(x) \quad \text{Grassmann variable} \tag{4.228}$$
$$\bar{c}_a(x) \;\leftrightarrow\; \zeta_a(x) \quad \text{Grassmann variable} \; . \tag{4.229}$$

Thereby we obtain for the normalized generating functional

$$\mathcal{Z}_0[J,\eta,\bar{\eta},\zeta,\bar{\zeta}] = \frac{1}{\mathcal{N}} \int \mathcal{D}A\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\mathcal{D}\bar{c}\,\mathcal{D}c\; e^{i\int d^4x \{\mathcal{L}_{eff} + A^\mu J_\mu + \bar{\psi}\eta + \bar{\eta}\psi + \bar{c}\zeta + \bar{\zeta}c\}} \; , \tag{4.230}$$

where

$$\mathcal{N} = \text{nominator}_{J=\eta=\bar{\eta}=\zeta=\bar{\zeta}=0} \; . \tag{4.231}$$

The Green functions are given by

$$\langle 0|T[\hat{A}^a_\mu(x)...\hat{\psi}_{s_j}(y)...\hat{\bar{\psi}}_{r_l}(z)...\hat{c}_b(u)...\hat{\bar{c}}_d(v)]|0\rangle \tag{4.232}$$
$$= \frac{1}{i}\frac{\delta}{\delta J^{a\mu}(x)}...\frac{1}{i}\frac{\delta}{\bar{\eta}_{s_j}(y)}...\frac{1}{-i}\frac{\delta}{\eta_{r_l}(z)}...\frac{1}{i}\frac{\delta}{\delta\bar{\zeta}_b(u)}...\frac{1}{-i}\frac{\delta}{\delta\zeta_d(v)}\mathcal{Z}|\text{sources}_{=0} \; . \tag{4.233}$$

The generating functional for connected Green functions is given by

$$W[J,\eta,\bar{\eta},\zeta,\bar{\zeta}] = \frac{1}{i}\ln \mathcal{Z}[J,\eta,\bar{\eta},\zeta,\bar{\zeta}] \; . \tag{4.234}$$

Example: We consider the Abelian $U(1)$ gauge theory with covariant gauge,

$$\partial^\mu A_\mu = 0 \; . \tag{4.235}$$

Thereby we have, as $f_{abc} = 0$,

$$M_{ab}(x,y) = -\partial^\mu \partial_\mu \delta^{(4)}(x-y) \; . \tag{4.236}$$

The generating functional for physical Green functions is then given by

$$\begin{aligned}\mathcal{Z}[J,\eta,\bar{\eta}] &= \int \mathcal{D}A\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\, e^{i\int d^4x\,\{\mathcal{L}_{class}-\frac{1}{2\xi}(\partial_\mu A^\mu)^2+JA+\bar{\psi}\eta+\bar{\eta}\psi\}} \\ &\times \int \mathcal{D}\bar{c}\,\mathcal{D}c\, e^{i\int d^4x\,\partial^\mu\bar{c}\partial_\mu c} \\ &\times \left\{\int \mathcal{D}A\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\, e^{i\int d^4x\,\{\mathcal{L}_{class}-\frac{1}{2\xi}(\partial_\mu A^\mu)^2\}} \int \mathcal{D}\bar{c}\,\mathcal{D}c\, e^{i\int d^4x\,\partial^\mu\bar{c}\partial_\mu c}\right\}^{-1} .\end{aligned} \tag{4.237}$$

Since there is no ghost-photon interaction, the ghost part factorizes and cancels against the denominator. If a non-linear gauge fixing condition is chosen, however, there are also ghosts in the Abelian case.

Non-Abelian gauge theory, axial gauges:

$$F[A_\mu^a] = n^\mu A_\mu^a = 0 , \tag{4.238}$$

where

$$n^\mu = \text{ const. } 4-\text{vector} \qquad \text{often } \; n^\mu = (0,0,0,1) . \tag{4.239}$$

The Faddeev-Popov determinant Δ is then independent of A_μ^a and the ghost part in the nominator of $\mathcal{Z}$ cancels against the denominator. This gauge, however, leads to a complicated gauge field propagator.

4.9 Green Function in Perturbation Theory

We will use in the following the covariant gauge fixing. We decompose the action, i.e.

$$S = \int d^4x\,\mathcal{L}_{eff} = \int d^4x\,\mathcal{L}_0 + \int d^4x\,\mathcal{L}_I . \tag{4.240}$$

Here is with

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f_{abc} A_\mu^b A_\nu^c \tag{4.241}$$

and after partial integration

$$\mathcal{L}_0 = \frac{1}{2} A_\mu^a \left(g^{\mu\nu}\partial^2 - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\right) A_\nu^a + \bar{\psi}_l(i\partial\!\!\!/ - m)\psi_l - \bar{c}_a\partial^2 c_a \tag{4.242}$$

and

$$\begin{aligned}\mathcal{L}_I &= \mathcal{L}_I(A,\bar{\psi},\psi,\bar{c},c) \\ &= \frac{g}{2} f_{abc}(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}) A_\mu^b A_\nu^c - \frac{g^2}{4} f_{abe} f_{cde} A^{a\mu} A^{b\nu} A_\mu^c A_\nu^d + g f_{abc}(\partial_\mu \bar{c}_a) c_b A^{c\mu} \\ &\quad - g\bar{\psi}_j T_{jl}^a \gamma^\mu \psi_l A_\mu^a , \qquad j,l = 1,...,N,\ a,b,c,d,e = 1,...,N^2-1 .\end{aligned} \tag{4.243}$$

Assuming that the coupling g is small we can expand e^{iS} and obtain

$$e^{i\int d^4x\,(\mathcal{L}_{eff}+\text{ source terms })} = \left\{1 + i\int d^4x\,\mathcal{L}_I(x) + \frac{i^2}{2}\left(\int d^4z\,\mathcal{L}_I(z)\right)^2 + ...\right\} e^{i\int d^4x\,(\mathcal{L}_0+\text{ source terms })} . \tag{4.244}$$

And thereby we have the central formula of functional perturbation theory

$$\mathcal{Z}[J,\eta,\bar\eta,\zeta,\bar\zeta] = \frac{e^{i\int d^4z\,\mathcal{L}_I(\frac{1}{i}\frac{\delta}{\delta J},-\frac{1}{i}\frac{\delta}{\delta\eta},\frac{1}{i}\frac{\delta}{\delta\bar\eta},\frac{1}{-i}\frac{\delta}{\delta\zeta},\frac{1}{i}\frac{\delta}{\delta\bar\zeta})}\,\mathcal{Z}_0[J,\eta,\bar\eta,\zeta,\bar\zeta]}{\text{nominator}|_{\text{sources}=0}} \,. \tag{4.245}$$

Here the normalized generating functional of the free theory is given by

$$\begin{aligned}\mathcal{Z}_0 &= \frac{\int\mathcal{D}A\,\mathcal{D}\bar\psi\,\mathcal{D}\psi\,\mathcal{D}\bar c\,\mathcal{D}c\,e^{i\int d^4x\,\{\mathcal{L}_0(x)+JA+\bar\psi\eta+\bar\eta\psi+\bar\zeta c+\bar c\zeta\}}}{\text{nominator}|_{\text{sources}=0}}\\ &= e^{-i\int d^4x\,d^4y\,\left\{\frac{1}{2}J^\mu_a(x)D^{ab}_{F\mu\nu}(x,y)J^\nu_b(y)+\bar\eta_{rl}(x)S^{lj}_{Frs}(x-y)\eta_{sj}(y)+\bar\zeta_a(x)\Delta^{ab}_F(x-y,m=0)\zeta_b(y)\right\}}\,,\end{aligned} \tag{4.246}$$

with the causal Green function $D_{F\mu\nu}$, for which holds

$$\begin{aligned}&\left(\partial^2 g^\mu_\alpha - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial_\alpha\right)_x \delta_{ac}D^{cb\,\alpha\nu}_F(x-y) = \delta_{ab}\delta^{(4)}(x-y)g^{\mu\nu}\\ &D^{ab}_{F\,\mu\nu}(x-y) = \lim_{\epsilon\to0+}\int\frac{d^4k}{(2\pi)^4}e^{-ik(x-y)}\left[-\frac{g_{\mu\nu}}{k^2+i\epsilon}+(1-\xi)\frac{k_\mu k_\nu}{(k^2+i\epsilon)^2}\right]\delta_{ab}\,.\end{aligned} \tag{4.247}$$

We have the following gauge parameters:

$$\begin{array}{ll}\xi=1: & \text{Feynman gauge}\\ \xi=0: & \text{Landau gauge}\,.\end{array} \tag{4.248}$$

For the ghost propagator we have

$$\begin{aligned}&\delta_{ac}\partial^2\Delta^{cb}_F(x-y,m=0) = -\delta_{ab}\delta^{(4)}(x-y) \quad\Rightarrow\\ &\Delta^{ab}_F(x-y) = \lim_{\epsilon\to0}\int\frac{d^4k}{(2\pi)^4}\frac{\delta_{ab}}{k^2+i\epsilon}e^{-ik(x-y)}\,.\end{aligned} \tag{4.249}$$

And for the fermion propagator

$$\begin{aligned}&(i\gamma^\mu\partial_\mu - m)_{rs'}\delta_{jl'}S^{l'l}_{F\,s's}(x-y) = \delta_{jl}\delta_{rs}\delta^{(4)}(x-y) \quad\Rightarrow\\ &S^{lj}_{F\,rs}(x-y) = \lim_{\epsilon\to0}\int\frac{d^4k}{(2\pi)^4}\frac{(\not{k}+m)_{rs}\delta_{lj}}{k^2-m^2+i\epsilon}e^{-ik(x-y)}\,.\end{aligned} \tag{4.250}$$

<u>Check:</u> The 2-point function of the free theory is given by

$$\langle 0|T[\hat A^a_\mu(x)\hat A^b_\nu(y)]|0\rangle_{\text{free}} = \frac{1}{i^2}\frac{\delta^2\mathcal{Z}_0}{\delta J^{a\mu}(x)\delta J^{b\nu}(y)}\Big|_{\text{sources}=0} = iD^{ab}_{F\,\mu\nu}(x-y) \tag{4.251}$$

$$\langle 0|T[\hat\psi_{rj}(x)\hat{\bar\psi}_{sl}(y)]|0\rangle_{\text{free}} = iS^{lj}_{F\,rs}(x-y) \tag{4.252}$$

$$\langle 0|T[\hat c_a(x)\hat{\bar c}_b(y)]|0\rangle_{\text{free}} = i\Delta^{ab}_F(x-y,m=0)\,. \tag{4.253}$$

In the following we describe how a T-matrix element $\mathcal{T}_{fi}$ is calculated. For this we consider the process $i\to f$ up to order g^n. We have

$$\langle f|(\hat S - I)|i\rangle = i\mathcal{T}_{fi}(2\pi)^4\delta^{(4)}(p_f-p_i)\,. \tag{4.254}$$

The procedure is as follows

1. Determine the functional $\mathcal{Z}$, respectively W, up to order g^n.

2. Determine the Green function corresponding to the process

$$G^{a...l...j}_{conn.\,\mu...r...s}(x_1,...,x_n) = \langle 0|T[A^a_\mu(x),...,\psi_{rl},...,\bar{\psi}_{sj}]|0\rangle_{\text{connected}} \tag{4.255}$$

through functional derivation of W.

3. Truncate, i.e. multiply with the inverse <u>free</u> propagators.
This delivers

$$G_{\text{connected,truncated}}(x_1,...,x_n)\ . \tag{4.256}$$

4. Fourier transformation

$$\tilde{G}(p_1,...,p_n)(2\pi)^4\delta(p_f-p_i) = \int \Pi^n_{j=1} d^4x_j e^{-i\sum p_j x^j} G(x_j)\ . \tag{4.257}$$

5. Multiplication with the external wave functions and going on-shell,

$$\Rightarrow i\mathcal{T}_{fi} = \lim_{p_j^2\to m_j^2} \tilde{G}_{\text{trunc}}\ . \tag{4.258}$$

4.10 The Feynman Rules of Quantum Chromodynamics

We have quarks q with spinor index $r=1,...,4$ and color index $l=1,...,3$, $q_{rl}(x)$. Quarks are triplets w.r.t. the $SU(3)$. We have gluons G with the Lorentz index $\mu=1,...,4$ and the color index $a=1,...,8$, $G^a_\mu(x)$. They are massless spin-1 fields. The Fourier decomposition is given by

$$G^a_\mu(x) = \int \frac{d^3k}{2k_0(2\pi)^4} \sum_\lambda \{e^{ikx}\epsilon^{*a}_\mu(\lambda)\alpha^{a\dagger}(k,\lambda) + e^{-ikx}\epsilon^a_\mu(\lambda)\alpha^a(k,\lambda)\}\ , \tag{4.259}$$

where ϵ_μ is the polarization vector and $\lambda=1,...,4$ denotes the polarization index. The $\alpha^{(\dagger)}$ are the annihilators (creators). The polarization indices $\lambda=1,2$ denote physical polarization states and the indices 3,4 unphysical polarization states. For the creators and annihilators it holds that

$$[\alpha^a(k,\lambda),\alpha^{\dagger b}(k',\lambda')] = -(2\pi)^3\, 2k_0\, g_{\lambda\lambda'}\delta_{ab}\delta^{(3)}(\vec{k}-\vec{k}')\ . \tag{4.260}$$

The polarization sum is given by

$$\sum_{\text{physical pol. }\lambda=1,2} \epsilon^{*a}_\mu(k,\lambda)\epsilon^b_\nu(k,\lambda) = \delta_{ab}\left(-g_{\mu\nu} + \frac{k_\mu\bar{k}_\nu + \bar{k}_\mu k_\nu}{k\bar{k}}\right)\ , \tag{4.261}$$

where $k=(k_0,\vec{k})$ und $\bar{k}=(k_0,-\vec{k})$. Because of the conservation of the fermion current we had in quantum electrodynamics (QED) (QED)

$$\sum_{\text{physical. pol. }\lambda=1,2} \epsilon^*_\mu(k,\lambda)\epsilon_\nu(k,\lambda) = -g_{\mu\nu} \qquad \text{for QED}\ . \tag{4.262}$$

In QCD this rule <u>does not</u> hold any more. For the ghost fields we have the Fourier decomposition

$$c^a(x) = \int \frac{d^3k}{2k_0(2\pi)^3}\left[f^a(k)e^{-ikx} + d^{\dagger a}(k)e^{ikx}\right] \ , \tag{4.263}$$

where $f^a(k)$ ($d^{\dagger a}$) destroys (generates) a ghost and $a = 1, ..., 8$. For the quark and antiquark spinors it holds (s, s' spinor indices, l colour index)

$$(\not{p} - m)_{ss'} u_{ls'}(p, r) \ = \ 0 \tag{4.264}$$
$$(\not{p} + m)_{ss'} v_{ls'}(p, r) \ = \ 0 \qquad \text{where } r = \pm\frac{1}{2} \ . \tag{4.265}$$

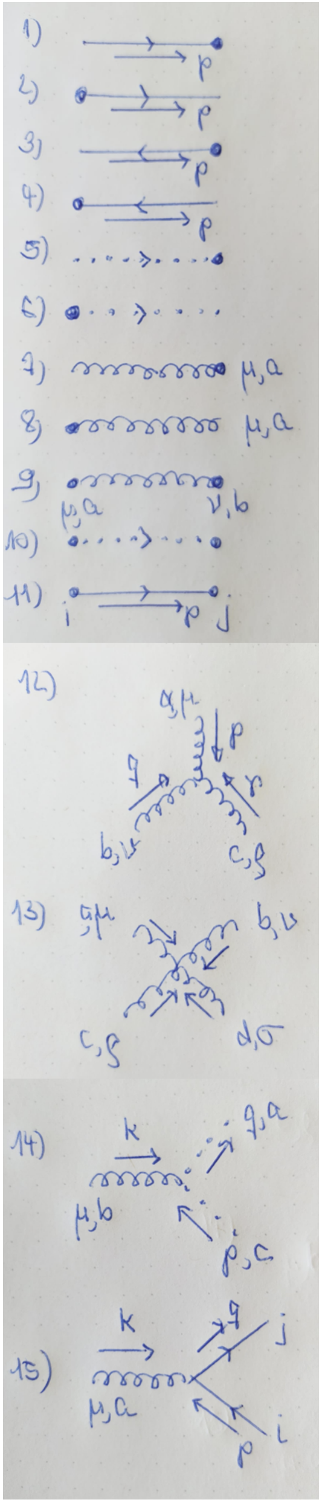
They are normalized as (r, r' spin indices, l, j color indices)

$$\bar{u}_l(p, r) \cdot u_j(p, r') \ = \ 2m\delta_{rr'}\delta_{lj} \tag{4.266}$$
$$\bar{v}_l(p, r) \cdot v_j(p, r') \ = \ -2m\delta_{rr'}\delta_{lj} \ , \quad l, j = 1, 2, 3 \quad r, r' = \pm\frac{1}{2} \ . \tag{4.267}$$

Furthermore it holds that

$$\sum_{r=\pm 1/2} \bar{u}_{ls}(p, r) u_{js'}(p, r) \ = \ \delta_{lj}(\not{p} + m)_{ss'} \tag{4.268}$$
$$\sum_{r=\pm 1/2} \bar{v}_{ls}(p, r) v_{js'}(p, r) \ = \ \delta_{lj}(\not{p} - m)_{ss'} \ . \tag{4.269}$$

Feynman rules: An example for the derivation of the Feynman rules will be treated in the exercise sheet. Altogether the Feynman rules of QCD are given by



$u(p)$ quark in the initial state

$\bar{u}(p)$ quar in the final state

$\bar{v}(p)$ antiquark in the initial state

$v(p)$ antiquark in the final state

1 ghost in the initial or antighost in the final state

1 ghost in the final or antighost in the initial state

$\epsilon^{\mu,a}$ gluon in the initial state

$\epsilon^{*\mu,a}$ gluon in the final state

$\delta^{ab}[-g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2+i\epsilon}]\frac{i}{p^2+i\epsilon}$

$\delta^{ab}\frac{i}{(p^2+i\epsilon)}$

$\delta^{lj}\frac{i}{(\not{p}-m+i\epsilon)_{sr}}$ (4.270)

$-g_s f^{abc}[(p-q)^\rho g^{\mu\nu} + (q-r)^\mu g^{\nu\rho} + (r-p)^\nu g^{\rho\mu}]$
3-gluon vertex (all momenta incoming), $p+q+r=0$]

$-ig_s^2 f^{xac} f^{xbd}[g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}]$
$-ig_s^2 f^{xad} f^{xbc}[g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}]$
$-ig_s^2 f^{xab} f^{xcd}[g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}]$
4-gluon vertex (all momenta incoming),
sum of all moment $= 0$]

$g_s f^{abc} q^\mu$
ghost-ghost-gluon vertex, momenta p, k of gluon and ghost incoming, momentum q of ghost outgoing

$-ig_s(t^a)_{jl}\gamma^\mu$
quark-quark-gluon vertex, momenta p, k of quark and gluon incoming, momentum q of quark outgoing
l, j color index of the incoming quark, of the outgoing quark

Remarks: For each closed fermion loop and each closed ghost loop a factor (-1) has to be added. For closed gluon loops a statistical factor has to be added. It is obtained by counting all possible contractions of field operators in perturbation theory, cf. Fig. 4.6.

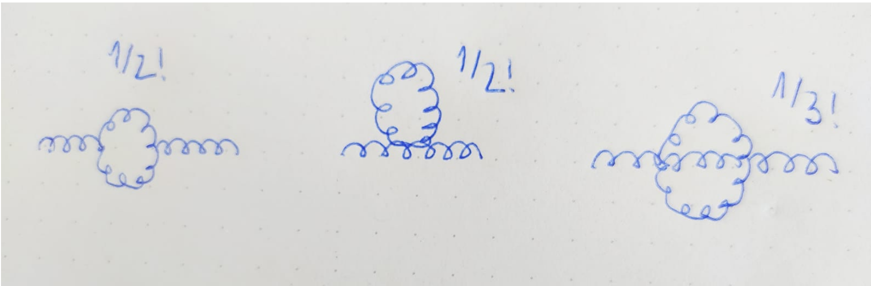


Figure 4.6: Statistical factors for gluon loops.

Chapter 5

Renormalisation

So far we have only considered diagrams at tree level, respectively Born level. We have to ask ourselves if the quantisation of non-abelian gauge theories and the derivation of the Feynman rules is still consistent when we look at diagrams which contain closed loops, i.e. when we look at the theory beyond Born level.

Actually, some of the loop integrals (cf. e.g. Fig. 7.1) exhibit ultraviolet (UV) divergences. This means that these diagrams have divergences for loop momenta $|l| \to \infty$, *i.e.* $|l_0| \to \infty$ (energy), This problem is solved through the process of renormalisation, which means nothing else but a substraction prescription for divergent amplitudes. Let us look at the diagrams in Fig. 7.1.

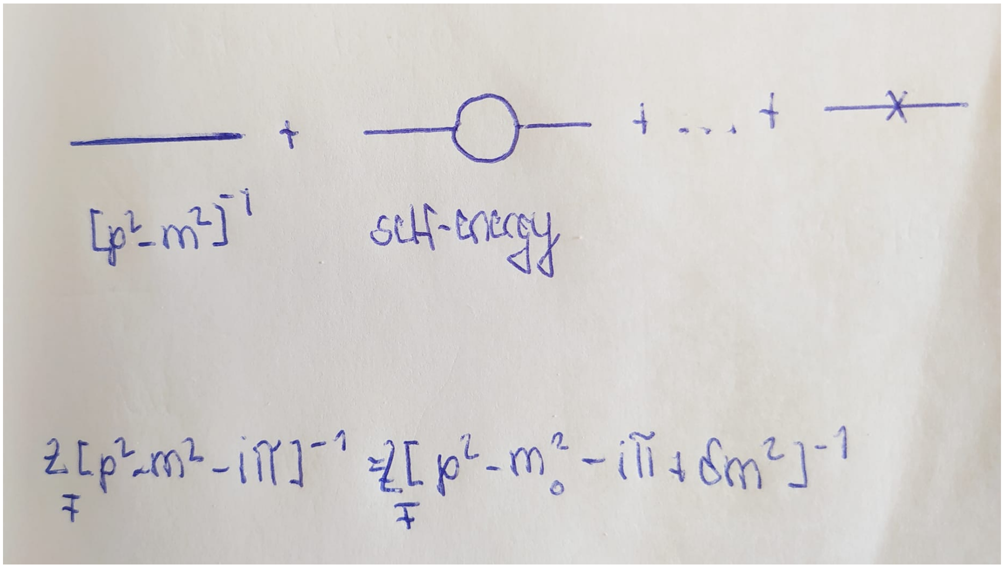


Figure 5.1: Higher-order corrected propagator.

(i) The self-energy (propagator) correction leads to a change of the mass.

(ii) The temporary splitting leads to a change of the wave function renormalisation and the coupling of the particle.

Renormalisation: The mass of the particle, however, is fixed through the experiments, which are defined at an energy scale μ. (Physicsal results are independent of the scale $\mu \leftarrow$ renormalisation group equations.) Renormalisation hence means to define with which prescription the parameters are measured. This is a consequence of the quantum fluctuations. Renormalisation means illustrated the following: Let us look at the photon-electron-positron interaction, cf. Fig. 5.2. In reality, we measure the full vertex, i.e. the tree-level vertex plus all diagrams of higher order. A possible renormalisation prescription is that the electric charge is defined as the complete photon-electron-positron coupling for on-shell particles (hence particles on the mass shell) in the Thomson limit.

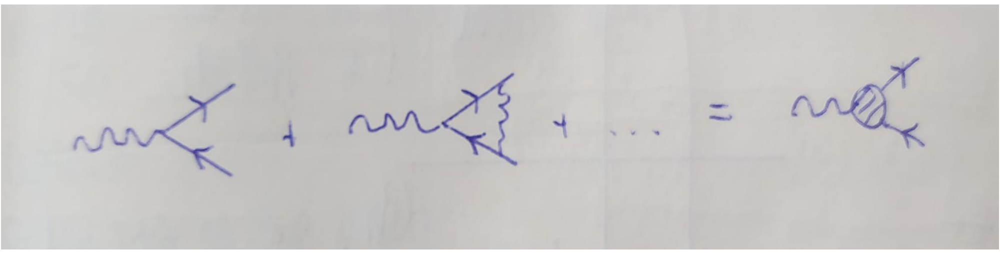


Figure 5.2: Loop corrected photon-electron-positron vertex. The dashed blob represents the full vertex.

We denote the classical parameters in the Lagrangian by m_0, g_0, ϕ_0, so that we have $\mathcal{L}(\phi_0; m_0, g_0)$. These parameters are called *bare parameters*. The *renormalised* parameters are denoted by ϕ_R, m_R, g_R and are related to the bare parameters through

$$\begin{aligned} m_0 &= m_R + \delta m \\ g_0 &= Z_g g_R = [1 + \delta Z_g] g_R \\ \phi_0 &= Z_\phi^{1/2} \phi_R = [1 + \delta Z_\phi]^{1/2} \phi_R \; , \end{aligned} \tag{5.1}$$

where the Z_i are dimensionless renormalisation constants. The quantumechanical Lagrangican can be written as

$$\begin{aligned} \mathcal{L}(\phi_0; m_0, g_0) &= \mathcal{L}(Z_\phi^{1/2} \phi_R; m_R + \delta m, Z_g g_R) \\ &= \mathcal{L}_R(\phi_R; m_R, g_R) + \mathcal{L}_{\text{counter}} \; . \end{aligned} \tag{5.2}$$

The former denotes the *renormalised Lagrangian* and the latter the so-called *counterterm Lagrangian.* The Feynman dagrams are calculated with the Feynman rules that are obtained from the renormalised Lagrangian $\mathcal{L}_R$ and the counterterm Lagrangian $\mathcal{L}_{\text{counter}}$. The kinetic part for the field ϕ looks e.g. like

$$\begin{aligned} \frac{1}{2} \partial_\mu \phi_0 \partial^\mu \phi_0 &= \frac{1}{2} Z_\phi \partial_\mu \phi_R \partial^\mu \phi_R \\ &= \frac{1}{2} \partial_\mu \phi_R \partial^\mu \phi_R + \frac{1}{2} (Z_\phi - 1) \partial_\mu \phi_R \partial^\mu \phi_R \equiv \mathcal{L}_R^{kin} + \mathcal{L}_{\text{counter}}^{\text{kin}} \; . \end{aligned} \tag{5.3}$$

Further remarks: Before we go into details, further remarks are at order: A theory is called renormailsable, if the appearing UV divergences can be cancelled through the process of renormalisation. This means that the number of *independent types* of UV divergences must be finite. And by setting the renormalised quantities equal to the measured values, the results of other experiments can be predicted. The proof of the renormalisability of a theory is highly non-trivial. The physicists 't Hooft und Veltman have solved this problem for spontaneously broken gauge theories. They showed in the early 70's how the Glashow-Salam-Weinberg (GSW) theory can be renormalised. and how this theory is to be used to perform precision calculations. They obtained the Nobel Prize in 1999 "for elucidating the quantum structure of electroweak interactions in physics".

GSW Parameter: There are several schemes that are used to express the parameters of the GSW theory. In the

(i) *On-shell scheme* the measured parameters are $\alpha, M_W, M_Z, m_f, m_H$. All other parameters are derived from these parameters.

(ii) G_F *scheme:* The input parameters are $\alpha, G_F, M_Z, m_f, m_H$.

(iii) $\overline{MS}$ *Schema:* This renormalisation scheme is often used in the renormalisation of QCD.

Regularisation: Regularisation is a prescription how to isolate the divergences that appear in higher-order corrections and which need to be cancelled through the process of renormalisation. The diagram shown in Fig. 5.3 leads e.g. according to simple power counting to a logarithmic divergence. Because

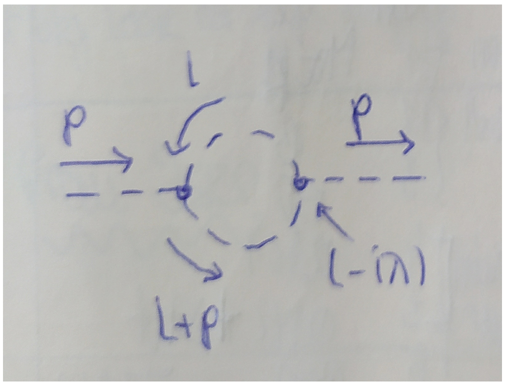


Figure 5.3: Scalar self-energy diagram.

$$\frac{1}{i}\Pi(p) = (-i\lambda)^2 \int \frac{d^4l}{(2\pi)^4} \frac{i^2}{[(p+l)^2 - m^2 + i\epsilon][l^2 - m^2 + i\epsilon]} \, . \tag{5.4}$$

We perform an analytic continuation into the Eucledian space,

$$\begin{aligned} l_0 &= il_4 \quad \Rightarrow \quad l^2 = l_0^2 - \vec{l}^2 = -l_E^2 \\ \int d^4l &= i \int d^4l_E = i\int_0^\infty |l_E|^3 d|l_E| d\Omega_4 \, , \end{aligned} \tag{5.5}$$

where $d\Omega_4$ denotes the 4-dimensional solid-angle-element. And

$$|l_E| = \sqrt{l_4^2 + \vec{l}^2} \, . \tag{5.6}$$

Thereby we have the

$$\text{integral} \sim \int_0^\infty \frac{|l_E|^3 d|l_E|}{|l_E|^4} \sim \int_0^\infty \frac{d|l_E|}{|l_E|} \,. \tag{5.7}$$

It is logarithmically divergent. The divergence appears for large $|l_E|$ and leads to UV divergence. In the following we give examples of regularisation schemes.

Momentum cut-off Λ (not suitable for gauge theories):

$$\int^\infty \to \int^\Lambda \frac{d|l_E|}{|l_E|} \sim \ln\Lambda \,. \tag{5.8}$$

Measurable physical quantities of course must not depend on the cut-off. The naive introduction of a cut-off breaks gauge invariance. Better suited is:

Dimensional regularisation: In dimensional regularisation the divergences are isolated by defining the theory in $D = 4 - 2\epsilon \neq 4$ dimensions. The divergences then appear as poles in ϵ: $\sim 1/\epsilon^{(n)}$, $\epsilon \to 0$.

Rules: The rules for performing a calculation in $D \neq 4$ dimensions are:

- The integral over the loop momentum q is replaced in the following way:

$$\int \frac{d^4 q}{(2\pi)^4} \to \mu^{4-D} \int \frac{d^D q}{(2\pi)^D} \,, \tag{5.9}$$

  where μ has the dimension of a mass. The introduction of the mass μ in D dimensions keeps the integral in the same mass dimension as in $D = 4$ dimensions. Thereby we get for the example above

$$\int d^4 l_E \to \int d^D l_E = \int |l_E|^{D-1} d|l_E| \int d\Omega_D \,, \tag{5.10}$$

  where Ω_D is the solid-angle-element in D dimensions. And for $D < 4$ we have

$$\int d|l_E| \frac{|l_E|^{D-1}}{|l_E|^4} \qquad \text{UV convergent} \,. \tag{5.11}$$

  Remark: Mathematically the origin of the UV divergences is the product of delta distributions with the same argument.

- The coupling constant g^2 is replaced by

$$g^2 \to g^2 \mu^{4-D} \tag{5.12}$$

  This is done in order to keep the dimension of the Green function unchanged.

- For the metric we have

$$g_{\mu\nu} \quad \text{is } D\text{-dimensional, i.e.} \quad g_{\mu\mu} = D \,. \tag{5.13}$$

- The Dirac matrices are generalised such that

$$\begin{aligned}\{\gamma_\mu, \gamma_\nu\} &= 2g_{\mu\nu}\mathbf{1} \qquad \gamma_\mu\gamma_\mu = D\mathbf{1} \\ \gamma_\rho\gamma_\mu\gamma_\rho &= 2g_{\mu\rho}\gamma_\rho - \gamma_\mu\gamma_\rho\gamma_\rho = (2-D)\gamma_\mu \quad \text{etc.} \\ \gamma_5 & \quad \text{is non-trivial in } D \text{ dimensions.}\end{aligned} \tag{5.14}$$

<u>1-point function:</u>
For illustration we compute the 1-point function (i.e. an integral that only contains one propagator integral over the loop momentum) in D dimensions. We hence have

$$\frac{i}{16\pi^2}A = \mu^{4-D}\int \frac{d^Dq}{(2\pi)^D}\frac{1}{q^2-m^2+i\epsilon} \tag{5.15}$$

We compute the integral in Eucledian spacetime dimensioins and for this perform a Wick rotation,

$$q^0 = iq_E^0 \qquad \vec{q} = \vec{q}_E \qquad d^Dq = id^Dq_E \;. \tag{5.16}$$

The integral becomes

$$\frac{iA}{16\pi^2} = -i\frac{\mu^{4-D}}{(2\pi)^D}\int \frac{d^Dq_E}{q_E^2+m^2} \;. \tag{5.17}$$

Explanations: We have poles in the q_0 plane:

$$\begin{aligned}0 &= q^2 - m^2 + i\epsilon = q_0^2 - \vec{q}^2 - m^2 + i\epsilon \\ q_0 &= \pm\sqrt{\vec{q}^2 + m^2 - i\epsilon} = \pm\sqrt{\vec{q}^2+m^2} \mp i\epsilon' \;.\end{aligned} \tag{5.18}$$

We look at the integral over the curve $\mathcal{C}$, cf. Fig. 5.4.

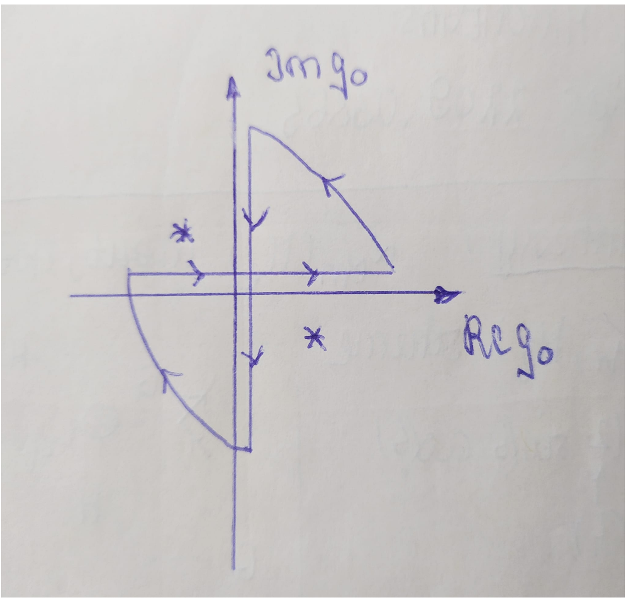


Figure 5.4: The curve $\mathcal{C}$ for the ntegration over the loop momentum.

$$\oint_{\mathcal{C}} dq_0(q^2 - m^2 + i\epsilon)^{-1} = 0 \tag{5.19}$$

The contributions over the circle segments disappear so that we have

$$\int_{-\infty}^{\infty} dq_0 ... = \int_{-i\infty}^{i\infty} dq_0 ... \,. \tag{5.20}$$

And thereby we have in Eucledian coordinates ($q^0 = iq_E^0, q^k = q_E^k$),

$$\int_{-i\infty}^{i\infty} dq_0 ... = i\int_{-\infty}^{\infty} dq_E^0 ... \,. \tag{5.21}$$

For the further computation of the integral we perform a transformation to spherical coordinates

$$\int d^D q_E = \int_{\Omega_D} d\Omega_D \int_0^\infty dq_E q_E^{D-1} = \int_{\Omega_D} d\Omega_D \int_0^\infty dq_E^2 \frac{1}{2}(q_E^2)^{D/2-1} \,, \tag{5.22}$$

where

$$\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)} \tag{5.23}$$

is the D-dimensional solid angle. And thereby we obtain for the integral

$$\begin{aligned}
\frac{i}{16\pi^2} A_{n-1} &= i\frac{\mu^{4-D}}{(2\pi)^D}\frac{1}{2}\int d\Omega_D \int_0^\infty d\rho\, \rho^{\frac{D}{2}-1}\frac{(-1)^n}{(\rho + L - i\epsilon)^n} \\
&= i(-1)^n \frac{\mu^{4-D}}{(2\pi)^D}\,\frac{1}{2}\frac{2\pi^{D/2}}{\Gamma(D/2)}\int_0^\infty d\rho\, \rho^{\frac{D}{2}-1}\frac{1}{(\rho + L - i\epsilon)^n} \\
&\overset{\rho=Ly}{=} i(-1)^n \frac{\mu^{4-D}}{(4\pi)^{\frac{D}{2}}}\frac{1}{\Gamma(D/2)}(L - i\epsilon)^{D/2-n}\underbrace{\int_0^\infty dy\, y^{D/2-1}(1+y)^{-n}}_{=B(\frac{D}{2}, n-\frac{D}{2})}
\end{aligned} \tag{5.24}$$

The factor $(-1)^n$ stems from $(q^2 - m^2 + i\epsilon)^{-n} = (-1)^n(q_E^2 + m^2 - i\epsilon)^{-n}$. With the Beta function

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{5.25}$$

we finally obtain

$$\frac{i}{16\pi^2} A_{n-1} = i(-1)^n \frac{\mu^{4-D}}{(4\pi)^{\frac{D}{2}}}(L - i\epsilon)^{D/2-n}\frac{\Gamma(n - \frac{D}{2})}{\Gamma(n)} \,. \tag{5.26}$$

The $\Gamma(x)$ function has poles in $x = 0, -1, -2, ...$ An expansion for small ϵ leads to

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + \mathcal{O}(\epsilon) \qquad \text{with the Euler } \gamma\,, \;\; \gamma = 0.577... \,. \tag{5.27}$$

We furthermore use the expansion

$$a^\epsilon = e^{\epsilon \ln a} = 1 + \epsilon \ln a + ... \,. \tag{5.28}$$

We obtain for $A^{n=1}$ with $4-D=2\epsilon$ and $L\equiv m^2$

$$A=-\frac{\mu^{2\epsilon}}{(4\pi)^{-\epsilon}}\frac{\Gamma(-1+\epsilon)}{\Gamma(1)}(m^2)^{1-\epsilon}\,. \tag{5.29}$$

With $\Gamma(1)=1$ and

$$\Gamma(1+x)=x\Gamma(x) \tag{5.30}$$

and the above expansions we obtain

$$\begin{aligned}\frac{A(m^2)}{m^2} &= -(1+\epsilon\ln(4\pi)+...)(1-\epsilon\ln(\frac{m^2}{\mu^2})+...)(-1)(1+\epsilon)(\frac{1}{\epsilon}-\gamma+...)\\ \frac{A(m^2)}{m^2} &= \frac{1}{\epsilon}-\gamma+\ln 4\pi-\ln\frac{m^2}{\mu^2}+1+\mathcal{O}(\epsilon)\end{aligned} \tag{5.31}$$

Thereby we find

$$\boxed{\begin{aligned}\frac{A(m^2)}{m^2} &= \Delta-\ln\frac{m^2}{\mu^2}+1\\ \Delta &= \frac{1}{\epsilon}-\gamma+\ln 4\pi \qquad (5.32)\end{aligned}}$$

Mass renormalisation: As example, we now look at the mass renormalisation for a scalar particle. The full propagator is given by (cf. Fig. 5.5)

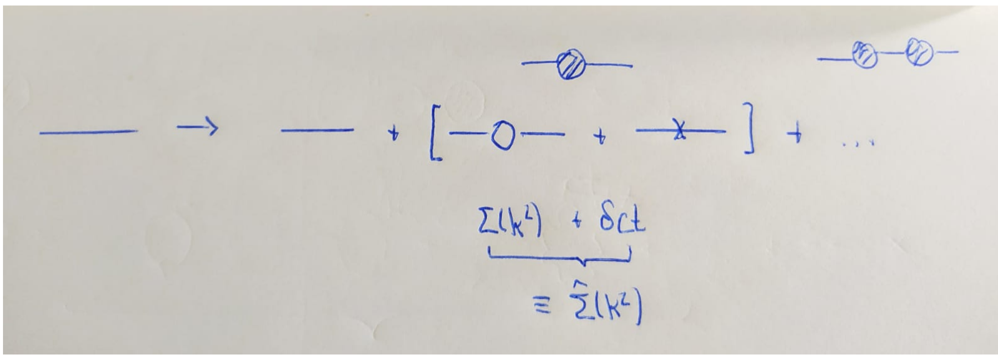


Figure 5.5: The propagator of the scalar particle including higher-order corrections.

$$\begin{aligned}\frac{1}{k^2-m^2} &\to \frac{1}{k^2-m^2}+\frac{1}{k^2-m^2}[\delta\mathrm{ct}+\Sigma]\frac{1}{k^2-m^2}+\frac{1}{k^2-m^2}\{[\delta\mathrm{ct}+\Sigma]\frac{1}{k^2-m^2}\}^2+...\\ &= \frac{1}{k^2-m^2}\{1+[\frac{\delta\mathrm{ct}+\Sigma}{k^2-m^2}]+[...]^2+...\}\\ &= \frac{1}{k^2-m^2}\frac{1}{1-\frac{\delta\mathrm{ct}+\Sigma}{k^2-m^2}}=\frac{1}{k^2-m^2-[\delta\mathrm{ct}+\Sigma]}=\frac{1}{k^2-m^2-\hat{\Sigma}(k^2)}\end{aligned} \tag{5.33}$$

We here used the formula for the geometric series,

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q} \; . \tag{5.34}$$

With the on-shell renormalisation condition

$$\begin{aligned} &\text{Renorm. Propagator } \frac{1}{k^2 - m^2 - \hat{\Sigma}(k^2)} \to \frac{1}{k^2 - m^2} \qquad \text{for} \quad k^2 \to m^2 \\ \rightsquigarrow \quad & \hat{\Sigma}(k^2)\Big|_{k^2=m2} = [\Sigma(k^2) - \delta m^2 + (k^2 - m^2)\delta Z]\big|_{k^2=m^2} = 0 \end{aligned} \tag{5.35}$$

we obtain (m is here the physical mass)

$$\delta m^2 - \Sigma(k^2 = m^2) = 0 \; . \tag{5.36}$$

5.1 Renormalisation of the GSW Theory

5.1.1 Renormalisation Constants

For the renormalisation of the GSW theory we have the following renormalisation constants (for simplicity we here assume the CKM matrix as unity matrix):

$$\begin{array}{lcllcl} e_0 & = & Z_e e = (1 + \delta Z_e) e & W_0^{\pm} & = & \sqrt{Z_W} W^{\pm} = (1 + \frac{1}{2}\delta Z_W) W^{\pm} \\ M_{W0}^2 & = & M_W^2 + \delta M_W^2 & H_0 & = & \sqrt{Z_H} H = (1 + \frac{1}{2}\delta Z_H) H \\ M_{Z0}^2 & = & M_Z^2 + \delta M_Z^2 & f_{L/R0} & = & \sqrt{Z_{L/R}} f_{L/R} = (1 + \frac{1}{2}\delta Z_{L/R}) f_{L/R} \\ M_{H0}^2 & = & M_H^2 + \delta M_H^2 & \begin{pmatrix} Z_0 \\ A_0 \end{pmatrix} & = & \begin{pmatrix} \sqrt{Z_{ZZ}} & \sqrt{Z_{ZA}} \\ \sqrt{Z_{AZ}} & \sqrt{Z_{AA}} \end{pmatrix} \begin{pmatrix} Z \\ A \end{pmatrix} \\ m_{f0} & = & m_f + \delta m_f & & & \end{array}$$

It is immediately evident that due to the higher-order corrections there appear mixings between the photon A and the Z boson. By choosing suitable renormalisation conditions, they are fixed to zero.

5.1.2 Renormalisation Conditions

The coefficients of the counterterms are completely determined by the renormalisation conditions. We start with the Higgs potential, which is given by

$$V = \frac{\lambda}{2}[|\phi|^2 - \frac{\mu^2}{\lambda}]^2 \; . \tag{5.37}$$

With the Higgs doublet

$$\phi = \begin{pmatrix} \phi^+ \\ \frac{1}{\sqrt{2}}(v + H + i\chi) \end{pmatrix} \tag{5.38}$$

we have

$$V = \frac{\lambda}{8}(v^2 - 2\frac{\mu^2}{\lambda})^2 + TH + ... \tag{5.39}$$

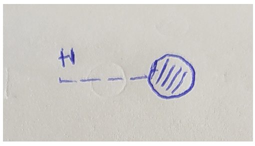
with

$$T = \frac{\lambda}{2} v (v^2 - 2\frac{\mu^2}{\lambda}) \tag{5.40}$$

From the minimisation of the Higgs potential $\frac{\partial V}{\partial H}\big|_{H=0} = 0$ we get the condition

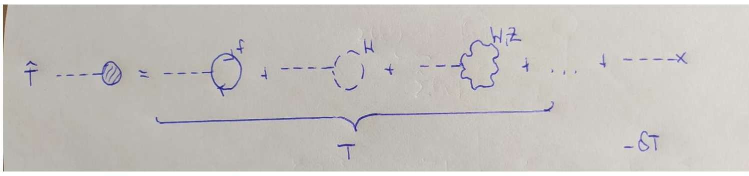
$$v^2 = 2\frac{\mu^2}{\lambda} \tag{5.41}$$

i.e. $T \overset{!}{\equiv} 0$.

$T =$ 

$= 0 \ . \qquad (5.42)$

Including the quantum fluctuations we have for the renormaised tapdole $\hat{T}$

$\hat{T} =$ 

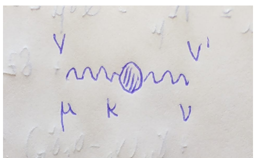
$= T - \delta T \qquad (5.43)$

We request that the renormalised tadpole $\hat{T} = T - \delta T$ is $\equiv 0$ and hence obtain the condition

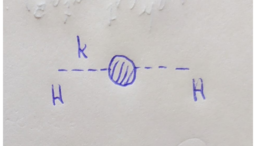
$$\delta T = T \ . \tag{5.44}$$

The tadpole diagrams are completely subtracted through the renormalisation of the Higgs potential and hence do not contribute to the physical observables.

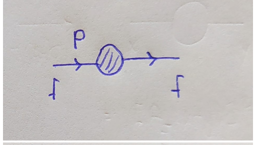
Definitions: We define the self-energies Σ and the photon-fermion-fermion vertex Γ as follows



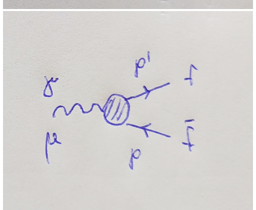
$= \quad \Sigma^{VV'}_{\mu\nu}(k) \qquad \text{for} \quad V = W^\pm, Z, A$



$= \quad \Sigma_H(k^2)$



$= \quad \Sigma_f(\not{p})$



$= \quad \Lambda^{ff\gamma}_{\mu}(q, p, p') \qquad (5.45)$

The vector boson self-energies can be decomposed into a transveral and a longitudinal contribution,

$$\Sigma_{\mu\nu}^{VV'}(k) = -i\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \underbrace{\Sigma_T^{VV'}(k^2)}_{\uparrow\,\text{transversal}} -i\frac{k_\mu k_\nu}{k^2} \underbrace{\Sigma_L^{VV'}(k^2)}_{\uparrow\,\text{longitudinal}} \quad . \tag{5.46}$$

The fermionic self-energies can be decomposed into a left-chiral, a right-chiral and a scalar part,

$$\Sigma_f(\not{p}) = i\left(\not{p}\left[\omega_- \underbrace{\Sigma_{f,L}(p^2)}_{\uparrow\,\text{left-chiral}} + \omega_+ \underbrace{\Sigma_{f,R}(p^2)}_{\uparrow\,\text{right-chiral}}\right] + m_f \underbrace{\Sigma_{f,S}(p^2)}_{\uparrow\,\text{scalar}}\right) \tag{5.47}$$

with

$$\omega_\pm = \frac{1 \pm \gamma_5}{2} \; . \tag{5.48}$$

Renormalisation conditions: With the condition that the renormalised mass parameters of the physical particles are equal to the physical masses, i.e. the real parts of the poles of the corresponding propagators, which are equivalent to the roots of the 1-particle irreducible 2-point functions, we have the following renormalisation conditions for the 2-point functions $\hat{\Gamma}$ for external on-shell fields,

$$\begin{aligned}
\mathrm{Re}\,\hat{\Gamma}_{\mu\nu}^{VV}(k)\,\epsilon^\nu(k)|_{k^2=M_V^2} &= 0 \qquad \text{for} \quad V = W, Z, A \\
\mathrm{Re}\,\hat{\Gamma}_H(k^2 = M_H^2) &= 0 \\
\mathrm{Re}\,\hat{\Gamma}_f(\not{p})u(p)|_{\not{p}=m_f} &= \mathrm{Re}\,\bar{u}(p)\hat{\Gamma}_f(\not{p})|_{\not{p}=m_f} = 0 \; .
\end{aligned} \tag{5.49}$$

In case of mass matrices we demand that the off-diagonal self-energies are zero, if the external lines are on their mass shells. For the diagonal entries we demand that the residua of the renormalised propagators are equal to 1. We obtain the renormalisation conditions

$$\begin{aligned}
\lim_{k^2\to M_V^2} \frac{1}{k^2 - M_V^2}\mathrm{Re}\hat{\Gamma}_{\mu\nu}^{VV}(k)\epsilon^\nu(k) &= -i\epsilon_\mu(k) \qquad \text{for} \quad V = W, Z \\
\lim_{k^2\to 0} \frac{1}{k^2}\hat{\Gamma}_{\mu\nu}^{AA}(k)\epsilon^\nu(k) &= -i\epsilon_\mu(k) \\
\hat{\Gamma}_{\mu\nu}^{AZ}(k)\epsilon^\nu(k)|_{k^2=0} &= \mathrm{Re}\,\hat{\Gamma}_{\mu\nu}^{AZ}\epsilon^\nu(k)|_{k^2=M_Z^2} = 0 \\
\lim_{k^2\to M_H^2} \frac{1}{k^2 - M_H^2}\mathrm{Re}\hat{\Gamma}^H(k) = i & \\
\lim_{p^2\to m_f^2} \left(\frac{\not{p} + m_f}{p^2 - m_f^2}\right)\mathrm{Re}\hat{\Gamma}_f(p)u(p) &= iu(p) \\
\lim_{p^2\to m_f^2} \bar{u}(p)\mathrm{Re}\hat{\Gamma}_f(p)\left(\frac{\not{p} + m_f}{p^2 - m_f^2}\right) &= i\bar{u}(p) \; .
\end{aligned} \tag{5.50}$$

And for the charge we have the renormalisation condition (e is the physical charge, which is measured in classical Thomson scattering $E_\gamma \to 0$)

$$\bar{u}(p')\hat{\Lambda}_\mu^{ff\gamma}(q,p,p')u(p)|_{p^2=p'^2=m_f^2, q^2=0} = -iee_f\bar{u}(p)\gamma_\mu u(p) \; . \tag{5.51}$$

$\hat{\Lambda}^{ff\gamma}$ denotes the renormalised photon-fermion-fermion vertex.

After plugging in the explicit decompositions of the self-energies we obtain the general expressions for the corresponding renormalisation conditions:

$$\begin{aligned}
\mathrm{Re}\ \hat{\Sigma}_T^{WW}(M_W^2) &= 0 \\
\mathrm{Re}\ \hat{\Sigma}_T^{ZZ}(M_Z^2) &= 0 \qquad\qquad \mathrm{Re}\ \hat{\Sigma}_T^{AZ}(M_Z^2) = 0 \\
\hat{\Sigma}_T^{AZ}(0) &= 0 \qquad\qquad \hat{\Sigma}_T^{AA}(0) = 0 \\
\mathrm{Re}\ \frac{\partial \hat{\Sigma}_T^{WW}(k^2)}{\partial k^2}|_{k^2=M_W^2} &= 0 \\
\mathrm{Re}\ \frac{\partial \hat{\Sigma}_T^{ZZ}(k^2)}{\partial k^2}|_{k^2=M_Z^2} &= 0 \qquad\qquad \mathrm{Re}\ \frac{\partial \hat{\Sigma}_T^{AA}(k^2)}{\partial k^2}|_{k^2=0} = 0 \\
\mathrm{Re}\ \hat{\Sigma}_H(M_H^2) &= 0 \qquad\qquad \mathrm{Re}\ \frac{\partial \hat{\Sigma}_H(k^2)}{\partial k^2}|_{k^2=M_H^2} = 0 \\
m_f \mathrm{Re}\ [\hat{\Sigma}_{f,L/R}(m_f^2) + \hat{\Sigma}_{f,S}(m_f^2)] &= 0 \\
\mathrm{Re}\ \{[\hat{\Sigma}_{f,L}(m_f^2) + \hat{\Sigma}_{f,R}(m_f^2) &+ 2m_f^2 \frac{\partial}{\partial p^2}[\hat{\Sigma}_{f,L}(p^2) + \hat{\Sigma}_{f,R}(p^2) + 2\hat{\Sigma}_{f,S}(p^2)]|_{p^2=m_f^2}\} = 0\ .
\end{aligned} \tag{5.52}$$

With the corresponding counterterms we have the explicit expressions for the renormalisation constants:

$$\begin{aligned}
\delta T &= T \\
\delta M_W^2 &= \mathrm{Re}\ \Sigma_T^{WW}(M_W^2) \qquad\qquad \delta Z_{WW} = -\mathrm{Re}\ \frac{\partial \Sigma_T^{WW}(k^2)}{\partial k^2}|_{k^2=M_W^2} \\
\delta M_Z^2 &= \mathrm{Re}\ \Sigma_T^{ZZ}(M_Z^2) \qquad\qquad \delta Z_{ZZ} = -\mathrm{Re}\ \frac{\partial \Sigma_T^{ZZ}(k^2)}{\partial k^2}|_{k^2=M_Z^2} \\
\delta Z_{AZ} &= -2\mathrm{Re}\ \frac{\Sigma_T^{AZ}(M_Z^2)}{M_Z^2} \qquad\qquad \delta Z_{ZA} = 2\frac{\Sigma_T^{AZ}(0)}{M_Z^2} \\
\delta Z_{AA} &= -\frac{\partial \Sigma_T^{AA}(k^2)}{\partial k^2}|_{k^2=0} \\
\delta M_H^2 &= \mathrm{Re}\Sigma_H(M_H^2) \qquad\qquad \delta Z_H = -\mathrm{Re}\frac{\partial \Sigma_H(k^2)}{\partial k^2}|_{k^2=M_H^2} \\
\delta m_f &= m_f \mathrm{Re}\ [\frac{\Sigma_{f,L}(m_f^2) + \Sigma_{f,R}(m_f^2)}{2} + \Sigma_{f,S}(m_f^2)] \\
\delta Z_{f,L} &= -\mathrm{Re}\ \Sigma_{f,L}(m_f^2) - m_f^2 \frac{\partial}{\partial p^2}\mathrm{Re}\ [\Sigma_{f,L}(p^2) + \Sigma_{f,R}(p^2) + 2\Sigma_{f,S}(p^2)]|_{p^2=m_f^2} \\
\delta Z_{f,R} &= -\mathrm{Re}\ \Sigma_{f,R}(m_f^2) - m_f^2 \frac{\partial}{\partial p^2}\mathrm{Re}\ [\Sigma_{f,L}(p^2) + \Sigma_{f,R}(p^2) + 2\Sigma_{f,S}(p^2)]|_{p^2=m_f^2}\ .
\end{aligned} \tag{5.53}$$

The full electromagnetic vertex is shown in Fig. 5.6. By using the Dirac equation and the Ward identitdy (which follows from the gauge invariance) we obtain the counterterm for the charge renormalisation (cf. A. Denner, Fortschr. Phys. 41 (1993) 307, arXiv:0709.1075 [hep-ph])

$$\delta Z_e = -\frac{1}{2}\delta Z_{AA} - \frac{s_W}{2c_W}\delta Z_{ZA} = \frac{1}{2}\frac{\partial \Sigma_T^{AA}(k^2)}{\partial k^2}|_{k^2=0} - \frac{s_W}{c_W}\frac{\Sigma_T^{AZ}(0)}{M_Z^2}\ . \tag{5.54}$$

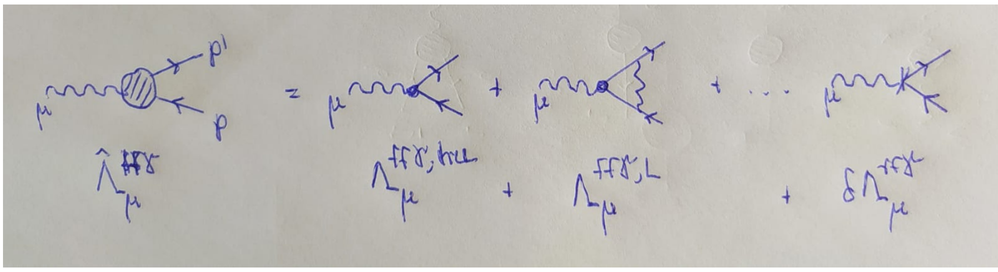


Figure 5.6: The full electromagnetic vertex.

It is independent of the fermion f. We hence have charge universality.

In the on-shell scheme the Weinberg angle is not a free parameter. A possible definition is (cf. A. Sirlin, Physical Review D 22 ('80) 971)

$$\sin^2\theta_W = s_W^2 = 1 - \frac{M_W^2}{M_Z^2}\,, \quad c_W^2 = \frac{M_W^2}{M_Z^2}\,. \tag{5.55}$$

From this directly follow the renormalisation constants for the Weinberg angle

$$\begin{aligned}
s_{W0} &= s_W + \delta s_W \qquad\qquad c_{W0} = c_W + \delta c_W \\
\frac{\delta c_W}{c_W} &= \frac{1}{2}\left(\frac{\delta M_W^2}{M_W^2} - \frac{\delta M_Z^2}{M_Z^2}\right) = \frac{1}{2}\mathrm{Re}\left[\frac{\Sigma_T^W(M_W^2)}{M_W^2} - \frac{\Sigma_T^Z(M_Z^2)}{M_Z^2}\right] \\
\frac{\delta s_W}{s_W} &= -\frac{c_W^2}{s_W^2}\frac{\delta c_W}{c_W} = -\frac{1}{2}\frac{c_W^2}{s_W^2}\mathrm{Re}\left[\frac{\Sigma_T^W(M_W^2)}{M_W^2} - \frac{\Sigma_T^Z(M_Z^2)}{M_Z^2}\right]\,.
\end{aligned} \tag{5.56}$$

5.2 1-Loop Integrals

One-loop integrals in general can be reduced to the following scalar integrals

- 1-point function:

$$A_0(m) = \frac{16\pi^2\mu^{4-D}}{i}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2-m^2} \tag{5.57}$$

- 2-point function:

$$B_0(p;m_0,m_1) = \frac{16\pi^2\mu^{4-D}}{i}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2-m_0^2)[(k+p)^2-m_1^2]} \tag{5.58}$$

- 3-point function:

$$\begin{aligned}
C_0(p_1,p_2;m_0,m_1,m_2) &= \frac{16\pi^2\mu^{4-D}}{i}\int\frac{d^Dk}{(2\pi)^D} \\
&\quad \frac{1}{(k^2-m_0^2)[(k+p_1)^2-m_1^2][(k+p_2)^2-m_2^2]}
\end{aligned} \tag{5.59}$$

- 4-point function:

$$D_0(p_1,p_2,p_3 \;\; ; \;\; m_0,m_1,m_2,m_3) = \frac{16\pi^2\mu^{4-D}}{i}\int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2-m_0^2)[(k+p_1)^2-m_1^2][(k+p_2)^2-m_2^2][(k+p_3)^2-m_3^2]} \tag{5.60}$$

All tensor integrals can be decomposed into tensors, which consist of the external 4-momenta. The corresponding coefficients can be expressed through the above scalar integrals and are symmetric. (See Passarino,Veltman, Nuclear Physics B 160 ('79), 151.)

$$\begin{aligned}
B_\mu(p;m_0,m_1) &= \frac{16\pi^2\mu^{4-D}}{i}\int \frac{d^Dk}{(2\pi)^D}\frac{k_\mu}{(k^2-m_0^2)[(k+p)^2-m_1^2]} = B_1 p_\mu \\
B_{\mu\nu}(p;m_0,m_1) &= \frac{16\pi^2\mu^{4-D}}{i}\int \frac{d^Dk}{(2\pi)^D}\frac{k_\mu k_\nu}{(k^2-m_0^2)[(k+p)^2-m_1^2]} = B_{00}g_{\mu\nu} + B_{11}p_\mu p_\nu \\
C_\mu(p_1,p_2;m_0,m_1,m_2) &= \frac{16\pi^2\mu^{4-D}}{i}\int \frac{d^Dk}{(2\pi)^D}\frac{k_\mu}{(k^2-m_0^2)[(k+p_1)^2-m_1^2][(k+p_2)^2-m_2^2]} \\
&= C_1 p_{1\mu} + C_2 p_{2,\mu} \\
C_{\mu\nu}(p_1,p_2;m_0,m_1,m_2) &= \frac{16\pi^2\mu^{4-D}}{i}\int \frac{d^Dk}{(2\pi)^D}\frac{k_\mu k_\nu}{(k^2-m_0^2)[(k+p_1)^2-m_1^2][(k+p_2)^2-m_2^2]} \\
&= C_{00}g_{\mu\nu} + C_{11}p_{1,\mu}p_{1,\nu} + C_{22}p_{2,\mu}p_{2,\nu} + C_{12}(p_{1,\mu}p_{2,\nu} + p_{1,\nu}p_{2,\mu}) \\
&= C_{00}g_{\mu\nu} + \sum_{i,j=1}^{2} C_{ij}p_{i\mu}p_{j\nu} \\
C_{\mu\nu\rho} &= \sum_{i=1}^{2}(g_{\mu\nu}p_{i\rho} + g_{\nu\rho}p_{i\mu} + g_{\mu\rho}p_{i\nu})C_{00i} + \sum_{i,j,k=1}^{2} C_{ijk}p_{i\mu}p_{j\nu}p_{k\rho} \\
D_\mu &= \sum_{i=1}^{3} D_i p_{i\mu} \\
D_{\mu\nu} &= D_{00}g_{\mu\nu} + \sum_{i,j=1}^{3} D_{ij}p_{i\mu}p_{j\nu} \\
D_{\mu\nu\rho} &= \sum_{i=1}^{3} D_{00i}(g_{\mu\nu}p_{i\rho} + g_{\nu\rho}p_{i\mu} + g_{\mu\rho}p_{i\nu}) + \sum_{i,j,k=1}^{3} D_{ijk}p_{i\mu}p_{j\nu}p_{k\rho} \\
D_{\mu\nu\rho\sigma} &= D_{0000}(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}) \\
&+ \sum_{i,j=1}^{3} D_{00ij}(g_{\mu\nu}p_{i\rho}p_{j\sigma} + g_{\nu\rho}p_{i\mu}p_{j\sigma} + g_{\mu\rho}p_{i\nu}p_{j\sigma} + g_{\mu\sigma}p_{i\nu}p_{j\rho} \\
&\quad + g_{\nu\sigma}p_{i\mu}p_{j\rho} + g_{\rho\sigma}p_{i\mu}p_{j\nu}) \\
&+ \sum_{i,j,k,l=1}^{3} D_{ijkl}p_{i\mu}p_{j\nu}p_{k\rho}p_{l\sigma}
\end{aligned} \tag{5.61}$$

$C_{ijk}, D_{ij}, D_{ijk}, D_{00ij}, D_{ijkl}$ are symmetric in the indices. The coefficients kann be determined through contractions with the tensors.

Examples:

$$
\begin{aligned}
(i)\ p^\mu B_\mu &= p^2 B_1 = \frac{16\pi^2\mu^{4-D}}{i}\int\frac{d^Dk}{(2\pi)^D}\frac{kp}{(k^2-m_0^2)[(k+p)^2-m_1^2]} \\
kp &= \frac{1}{2}\{[(k+p)^2-m_1^2]-(k^2-m_0^2)-p^2-m_0^2+m_1^2\} \qquad (5.62)
\end{aligned}
$$

$$
\boxed{B_1 = \frac{1}{2p^2}\{A_0(m_0)-A_0(m_1)-(p^2+m_0^2-m_1^2)B_0(p;m_0,m_1)\} \qquad (5.63)}
$$

$$
\begin{aligned}
(ii)\ g^{\mu\nu}B_{\mu\nu} &= DB_{00}+p^2B_{11} = \frac{16\pi^2\mu^{4-D}}{i}\int\frac{d^Dk}{(2\pi)^D}\frac{k^2-m_0^2+m_0^2}{(k^2-m_0^2)[(k+p)^2-m_1^2]} \\
&= A_0(m_1)+m_0^2B_0 \\
p^\mu p^\nu B_{\mu\nu} &= p^2(B_{00}+p^2B_{11}) = \frac{16\pi^2\mu^{4-D}}{i}\int\frac{d^Dk}{(2\pi)^D}\frac{(kp)^2}{(k^2-m_0^2)[(k+p)^2-m_1^2]} \\
(kp)^2 &= \underbrace{\frac{kp}{2}[(k+p)^2-m_1^2]}_{\to 0}-\underbrace{\frac{kp}{2}(k^2-m_0^2)}_{\to\frac{p^2}{2}(k^2-m_0^2)}-\frac{kp}{2}(p^2+m_0^2-m_1^2) \\
\text{because} &\quad \int\frac{d^Dk}{(2\pi)^D}\frac{k^\mu}{k^2-m^2} = 0 \\
\frac{1}{p^2}p^\mu p^\nu B_{\mu\nu} &= B_{00}+p^2B_{11} = \frac{1}{2}\{A_0(m_1)-(p^2+m_0^2-m_1^2)B_1\} \\
\text{and} &\quad DB_{00}+p^2B_{11} = A_0(m_1)+m_0^2B_0 \qquad (5.64)
\end{aligned}
$$

We hence have

$$
\boxed{
\begin{aligned}
B_{00} &= \frac{1}{2(D-1)}\{A_0(m_1)+2m_0^2B_0+(p^2+m_0^2-m_1^2)B_1\} \\
B_{11} &= \frac{1}{2(D-1)p^2}\{(D-2)A_0(m_1)-2m_0^2B_0-D(p^2+m_0^2-m_1^2)B_1\} \qquad (5.65)
\end{aligned}
}
$$

$$
\begin{aligned}
(iii)\quad p_1^\mu C_\mu &= p_1^2C_1+p_1p_2C_2 = \frac{16\pi^2\mu^{4-D}}{i}\int\frac{d^Dk}{(2\pi)^D}\frac{kp_1}{(k^2-m_0^2)[(k+p_1)^2-m_1^2][(k+p_2)^2-m_2^2]} \\
p_1k &= \frac{1}{2}\{[(k+p_1)^2-m_1^2]-(k^2-m_0^2)-(p_1^2-m_1^2+m_0^2)\} \\
p_1^2C_1+p_1p_2C_2 &= \frac{1}{2}\{B_0(p_2;m_0,m_2)-B_0(p_2-p_1;m_1,m_2)-(p_1^2-m_1^2+m_0^2)C_0\} \\
p_1p_2C_1+p_2^2C_2 &= \frac{1}{2}\{B_0(p_1;m_0,m_1)-B_0(p_2-p_1;m_1,m_2)-(p_2^2-m_2^2+m_0^2)C_0\}
\end{aligned}
\qquad (5.66)
$$

In matrix form $[f_i = p_i^2-m_i^2+m_0^2]$:

$$
\begin{pmatrix} p_1^2 & p_1p_2 \\ p_1p_2 & p_2^2 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} B_0(p_2;m_0,m_2)-B_0(p_2-p_1;m_1,m_2)-f_1C_0 \\ B_0(p_1;m_0,m_1)-B_0(p_2-p_1;m_1,m_2)-f_2C_0 \end{pmatrix} \qquad (5.67)
$$

Thereby we have as solution

$$\boxed{\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} p_1^2 & p_1p_2 \\ p_1p_2 & p_2^2 \end{pmatrix}^{-1} \frac{1}{2} \begin{pmatrix} B_0(p_2; m_0, m_2) - B_0(p_2 - p_1; m_1, m_2) - f_1 C_0 \\ B_0(p_1; m_0, m_1) - B_0(p_2 - p_1; m_1, m_2) - f_2 C_0 \end{pmatrix} \quad (5.68)}$$

5.2.1 Useful Formulae

For the calculation of the scalar integrals we need some formulae and/or relations, that are given in the following.

Feynman Parametrisation:

$$\begin{aligned} \frac{1}{a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_n^{\alpha_n}} &= \frac{\Gamma(\alpha_1 + \ldots + \alpha_n)}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \ldots \int_0^{1-x_1\ldots-x_{n-2}} dx_{n-1} \\ & \frac{(1 - x_1 - \ldots - x_{n-1})^{\alpha_1 - 1} x_1^{\alpha_2 - 1} \ldots x_{n-1}^{\alpha_n - 1}}{[a_1(1 - x_1 - \ldots x_{n-1}) + a_2 x_1 + \ldots + a_n x_{n-1}]^{\sum \alpha_i}} \end{aligned} \tag{5.69}$$

D-dimensional integrals

$$\begin{aligned} \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2 - M^2 + i\epsilon)^N} &= i \frac{(-1)^N}{(4\pi)^{D/2}} \frac{\Gamma(N - \frac{D}{2})}{\Gamma(N)} \frac{1}{(M^2 - i\epsilon)^{N - \frac{D}{2}}} \\ \int \frac{d^Dk}{(2\pi)^D} \frac{k^2}{(k^2 - M^2 + i\epsilon)^N} &= \frac{i}{2} \frac{(-1)^{N-1}}{(4\pi)^{D/2}} \frac{\Gamma(N - 1 - \frac{D}{2})}{\Gamma(N)} \frac{D}{(M^2 - i\epsilon)^{N - 1 - \frac{D}{2}}} \\ \int \frac{d^Dk}{(2\pi)^D} \frac{k_\mu k_\nu}{(k^2 - M^2 + i\epsilon)^N} &= \frac{i}{2} \frac{(-1)^{N-1}}{(4\pi)^{D/2}} \frac{\Gamma(N - 1 - \frac{D}{2})}{\Gamma(N)} \frac{g_{\mu\nu}}{(M^2 - i\epsilon)^{N - 1 - \frac{D}{2}}} \\ \int \frac{d^Dk}{(2\pi)^D} k_\mu k_\nu f(k^2) &= \frac{1}{D} g_{\mu\nu} \int \frac{d^Dk}{(2\pi)^D} k^2 f(k^2) \end{aligned} \tag{5.70}$$

Expansions of the Gamma- and of the Beta-functions

$$\begin{aligned} \Gamma(N - \frac{\epsilon}{2}) &= \Gamma(N)(1 - \frac{\epsilon}{2}\Psi(N)) + \mathcal{O}(\epsilon^2) \qquad \text{for } \epsilon = 4 - D \\ \text{with } \Psi(N) &= S_{N-1} - \gamma_E \\ S_N &= \sum_{j=1}^{N} \frac{1}{j} \qquad \text{and } \gamma_E = 0.5772\ldots \\ \Gamma(A) &= \frac{\Gamma(A+1)}{A} \\ B(A_1, A_2) &= \frac{\Gamma(A_1)\Gamma_(A_2)}{\Gamma(A_1 + A_2)} \\ \Gamma(\frac{\epsilon}{2}) &= \frac{2}{\epsilon} - \gamma_E + (\epsilon) \\ B(N - \frac{\epsilon}{2}, 1 - \frac{\epsilon}{2}) &= \frac{1}{N}[1 + \epsilon S_N - \frac{\epsilon}{2} S_{N-1}] + \mathcal{O}(\epsilon^2) \\ B(N - \frac{\epsilon}{2}, 2 - \frac{\epsilon}{2}) &= \frac{1}{N(N+1)}[1 - \frac{\epsilon}{2} S_{N-1} - \frac{\epsilon}{2} + \epsilon S_{N+1}] + \mathcal{O}(\epsilon^2) \end{aligned} \tag{5.71}$$

5.2.2 Calculation of the Scalar Integrals

We have already found

$$A_0(m) = m^2\{\Delta + \ln\frac{\mu^2}{m^2} + 1\} \tag{5.72}$$

with

$$\Delta = \frac{1}{\epsilon} - \gamma_E + \ln 4\pi \tag{5.73}$$

Next, we calculate the B_0 function

$$B_0(p; m_0, m_1) = \frac{16\pi^2\mu^{4-D}}{i}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2 - m_0^2 + io)[(k+p)^2 - m_1^2 + io]} \tag{5.74}$$

With the Feynman parametrisation (cf. Eq. (5.69)) we have

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[Ax + B(1-x)]^2} = -\frac{1}{A-B}\left(\frac{1}{Ax + B(1-x)}\Bigg|_0^1\right) = -\frac{1}{A-B}\left(\frac{1}{A} - \frac{1}{B}\right) \qquad \text{q.e.d.} \tag{5.75}$$

After the Feynman parametrisation $[A = (k+p)^2 - m_1^2$ and $B = k^2 - m_0^2]$ we have for the B_0 function

$$\begin{aligned} B_0 &= \frac{16\pi^2\mu^{4-D}}{i}\int_0^1 dx\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2 + 2kQ - M^2 + io)^2} \qquad \text{with} \\ Q &= xp \qquad M^2 = m_0^2(1-x) - (p^2 - m_1^2)x\ . \end{aligned} \tag{5.76}$$

We redefine k as $k = k' - Q$ and then write k' as k. We then have

$$\begin{aligned} B_0 &= \frac{16\pi^2\mu^{4-D}}{i}\int_0^1 dx\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2 - R^2 + io)^2} \qquad \text{with} \\ R^2 &= Q^2 + M^2 = m_0^2(1-x) + m_1^2 x - p^2 x(1-x)\ . \end{aligned} \tag{5.77}$$

For the integral over k we use the integration formula (see also above)

$$\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2 - R^2)^N} = (-1)^N\frac{i\Gamma(N - \frac{D}{2})}{(4\pi)^{D/2}\Gamma(N)}(R^2)^{\frac{D}{2}-N} \tag{5.78}$$

and obtain

$$\begin{aligned} B_0 &= \frac{16\pi^2\mu^{2\epsilon}}{i}\frac{i\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}\Gamma(2)}\int_0^1 dx(R^2 - io)^{-\epsilon} \\ &= \Gamma(\epsilon)\left(\frac{4\pi\mu^2}{\bar{m}_0^2}\right)^{\epsilon}\int_0^1 dx\{1 - \epsilon\ln\frac{R^2 - io}{\bar{m}_0^2} + \mathcal{O}(\epsilon^2)\} \qquad [\bar{m}_i^2 = m_i^2 - io] \end{aligned} \tag{5.79}$$

An expansion in ϵ leads to

$$B_0 = \Delta + \ln\frac{\mu^2}{m_0^2} - \int_0^1 dx \ln\underbrace{\left\{\frac{\bar{m}_0^2(1-x) + \bar{m}_1^2 x - p^2 x(1-x)}{\bar{m}_0^2}\right\}}_{\left(1-\frac{x}{x_+}\right)\left(1-\frac{x}{x_-}\right)} \tag{5.80}$$

with

$$x_{\pm} = \frac{1}{2p^2}\{p^2 + m_0^2 - m_1^2 \pm \sqrt{(p^2 + \bar{m}_0^2 - \bar{m}_1^2)^2 - 4\bar{m}_0^2 p^2}\} \ . \tag{5.81}$$

For the complex logarithm, which appears in the integral, we need the following theorem,

$$\begin{aligned} \ln ab &= \ln a + \ln b + \eta(a,b) \qquad \text{mit} \\ \eta(a,b) &= \begin{cases} 2\pi i & \text{if } \mathrm{Im}a > 0, \mathrm{Im}b > 0, \mathrm{Im}ab < 0 \\ & \text{or } \mathrm{Im}a < 0, \mathrm{Im}b < 0, \mathrm{Im}ab > 0 \\ 0 & \text{otherwise} \end{cases} \ . \end{aligned} \tag{5.82}$$

With this we have for

$$\ln[(a + io)(b - io')] = \ln(a + io) + \ln(b - io') \ . \tag{5.83}$$

And thereby we finally find for B_0

$$\boxed{B_0 = \Delta + \ln \frac{\mu^2}{m_0^2} + 2 + (x_+ - 1) \ln \left(\frac{x_+ - 1}{x_+} \right) + (x_- - 1) \ln \left(\frac{x_- - 1}{x_-} \right)} \tag{5.84}$$

Analogous procedures lead to the analytic results for the 3- and 4-point functions.

5.3 Classification of Local Interactions

(Content: superficial degree of divergence; super-renormalisable, renormalisable and non-renormalisable interactions.)

For non-Abelian gauge theories the only regularisation of practical relevance is dimensional regularisation:

$$D = 4 \to D \neq 4 \ (< 4) \qquad \text{with} \quad D - 1 \text{ space dimension and 1 time dimension} \ . \tag{5.85}$$

Naive dimensional analysis: $\hbar = c = 1$. Thereby

$$S = \int d^D x \mathcal{L}(x) \Rightarrow [S] = [\hbar] = \text{dimensionless} \ . \tag{5.86}$$

Be M an arbitrary mass scale. We then have ($\hbar \sim xp$)

$$[x_\mu] = M^{-1} \quad [\partial_\mu] = M \ . \tag{5.87}$$

The mass dimension of the fields is

(a) Scalar fields

$$[\phi, \dot{\phi}]_{t=t'} = i \underbrace{\delta^{(D-1)}(\vec{x} - \vec{x'})}_{[.]=M^{D-1}} \tag{5.88}$$

and hence

$$[\phi] = M^{\frac{D-2}{2}} \ . \tag{5.89}$$

Analogous

$$[A_\mu] = M^{\frac{D-2}{2}} \ . \tag{5.90}$$

(b) Spinor fields

$$\{\psi_l, \psi_s^\dagger\}_{t=t'} = \delta_{ls}\delta^{(D-1)}(\vec{x}-\vec{x}') \ . \tag{5.91}$$

Thereby

$$[\psi] = M^{\frac{D-1}{2}} \ . \tag{5.92}$$

(c) Ghost fields

$$\mathcal{L}_0^{\text{ghost}} = \partial_\mu \bar{c}\partial^\mu c \tag{5.93}$$
$$M^D \qquad M^2[c]^2 \ . \tag{5.94}$$

Thereby

$$[c] = M^{\frac{D-2}{2}} \ . \tag{5.95}$$

Let us now look at the coupling constant. We consider e.g. QED. The coupling term is

$$-e\bar{q}\gamma^\mu A_\mu q \ . \tag{5.96}$$

Thereby is

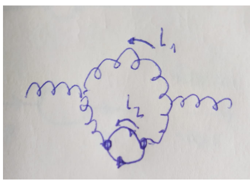
$$[e_{\text{QED}}] = [g_{\text{QCD}}] = M^{\frac{4-D}{2}} \ . \tag{5.97}$$

This means that in four dimensions the coupling is dimensionless. In $D < 4$ dimensions the coupling has a positive mass dimension. Furthermore, the gauge parameter is dimensionless,

$$[\xi] = M^0 \ . \tag{5.98}$$

Above, we have already looked at the γ algebra in D dimensions. Remark: γ^5 in D dimensions is problematic, as the Levi-Civitá-Tensor is only defined in $D = 4$ dimensions.

Degree of divergence of a diagram Let us look at the following diagram



The Feynman integral I_G is for large loop momenta l

$$I_G \sim \int d^D l_1 \, d^D l_2 \, \frac{1}{(l_1^2)^3} \frac{1}{l_2} \frac{1}{l_2} l_1 \, l_1 \ . \tag{5.99}$$

The "superficial degree of divergence" thereby is

$$d = 2D + 2 - 6 - 2 = 2D - 6 \ . \tag{5.100}$$

In general the superficial degree of divergence of a 1-particle irreducible diagram G is

$$d_G = Dl + \sum_v \delta_v - 2n_B - n_F \ , \tag{5.101}$$

where l is the number of loops, δ_v is the number of momentum factors at the vertex, n_B denotes the number of the inner boson lines and n_F the number of the inner fermion lines. The above, however, does not hold for massive gauge bosons. They have the Feynman propagator

$$D^{\mu\nu} = \frac{i}{k^2 - m_V^2}\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{m_V^2}\right) \ . \tag{5.102}$$

The convergence theorem (by Weinberg) reads:
I_G is absolutely convergent, if the degree of divergence $d_H < 0$ for all subdiagrams $H \subset G$.

We now look at the classification of renormalisable interactions. Be

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I \ , \tag{5.103}$$

with

$$\begin{aligned} b &= \text{numnber of bosonic fields in } \mathcal{L}_I && (5.104)\\ f &= \text{number of fermionic fields in } \mathcal{L}_I && (5.105)\\ \delta &= \text{number of derivatives in } \mathcal{L}_I \ . && (5.106)\end{aligned}$$

We consider the Feynman diagram denoted by G. Be

$$\begin{aligned} n &= \text{number of vertices (generated through a specific } \mathcal{L}_I) && (5.107)\\ N_B &= \text{nunmber of external bosonic lines} && (5.108)\\ N_F &= \text{number of external fermionic lines} && (5.109)\\ l &= \text{number of loops (after application of the energy-momentum conservation)} \\ &\quad\ \text{at each vertex .} && (5.110)\end{aligned}$$

The latter l is given by

$$l = n_B + n_F - n + 1 \ . \tag{5.111}$$

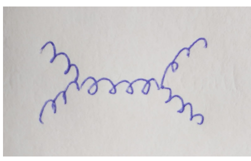
Because each vertex generates δ functions for the energy-momentum conservation and implies a reduction in l by $n-1$ momentum integrals. Furthermore, we have

$$n\delta = \sum_v \delta_v \tag{5.112}$$

and

$$nb = 2n_B + N_B \ . \tag{5.113}$$

Because e.g.



$$2 \cdot 3 = 2 \cdot 1 + 4 \ . \tag{5.114}$$

The first term on the right-hand side is generated by the fact that an inner line participates in 2 vertices. Furthermore, we have

$$nf = 2n_F + N_F \ . \tag{5.115}$$

Insertion of the equations (5.113), (5.115) in (5.111) leads to

$$l = \frac{b+f}{2}n - n - \frac{N_B + N_F}{2} + 1 \ . \tag{5.116}$$

Insertion of (5.112), (5.116) in (5.101) leads to

$$d_G = rn - \frac{D-2}{2}N_B - \frac{D-1}{2}N_F + D \ . \tag{5.117}$$

We call r the degree of divergence of $\mathcal{L}_I$. It is given by

$$r = \frac{D-2}{2}b + \frac{D-1}{2}f + \delta - D \ . \tag{5.118}$$

The degree of divergence characterises $\mathcal{L}_I$. For the dimension of the coupling g_I it holds that

$$[g_I] = -r \ . \tag{5.119}$$

Because be schematically

$$\mathcal{L}_I \sim g_I(\partial)^\delta(\phi)^b(\psi)^f \ . \tag{5.120}$$

Then the dimension of $\mathcal{L}_I$ is

$$D = [\mathcal{L}_I] = [g_I] + \delta + b\frac{D-2}{2} + f\frac{D-1}{2} \ . \tag{5.121}$$

With the generalisation to several vertices we have for

$$\mathcal{L}_I = \sum_{i=1}^{k} \mathcal{L}_I^{(i)} \tag{5.122}$$

and thereby

$$d_G = \sum_{i=1}^{k} r_i n_i - \frac{D-2}{2}N_B - \frac{D-1}{2}N_F + D \ , \tag{5.123}$$

with the divergence indices

$$r_i = \frac{D-2}{2}b_i + \frac{D-1}{2}f_i + \delta_i - D \ . \tag{5.124}$$

The integral I_G is UV-divergent, if $d_H > 0$ for at least one subdiagramn $H \subset G$.

Classification

1) If $r_i > 0$ $(i = 1, ..., k)$ for an arbitrary i, then for sufficiently large n_i the degree of divergence d_G grows continuously without limit so that the theory is “non-renormalisable”.

2) $r_i = 0$ for all i. Then there is a finite number of types of Feynman diagrams (i.e. n-point functions), which have $d_G \geq 0$ so that the theory is “renormalisable”.

3) $r_i < 0$ for all i. Then the theory is “super-renormalisable”.

We look at some examples

$$\text{A) } \mathcal{L}_I = \lambda\phi^3 \ . \tag{5.125}$$

In D dimensions we have

$$[\lambda] = -r = D - 3\frac{D-2}{2} = -\frac{D-6}{2} \ . \tag{5.126}$$

The theory is

$$\begin{array}{rcl} \text{renormalisable} & \text{for} & D = 6 \\ \text{non-renormalisable} & \text{for} & D > 6 \\ \text{super-renormalisable} & \text{for} & D < 6 \ . \end{array}$$

$$\text{B) } \mathcal{L}_I = h(\bar{\psi}\psi)^2 \quad \text{4-Fermi interaction.} \tag{5.127}$$

Thereby is

$$[h] = -r = D - 4\frac{D-1}{2} = 2 - D \ . \tag{5.128}$$

The theory hence is renormalisable in $D = 2$ dimensions (Thirring-Modell) and non-renormalisable in $D > 2$ (in particular in $D = 4$).

We have the following rule:

Coupl. constants have	pos.	mass dim.	$\leftrightarrow$	theory	super-renormalisable
Coupl. constants have	0	mass dim.	$\leftrightarrow$	theory	renormalisable
Coupl. constants have	neg.	mass dim.	$\leftrightarrow$	theorie	non-renormalisable

This rule does not hold for massive vector fields. Their Feynman propagator (see above) approaches $\mathcal{O}(1)$ in the limit $k \to \infty$. We consider as further example QCD. The interaction Lagrangian reads

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I \ , \tag{5.129}$$

where

$$\begin{aligned} \mathcal{L}_I \ = \ & g f_{abc}\partial_\mu \bar{c}^a G^{\mu b} c^c - g\bar{q}T^a\gamma^\mu q G^a_\mu \\ & + \frac{g}{2} f_{abc}(\partial_\mu G^a_\nu - \partial_\nu G^a_\mu) G^{b\mu} G^{c\nu} \\ & - \frac{g^2}{4} f_{abe} f_{cde} G^a_\mu G^b_\nu G^{c\mu} G^{d\nu} \ . \end{aligned} \tag{5.130}$$

We determine the divergence indices

$$r_i = \frac{D-2}{2} b_i + \frac{D-1}{2} f_i + \delta_i - D \tag{5.131}$$

of the four interaction terms:

$$\begin{aligned}
r_{ghost} &= 3\frac{D-2}{2} + 1 - D = \frac{D-4}{2} &(5.132)\\
r_{quark} &= \frac{D-2}{2} + \frac{D-1}{2} 2 - D = \frac{D-4}{2} &(5.133)\\
r_{3G} &= 3\frac{D-2}{2} + 1 - D = \frac{D-4}{2} &(5.134)\\
r_{4G} &= 4\frac{D-2}{2} - D = D - 4\,. &(5.135)
\end{aligned}$$

The coupling constant has the dimension

$$[g] = \frac{4-D}{2} \tag{5.136}$$

and is dimensionless in $D = 4$ dimensions. For $D = 4$, $r_i = 0$. Thereby QCD is renormalisable according to the counting of the powers. (If the counterterms are added $\mathcal{L}$ does not change structurally.)

Remark: For ghosts, the following replacement has to be done,

$$N_B \to N_{\text{gluon}} + \frac{3}{2} N_{\text{ghost}}\,. \tag{5.137}$$

Chapter 6

Spontaneous Symmetry Breaking

In the last semester you have learnt in theoretical particle physics I what is spontaneous symmetry breaking. It is therefore repeated here only very briefly. It is essential for the Higgs mechanism which is one of the four pillars of the Standard Model of particle physics.

The symmetry of a Lagrangian is called *spontaneously broken*, if the Lagrangian is symmetric, but the physical vacuum *does not* obey the symmetry. If the Lagrangian of a theory is invariant under an exact continous symmetry, which is not the symmetry of the physical vacuum, then one or several massless spin-0 particles will appear. These particles are called Goldstone bosons. If the spontaneously broken symmetry is a local gauge theory, then the interplay (induced through the Higgs mechanism) between the would-be Goldstone bosons and the massless gauge bosons leads to the masses of the gauge bosons and removes the Goldstone bosons from the spectrum.

6.1 The Goldstone Theorem

Be

N = dimension of the algebra of the symmetry group of the complete Lagrangian.
M = dimension of the algebra of the group under which the vacuum is invariant after spontaneious symmetry breaking.

$\Rightarrow$ There are N-M Goldstone bosons without mass in the theory.

The Goldstone theorem states that for each spontaneously broken degree of freedom of the symmetry there is a massless Goldstone boson.

For gauge theories the Goldstone theorem does not hold: Massless scalar degrees of freedom are absorbed by the gauge bosons to give them mass. The Goldstone phenomenon leads to the Higgs phenomenon.

6.2 Spontaneously Broken Gauge Theories

In gauge theories there are no Goldstone bosons in the physical spectrum. They are *would-be* Goldstone bosons. In spontaneous symmetry breaking (SSB) they are directly absorbed by the longitudinal degrees of freedom of the massive gauge bosons. For gauge theories the following holds: Be

N	$=$	dimension of the algebra of the symmetry group of the complete Lagrangian..
M	$=$	dimension of the algebra of the group under which the vacuum is invariant after spontaneious symmetry breaking.
n	$=$	the number of the scalar fields

$$\Rightarrow$$

There are M massless vector fields. (M is the dimension of the symmetry of the vacuum.)
There are $N - M$ massive vector fields. ($N - M$ is the number of the broken generators.)
There are $n - (N - M)$ scalar Higgs fields

Chapter 7

The Standard Model of Particle Physics

The Standard Model of particle physics describes the today known basic building blocks of matter and (except for gravity) their interactions among each other. These are the electromagnetic and the weak forces (combined in the Glashow-Salam-Weinberg theory into the electroweak force) and the strong interaction.

Before going into details we give a short historical overview of the steps towards the development of the electroweak theory by Sheldon Glashow, Abdus Salam and Steven Weinberg (1967).

7.1 A Short History of the Standard Model of Particle Physics

- Weak interaction: β decay [*A. Becquerel 1896, Nobel Prize 1903*[1]]

  Antoine Henri Becquerel (15.12.1852 - 25.8.1908) was a French physicist, Nobel Prize winner and one of the discoverers of radioactivity.

  In 1896, while investigating fluorescence in uranium salts, Becquerel discovered radioactivity accidentally. Investigating the work of Wilhelm Conrad Röntgen, Becquerel wrapped a fluorescent mineral, potassium uranyl sulfate, in photographic plates and black material in preparation for an experiment requiring bright sunlight. However, prior to actually performing the experiment, Becquerel found that the photographic plates were fully exposed. This discovery led Becquerel to investigate the sponaneous emission of nuclear radiation.

  In 1903, he shared the Nobel Prize with Marie and Pierre Curie "in recognition of the extraordinary services he has rendered by his discovery of spontaneous radioactivity".

$N \to N' + e^-$ violates energy and angular momentum conservation

  Lise Meitner and Otto Hahn showed in 1911 that the energy of the emitted electrons is continuous. Since the released energy is constant, a discrete spectrum had been expected. In order to explain this obvious energy loss (and also the violation of angular

[1]shared with Marie and Pierre Curie

momentum conservation) Wolfgang Pauli proposed in 1930 in his letter of Dec 4 to the "Dear radioactive ladies and gentlemen" (Lise Meitner et al.) the participation of a neutral, extremely light elementary particle (with a mass no greater than 1% the mass of a proton) in the decay process, which he called "neutron". Enrico Fermi changed this name 1931 in "neutrino", as a diminuation form of the nearly at the same time discovered heavy neutron.

Lise Meitner (7. 11.1878 - 27.10.1968) was an Austrian physicist who investigated radioactivity and nuclear physics. Otto Hahn (8.3.1879 - 28.7.1968) was a German chemist and received in 1944 the Nobel Prize in chemistry. Wolfgang Ernst Pauli (25.4.1900 - 15.12.1958) was an Austrian physicist.

- The neutrino hypothesis: [*W. Pauli 1930, Nobel Prize 1945*]

$N \to P + e^- + \bar{\nu}_e$
Spin = 1/2, Mass ≈ 0

In 1956, Clyde Cowan and Frederick Reines suceeded in the first experimental proof of the neutrino in one of the first big nuclear reactors.

Clyde Lorrain Cowan Jr (6.121919 - 24.5.1974) discoverd together with Frederick Reines the neutrino. Frederick Reines (6.3.1918 - 26.8.1998) was an American physicist and won in 1995 the Nobel Prize of physics in the name of the two of them

- The Fermi Theory [*E. Fermi, Nobel Prize 1938*]

Enrico Fermi developed a theory of weak interactions in analogy to quantum electrodynamics (QED), where four fermions directly interact with each other:
$\mathcal{L}_{\text{eff}} = \frac{G_F}{\sqrt{2}} J_\mu J^\mu$
[For small momentum tranfers the reactions can be approximated by a point-like interaction.]

Enrico Fermi (29.9.1901 - 28.11.1954) was an Itaiian physicist He received the Nobel Prize for physics in 1938 for his work on 'induced radioactivity'.

The Fermi interaction consists of 4 fermions directly interacting with each other. For example a neutron (or down quark) can split into an electron, anti-neutrino and proton (or up quark). Tree-level Feynman diagrams describe this interaction remarkably well. However, no loop diagrams can be taken into account, since the Fermi interaction is *not renormalizable*. The solution consists in replacing the 4-fermion interaction by a more complete theory - with an exchange of a W or Z boson like in the electroweak theory. This is then renormalizable. Before the electroweak theory was constructed George Sudarshan and Robert Marshak, and independently also Richard Feynman and Murray Gell-Mann, were able to determine the correct tensor structure (vector minus axial vector $V - A$) of the 4-Fermi interaction.

- The Yukawa Hypothesis: [*H. Yukawa, Nobel Prize 1949 for 'his prediction of mesons based on the theory of nuclear forces'*]

The point-like Fermi coupling is the limiting case of the exchange of a "heavy photon" $\to$ W boson.

$$\frac{G_F}{\sqrt{2}} \text{ pointlike coupling } \approx \frac{g^2}{Q^2 - m_W^2} \approx \frac{-g^2}{m_W^2} \text{ with exchange of a } W-\text{boson}$$

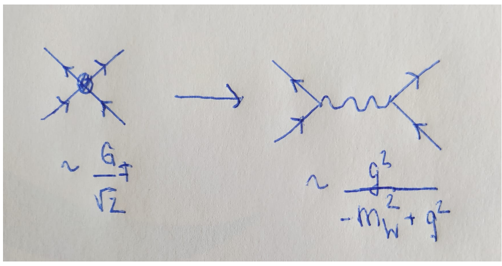


Figure 7.1: Fermi coupling: limiting case of a heavy gauge boson.

Hideki Yukawa (23.1.1907 - 8.9.1981) was a Japanese theoretical physicist and the first Japanese to win the Nobel Prize.

Hideki Yukawa established the hypothesis, that nuclear forces can be explained through the exchange of a new hypothetic particle between the nucleons, in the same manner as the electromagnetic force between two electrons can well be described by the exchange of photons. However, this particle exchanging the nuclear force should not be massless (as are the photons), but have a mass of 100 MeV. This value can be estimated from the range of the nuclear forces: the bigger the mass of the particle, the smaller the range of the interaction transmitted by the particle. A plausible argument for this connection is given by the energy-time uncertainty principle.

- Parity violation in the weak interaction [*T.D. Lee, C.N. Yang, Nobel Prize 1957, und C.-S. Wu*]

The $\tau - \theta$ puzzle: Initially, two different positively charged mesons with strangeness ($S \neq 0$) were known. These were distinguished based on their decay processes:

$$\begin{array}{lll} \Theta^+ & \to \pi^+\pi^0 & P_{2\pi} = +1 \\ \tau^+ & \to \pi^+\pi^+\pi^- & P_{3\pi} = -1 \end{array}$$

The final states of these two reactions have different parity. Since at that time it was supposed that parity is conserved in all reactions, the τ and θ would have had to be two different particles. However, precision measurements of mass and lifetime showed no difference between both particles. They seemed to be identical. The solution of this this $\theta - \tau$ puzzle was the parity violation of the weak interaction. Since both mesons decay via the weak interaction, this reaction does not conserve parity contrary to the initial assumption. Hence, both decays could stem from the same particle, which was then named K^+.
$\Theta^+ = \tau^+ = K^+ \Rightarrow \mathcal{P}$ violated. (π has negative parity.)

Tsung-Dao Lee (born November 24, 1926) is a Chinese American physicist, well known for creating the Lee Model, the field of relativistic heavy ion physics, and that of non-topological solitons and soliton stars in quantum field theory, as well as the solution for the theta-tau puzzle in particle physics. In 1957, Lee with C. N. Yang received the Nobel Prize in Physics for their work on the violation of parity law in weak interaction, which Chien-Shiung Wu experimentally verified. Lee and Yang were the first Chinese Nobel Prize winners. Mrs Chien-Shiung Wu (31. Mai 1912 in Liuho, Province Jiangsu, China ; - 16. Februar 1997 in New York, USA) was a Chinese-American physicist.*

When Lee and Yang received the Nobel Prize in physics in the same year of her experiment, many specialists thought, that it was unjustified that Wu did not receive the Prize as well.

$V-A$ theory: Actually, parity is not only violated, it is maximally violated. This means that the axial coupling has the same strength as the vectorial coupling: $|c_V| = |c_A|$. Since, as was shown in the Goldhaber experiment, there are only left-handed neutrinos and right-handed antineutrinos, one has rather: $c_V = -c_A$. This is why one calls the theory "$V-A$ theory".

- Proof of the existence of the neutrino:

$$N \to P + e^- + \bar{\nu}_e \qquad \bar{\nu}_e + P \to N + e^+$$

The neutrino could be verified experimentally 1956 by Clyde L. Cowan and Frederick Reines in the inverse β decay ($\bar{\nu}_e + P \to e^+ + N$) at a nuclear reactor, which causes a much higher neutrino flux as radioactive elements in the β decay. (Nobel prize to Reines alone 1995, since Cowan died 1974.)
The muon neutrino was discovered 1962 by Jack Steinberger, Melvin Schwartz and Leon Max Lederman with the first produced neutrino beam at an accelerator. All three physicists received 1988 the Nobel Prize for their basic experiments about neutrinos - weakly interacting elementary particles with vanishing or very small rest mass.
In 2000, the tau-neutrino was found in the DONUT-experiment.

- $\mathcal{CP}$ violation [*Cronin, Fitch, Nobel Prize 1980*]

$$\begin{array}{lcll} K_L^0 & \to & 3\pi & \mathcal{CP} = - \\ K_S^0 & \to & 2\pi & \mathcal{CP} = + \end{array}$$

Details: After the discovery of parity violation it was widely supposed that $\mathcal{CP}$ is conserved. Assuming $\mathcal{CP}$ symmetry, the physical Kaon states are given by the $\mathcal{CP}$ eigenstates. The K^0, $\bar{K}^0$ would be mass eigentstates w.r.t. the strong (also the electromagnetic) interaction alone. However, they mix through the weak interaction. The physical Kaon states are hence linear combinations of these two states with the following behaviour under CP transformations ($\mathcal{CP}|K^0>=-|\bar{K}^0>$, $\mathcal{CP}|\bar{K}^0>=-|K^0>$):

$$|K_1^0>=\frac{1}{\sqrt{2}}(|K^0>-|\bar{K}^0> \quad \text{with} \quad \mathcal{CP}|K_1^0>=|K_1^0> \tag{7.1}$$

$$|K_2^0>=\frac{1}{\sqrt{2}}(|K^0>+|\bar{K}^0> \quad \text{with} \quad \mathcal{CP}|K_2^0>=-|K_2^0> \tag{7.2}$$

Assuming $\mathcal{CP}$ symmetry these states can only decay under $\mathcal{CP}$ conservation. For the neutral Kaons this leads to two different decay channels for K_1 and K_2, with very different phase spaces and hence very different lifetimes:

$$K_1^0 \to 2\pi \quad \text{(quick, since big phase space)} \tag{7.3}$$

$$K_2^0 \to 3\pi \quad \text{(slow, since small phase space)} \tag{7.4}$$

In fact, two different species of neutral Kaons were found, which are very different in their lifetimes. These were named K_L^0 (long-lived, average lifetime $(5.16 \pm 0.04) \cdot 10^{-8}$ s) and K_S^0

(short-lived, average lifetime $(8.953 \pm 0.006) \cdot 10^{-11}$ s). The average lifetime of the long-lived Kaon is about a factor 600 larger than the one of the short-lived Kaon.
$\mathcal{CP}$ violation: Due to the assumed $\mathcal{CP}$ symmetry it was natural to identify the K_1^0, K_2^0 with K_S^0, K_L^0. Hence, the K_L^0 would always decay into three and never into two pions. But in reality James Cronin and Val Fitch found out 1964, that the K_L^0 decays with a small probaility (about 10^{-3}) also into two pions. This leads to the fact, that the physical states are no pure $\mathcal{CP}$ eigenstates, but contain a small amount ϵ of the other $\mathcal{CP}$ eigenstate, respectively. We have

$$|K_S^0 >= \frac{1}{\sqrt{1+|\epsilon|^2}}(|K_1^0 > +\epsilon|K_2^0 >) \tag{7.5}$$

$$|K_L^0 >= \frac{1}{\sqrt{1+|\epsilon|^2}}(|K_2^0 > +\epsilon|K_1^0 >) \tag{7.6}$$

This phenomenon has been checked very carefully in experiments and is called *$\mathcal{CP}$ violation through mixing*, since it is given by the mixing of the $\mathcal{CP}$ eigenstates to the physical eigenstate. Cronin and Fitch received 1980 the Nobel prize for their discovery. Since this $\mathcal{CP}$ violation can only be concluded indirectly through the observation of the decay, it is also called **indirect $\mathcal{CP}$ violation**. Also **direct $\mathcal{CP}$ violation**, hence a violation directly in the observed decay itself, has been found. The direct $\mathcal{CP}$ violation is for Kaons another factor of 1000 smaller than the indirect one and was found experimentally only three decades later at CERN.

Val Logsdon Fitch (10. March 1923 in Merriman, Nebraska), American physicist. Fitch received 1980 together with James Cronin the physics Nobel Prize. James Watson Cronin (* 29. September 1931 in Chicago), US-American physicist.*

- Glashow-Salam-Weinberg Theory (GSW): [*S.L. Glashow, A. Salam, S. Weinberg, Nobel Prize 1979*]

Sheldon Lee Glashow (5. December 1932 in New York) is a US-American physicist and Nobel prize winner. He received 1979 together with Abdus Salam and Steven Weinberg the physics Nobel prize for their work on the theory of the unification of the weak and electromagnetic interaction between elementary particles, including among others the prediction of the Z boson and the weak neutral currents. Abdus Salam (* 29. Januar 1926 in Jhang, Pakistan; - 21. November 1996 in Oxford, England) was a Pakistanian physicist and Nobel prize winner. Steven Weinberg (* 3. Mai 1933 in New York City - 23rd July 2021 in Austin, Texas) is a US-American physicist and Nobel prize winner.*

The electromagnetic interaction is the unified theory of quantum electrodynamics and the weak interaction. Together with quantum chromodynamics it is a pillar of the Standard Model of particle physics. This unification was initially described theoretically by S.L. Glashow, A. Salam and S. Weinberg, in 1967. Experimentally, the theory was confirmed 1973 indirectly through the discovery of the neutral currents and 1983 through the experimental proof of the W and Z bosons. A peculiarity is the parity violation through the electroweak interaction.

7.2 Unitarity: The Path to Gauge Theories

Fermi theory: describes μ, β decays, charged current (CC) reactions at small energies.

$$\mathcal{L}_{\text{eff}} = \frac{G_F}{\sqrt{2}} j^*_\lambda j^\lambda \qquad \begin{aligned} j_\lambda &= \bar{e}\gamma_\lambda(1-\gamma_5)\nu_e + (\mu) + (q) \\ G_F &= 1.16 \cdot 10^{-5}/\text{GeV}^2 \end{aligned}$$

CC scattering at high energies:

$$\left.\begin{aligned} &\sigma_{LL}(\nu_\mu e^- \to \mu^- \nu_e) = \tfrac{G_F^2 s}{\pi} \\ &s\text{-wave unitarity} \quad \sigma_{LL} < \tfrac{4\pi}{s} \end{aligned}\right\} \quad \begin{aligned} &\text{Im} f = |f|^2 \\ &|f| < 1 \end{aligned}$$

[Partial-wave unitarity constrains the modulus of an inelastic partial-wave amplitude to be $|\mathcal{M}| < 1$. Make a partial-wave expansion of the scattering amplitude. The constraint is equivalent to $\sigma < \pi/p^2_{c.m.}$ for inelastic s-wave scattering.]

Domain of validity/unitarity constraint: $\quad \sqrt{s} < (2\pi/G_F)^{\frac{1}{2}} \sim 600$ GeV

$\Rightarrow$ 4 steps are necessary to construct out of the Fermi theory a consistent field theory with attenuation of the 4-point coupling.

Although Fermi's phenomenological interaction was inspired by the theory of electromagnetism, the analogy was not complete, and one may hope to obtain a more satisfactory theory by pushing the analogy further. An obvious guideline is to assume that the weak interaction, like quantum electrodynamics, is mediated by vector boson exchange. The intermediate weak boson must have the following three properties:

(i) It carries charge ± 1, because the familiar manifestations of the weak interactions (such as β-decay) are charge-changing.

(ii) It must be rather massive, to reproduce the short range of the weak force.

(iii) Its parity must be indefinite.

1.) Introduction of charged $W^\pm$ bosons [*Yukawa*]:
Interaction range $\sim m_W^{-1} \Rightarrow$

$E \to \infty : \sigma \sim \frac{G_F^2 m_W^2}{\pi} \to$ partial-wave unitarity is fulfilled; $G_F = g_W^2/m_W^2$.
2.) Introduction of a neutral vector boson W^3 [*Glashow*]:

The introduction of the intermediate boson softens the divergence of the s-wave amplitude for the above process, it gives rise, however, to new divergences in other processes:

Production of longitudinally polarized W's in $\nu\bar{\nu}$ collisions, cf. Fig. 7.3:

$$\begin{aligned} &\epsilon^L_\lambda = \left(\tfrac{k_\lambda}{m_W}, 0, 0, \tfrac{E}{m_W}\right) \approx \tfrac{E}{m_W} \text{ in the limit of high energies} \\ &\sigma(\nu\bar{\nu} \to W_L W_L) \sim \tfrac{g_W^4}{s}\left(\tfrac{\sqrt{s}}{m_W}\right)^4 \sim \tfrac{g_W^4 s}{m_W^4} \end{aligned} \qquad (7.7)$$

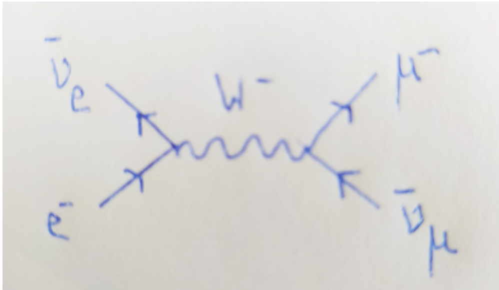


Figure 7.2: Introduction of the W boson in the process $e^-\bar{\nu}_e \to \mu^-\bar{\nu}_\mu$.

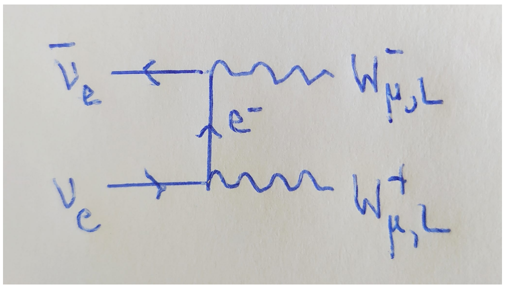


Figure 7.3: Production of longitudinally polarized W's in $\nu\bar{\nu}$ collisions.

It violates unitarity for $\sqrt{s} \gtrsim 1$ TeV.

Solution: Introduction of a neutral W^3, coupled to fermions and $W^\pm$ (cf. Fig. 7.4): Condition for the disappearance of the linear s singularity:

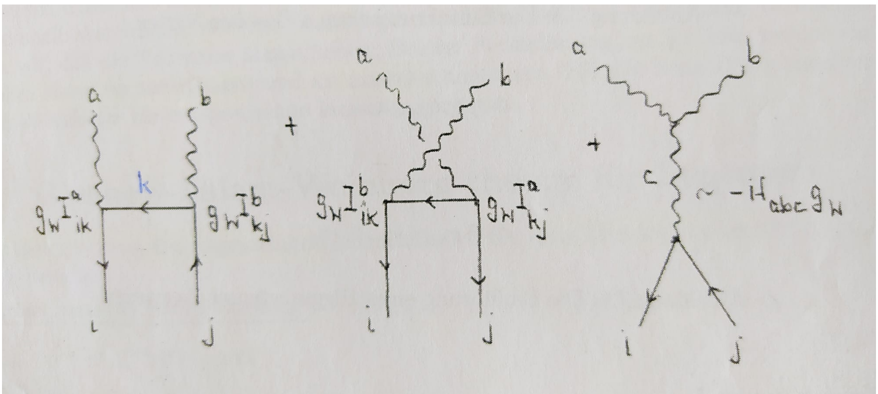


Figure 7.4: Introduction of the W^3 in $\nu\bar{\nu} \to W_L W_L$.

$$I^a_{ik} I^b_{kj} - I^b_{ik} I^a_{kj} - i f_{abc} I^c_{ij} = 0$$

$\boxed{[I^a, I^b] = i f_{abc} I^c}$ The fermion-boson couplings form a Lie algebra [associated to a non-abelian group].

$$\left.\begin{array}{llll}\text{Fermion-boson coupling} & \sim & g_W \times & \text{representation matrix} \\ \text{Boson-boson coupling} & \sim & g_W \times & \text{structure constants}\end{array}\right\} g_W \text{ universal.}$$

3.) 4-point coupling:

$W_L W_L \to W_L W_L$ (cf. Fig. 7.5)

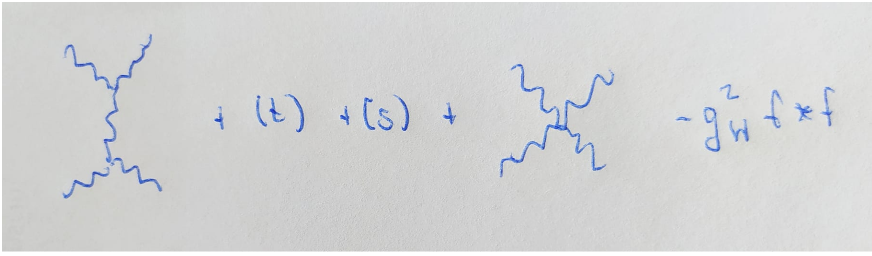


Figure 7.5: Introduction of the four-point coupling.

Amplitude $\sim g_W^2 f^2 \frac{s^2}{m_W^4} + ...$ compensated by: $-g_W^2 f^2 \frac{s^2}{m_W^4}$:

4-boson vertex: $\sim g_W^2 f \star f$

4.) Higgs particle: [*Weinberg, Salam*]

The remaining linear s divergence is canceled by the exchange of a scalar particle with a coupling $\sim$ mass of the source, cf. Fig. 7.6.

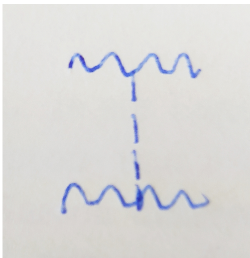


Figure 7.6: Introduction of the Higgs exchange.

Amplitude $\sim -(g_W m_W)^2 \frac{1}{s} \left(\frac{\sqrt{s}}{m_W}\right)^4 \sim -g_W^2 \frac{s}{m_W^2}$

The same mechanism cancels the remaining singularity in $f\bar{f} \to W_L W_L$ (f massive!), cf. Fig. 7.7.

Adding up the gauge diagrams we are left with $\sim g_W^2 \frac{m_f \sqrt{s}}{m_W^2}$

scalar diagram $\sim \sqrt{s}\left(g_W \frac{m_f}{m_W}\right) \frac{1}{s} (g_W m_W) \left(\frac{\sqrt{s}}{m_W}\right)^2 \sim g_W^2 \frac{\sqrt{s} m_f}{m_W^2}$

Summary:

A theory of massive gauge bosons and fermions that are weakly coupled up to very high energies, requires, by unitarity, the existence of a Higgs particle; the Higgs particle is a scalar 0^+ *particle that couples to other particles proportional to the masses of the particles.*

$\Rightarrow$ Non-abelian gauge field theory with spontaneous symmetry breaking.

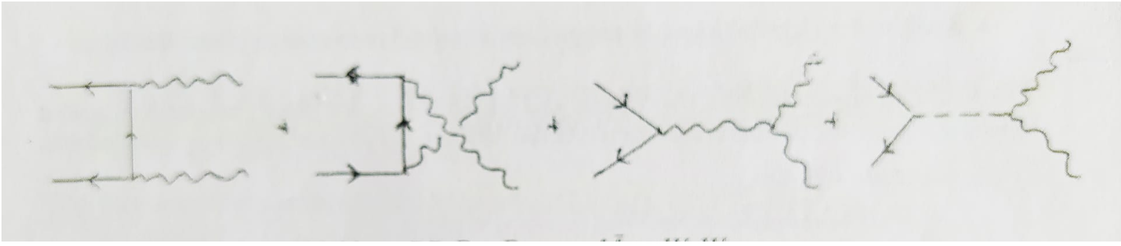


Figure 7.7: The process $f\bar{f} \to W_L W_L$.

7.3 Gauge Symmetry and Particle Content

The underlying gauge symmetry of the SM is the $SU(3)_C \times SU(2)_L \times U(1)_Y$. The $SU(3)_C$ describes QCD, and $SU(2)_L \times U(1)_Y$ the electroweak sector. The conserved charge associated with QCD is the colour charge. The conserved charges associated with the electroweak sector are the weak isospin and the weak hypercharge. The corresponding gauge bosons are in QCD the 8 massless gluons and in the electroweak sector the massive charged $W^\pm$ bosons, the massive neutral Z boson and the massless photon γ. These particles are also called interaction particles and carry spin 1.

The particle content is given by the matter particles and the interaction particles. The matter particles are fermions with spin 1/2 and are subdivided into three families. They comprise 6 quarks and 6 leptons. We know three up-type (up, charm, top) and three down-type (down, strange, bottom) quarks. The leptons consist of three charged (e, μ, τ) and three neutral leptons, the neutrinos (ν_e, ν_μ, ν_τ), *cf.* Table 7.1.

The three lepton and quark families have identical quantum numbers, respectively, and are only distinguished through their masses. Therefore, when discussing the gauge interaction is is sufficient to consider only one family. The transformation behaviour of the quark and lepton fields under the SM gauge groups is summarized (for one generation) in Table 7.2.

$$\begin{array}{cccl} u & c & t & \multirow{2}{*}{$\Big\}$ Quarks} \\ d & s & b & \\ \nu_e & \nu_\mu & \nu_\tau & \multirow{2}{*}{$\Big\}$ Leptons} \\ e & \mu & \tau & \\ 1. & 2. & 3. & \text{Family} \end{array}$$

Table 7.1: Matter particles of the Standard Model.

The masses of the particles are generated through spontaneous symmetry breaking (SSB). For this a complex Higgs doublet ($d_D = 4$ degrees of freedom) is added together with the Higgs potential V. The SSB breaks down the $SU(2)_L \times U(1)_Y$ ($d_{EW} = 4$) to the electromagnetic $U(1)_{em}$ ($d_{em} = 1$). The electromagnetic charge hence remains conserved. Associated with this SSB are $d_{EW} - d_{em} = 4 - 1 = 3$ would-be Goldstone bosons that are absorbed to give masses to the $W^\pm$ and Z bosons. The photon remains massless. Furthermore, after SSB there is $d_D - (d_{EW} - d_{em}) = 4 - (4 - 1) = 4 - 3 = 1$ Higgs particle in the spectrum.

One last remark is at order: We know that the neutrinos have mass. When we formulate

Field	$U(1)_Y \times SU(2)_L \times SU(3)_C$
$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$(\frac{1}{3}, \mathbf{2}, \mathbf{3})$
u_R	$(\frac{2}{3}, \mathbf{1}, \bar{\mathbf{3}})$
d_R	$(-\frac{4}{3}, \mathbf{1}, \bar{\mathbf{3}})$
$L_L = \begin{pmatrix} e_L \\ \nu_{e_L} \end{pmatrix}$	$(-1, \mathbf{2}, \mathbf{1})$
e_R	$(2, \mathbf{1}, \bar{\mathbf{1}})$

Table 7.2: Transformation behaviour under the SM gauge groups.

the SM in the following we will neglect the neutrino masses and assume neutrinos to be massless. For the treatment of massive neutrino we refer to the literature.

7.4 Glashow-Salam-Weinberg Theory for Leptons

For simplicity, we only consider the first lepton generation, *i.e.* e, ν_e. The generalization to the other generations is trivial. We have the
electromagnetic interaction:

$$\mathcal{L}_{int} = -e_0 j_\mu^{elm} A^\mu \qquad \text{with} \tag{7.8}$$

$$j_\mu^{elm} = -\bar{e}\gamma_\mu e\ , \tag{7.9}$$

where e_0 denotes the elementary charge with $\alpha = e_0^2/4\pi$. And we have the
weak interaction:

$$\mathcal{L}_W = -\frac{4G_F}{\sqrt{2}} j_\mu^- j^{\mu+} \tag{7.10}$$

in the Fermi notation for charged currents, with

$$j_\mu^+ = \bar{\nu}_e \gamma_\mu \frac{1-\gamma_5}{2} e = \bar{\nu}_{eL}\gamma_\mu e_L \qquad \text{(left-chiral)} \tag{7.11}$$

$$j_\mu^- = (j_\mu^+)^* \tag{7.12}$$

G_F denotes the Fermi constant, $G_F = 10^{-5}/m_P^2$.

The next steps are:

- Resolve the 4-Fermi coupling through the exchange of a heavy vector boson. Apart from the vector boson mass the structure of the weak interaction is similar to the one of electrodynamics.
- Construction of the theory as gauge field theory with spontaneous symmetry breaking, to guarantee renormalizability.
- Analysis of the physical consequences of the symmetry and its breaking.

The free Lagrangian for the electrons and left-handed neutrinos[2] is given by the following expression that takes into account that the particles are massless in case of chiral invariance,

$$\begin{aligned}\mathcal{L}_0 &= \bar{e} i\partial\!\!\!/ e + \bar{\nu}_{eL} i\partial\!\!\!/ \nu_{eL} \\ &= \bar{e}_L i\partial\!\!\!/ e_L + \bar{e}_R i\partial\!\!\!/ e_R + \bar{\nu}_{eL} i\partial\!\!\!/ \nu_{eL} \ ,\end{aligned} \tag{7.13}$$

where

$$f_{R,L} = \frac{1}{2}(1 \pm \gamma_5) f. \tag{7.14}$$

The free Lagrangian $\mathcal{L}_0$ is $SU(2)_L$ symmetric. The associated conserved charge is the weak isospin:

$$\begin{array}{lll} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L & : \text{ iso-doublet with } \quad I(\nu_{eL}) = I(e_L) = \dfrac{1}{2} \quad \text{and} & I_3(\nu_{eL}) = +\dfrac{1}{2} \\ & & I_3(e_L) = -\dfrac{1}{2} \\ e_R & : \text{ Iso-singlet with } \quad I(e_R) = I_3(e_R) = 0 & \end{array} \tag{7.15}$$

The Lagrangian

$$\mathcal{L}_0 = \overline{\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L} i\partial\!\!\!/ \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L + \bar{e}_R i\partial\!\!\!/ e_R \tag{7.16}$$

is invariant under the global isospin transformation

$$\begin{aligned}\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L &\to e^{-\frac{i}{2} g\vec{\alpha}\vec{\tau}} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \\ e_R &\to e_R\end{aligned} \tag{7.17}$$

The theory becomes locally $SU(2)_L$ invariant through the introduction of an isovector $\vec{W}_\mu$ of vector fields with minimal coupling:

$$\left.\begin{array}{lll} \text{doublet} & : & i\partial\!\!\!/ \to i\partial\!\!\!/ - \frac{g}{2}\vec{\tau}\vec{W\!\!\!/} \\ \text{singlet} & : & i\partial\!\!\!/ \to i\partial\!\!\!/ \end{array}\right\} \qquad \text{from:} \quad i\partial\!\!\!/ \to i\partial\!\!\!/ - g\vec{I}\vec{W\!\!\!/} \tag{7.18}$$

The resulting interaction Lagrangian for the lepton-W coupling reads:

$$\begin{aligned}\mathcal{L}_{int} &= -\frac{g}{2} \overline{\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L} \gamma_\mu \vec{\tau} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \vec{W}^\mu \\ &= -\frac{g}{2\sqrt{2}} \bar{\nu}_e \gamma_\mu (1-\gamma_5) e W^{+\mu} + h.c. - \frac{g}{4}\{\bar{\nu}_e \gamma_\mu (1-\gamma_5)\nu_e - \bar{e}\gamma_\mu(1-\gamma_5)e\} W^{3\mu}\end{aligned} \tag{7.19}$$

where we have introduced

$$W^\pm = \frac{1}{\sqrt{2}}(W^1 \mp iW^2) \ . \tag{7.20}$$

From Eq. (7.19) we can read off

[2] The Goldhaber experiment (1957) has shown, that neutrinos appear in nature only as left-handed particles. This is a confirmation of the $V-A$ theory that predicts the parity violation of the weak interaction.

- The charged lepton current has per construction the correct structure.
- W_μ^3, the neutral isovector field cannot be identified with the photon field A_μ since the electromagnetic current does not contain any ν's and furthermore has a pure vector character (and hence does not contain a γ_5).

This leads to the formulation of the minimal $SU(2)_L \times U(1)_Y$ gauge theory:

The Lagrangian $\mathcal{L}_0$, Eq. (7.19), has an additional $U(1)$ gauge symmetry (after coupling $\vec{W}$) and associated with this the weak hypercharge. The quanum numbers are defined in such a way that we obtain the correct electromagnetic current:
(In order to include electromagnetism we define the "weak hypercharge".)

$$
\begin{aligned}
j_\mu^{elm} = -\bar{e}\gamma_\mu e &= -\bar{e}_L\gamma_\mu e_L - \bar{e}_R\gamma_\mu e_R \\
&= \underbrace{\frac{1}{2}\overline{\begin{pmatrix}\nu_e\\ e\end{pmatrix}}_L \gamma_\mu\tau_3 \begin{pmatrix}\nu_e\\ e\end{pmatrix}_L}_{\substack{\text{Isovector current,}\\ \text{couples only to } W_\mu^3}} - \underbrace{\frac{1}{2}\overline{\begin{pmatrix}\nu_e\\ e\end{pmatrix}}_L \gamma_\mu 1 \begin{pmatrix}\nu_e\\ e\end{pmatrix}_L - \bar{e}_R\gamma_\mu e_R}_{\substack{\text{Iso-singlets, for the construction}\\ \text{of the hypercharge current}}}
\end{aligned}
\tag{7.21}
$$

The hypercharge quantum numbers are

$$
Y(\nu_{eL}) = Y(e_L) = -1 \tag{7.22}
$$

$$
Y(e_R) = -2\,. \tag{7.23}
$$

This follows from the requirement that the Gell-Mann Nishijima relation[3] holds

$$
Q = I_3 + \frac{1}{2}Y \tag{7.24}
$$

Local gauge invariance is achieved through the minimal coupling of the gauge vector field,

$$
i\partial\!\!\!/ \rightarrow i\,\partial\!\!\!/ - \frac{g'}{2}Y B\!\!\!/\,. \tag{7.25}
$$

This leads to the Lagrangian

$$
\begin{aligned}
\mathcal{L}_{int} = -\frac{g}{\sqrt{2}}\bar{\nu}_{eL}\gamma_\mu e_L W^{+\mu} + h.c. \quad &- \quad \frac{g}{2}\{\bar{\nu}_{eL}\gamma_\mu\nu_{eL} - \bar{e}_L\gamma_\mu e_L\}W^{3\mu} \\
&+ \quad g'\{\frac{1}{2}\bar{\nu}_{eL}\gamma_\mu\nu_{eL} + \frac{1}{2}\bar{e}_L\gamma_\mu e_L + \bar{e}_R\gamma_\mu e_R\}B^\mu
\end{aligned}
\tag{7.26}
$$

From the Lagrangian Eq (7.26) we can read off:

- The charged currents remain unchanged.
- We can introduce a mixture between W^3 and B in such a way that the pure parity-invariant electron photon interaction is generated. We are left with a neutral current interaction with the orthogonal field combination:

$$
\left.\begin{aligned}
A_\mu &= \cos\theta_W B_\mu + \sin\theta_W W_\mu^3 \\
Z_\mu &= -\sin\theta_W B_\mu + \cos\theta_W W_\mu^3
\end{aligned}\right\}
\qquad
\left.\begin{aligned}
B_\mu &= \cos\theta_W A_\mu - \sin\theta_W Z_\mu \\
W_\mu^3 &= \sin\theta_W A_\mu + \cos\theta_W Z_\mu
\end{aligned}\right\}
\tag{7.27}
$$

[3] Originally this equation was derived from empiric observations. Nowadays it is understood as result of the quark model.

Here θ_W denotes the Weinberg angle. Rewriting the Lagrangian in terms of A_μ and Z_μ leads to the A_μ coupling

$$A_\mu\{\bar{\nu}_{eL}\gamma_\mu\nu_{eL}\{-\frac{g}{2}\sin\theta_W+\frac{g'}{2}\cos\theta_W\}+\bar{e}_L\gamma_\mu e_L\{\frac{g}{2}\sin\theta_W+\frac{g'}{2}\cos\theta_W\}+\bar{e}_R\gamma_\mu e_R g'\cos\theta_W\}\ . \tag{7.28}$$

The neutrino ν can be eliminated through

$$\underline{\underline{\tan\theta_W=\frac{g'}{g}}}\ . \tag{7.29}$$

(The photon only couples to charged particles!) The correct e-coupling is obtained by

$$\left.\underline{\underline{\begin{array}{rcl} g'\cos\theta_W &=& e_0 \\ g\sin\theta_W &=& e_0 \end{array}}}\right\}\frac{1}{e_0^2}=\frac{1}{g^2}+\frac{1}{g'^2} \tag{7.30}$$

The lepton-boson interaction hence reads

$$\begin{aligned}\mathcal{L}_{int} &= -\frac{g}{2\sqrt{2}}\bar{\nu}_e\gamma_\mu(1-\gamma_5)eW^{+\mu}+h.c.\\ &- \frac{g}{4\cos\theta_W}\{\bar{\nu}_e\gamma_\mu(1-\gamma_5)\nu_e-\bar{e}\gamma_\mu(1-\gamma_5)e+4\sin^2\theta_W\bar{e}\gamma_\mu e\}Z^\mu\\ &+ e_0\bar{e}\gamma_\mu eA^\mu\end{aligned} \tag{7.31}$$

The first line describes the charged current interactions, the second line the neutral current interaction and the third line the electromagnetic ineractions. The coupling constants of the theory are: $[g,g']$ or $[e_0,\sin\theta_W]$.

- The coupling $e_0=\sqrt{4\pi\alpha}\sim\frac{1}{3}$ is fixed within electromagnetism.
- The second parameter is not fixed through the weak interactions as the charged current only fixes the relation $\frac{G_F}{\sqrt{2}}=\frac{g^2}{8m_W^2}$.

With the notation

$$\begin{aligned} j_\mu^+ &= \bar{\nu}_e\gamma_\mu\frac{1-\gamma_5}{2}e\\ j_\mu^3 &= \overline{\begin{pmatrix}\nu_e\\ e\end{pmatrix}}_L\gamma_\mu\frac{\tau^3}{2}\begin{pmatrix}\nu_e\\ e\end{pmatrix}_L\\ j_\mu^{em} &= -\bar{e}\gamma_\mu e\end{aligned} \tag{7.32}$$

the interaction Lagrangian can be written as

$$\boxed{\begin{aligned}\mathcal{L}_{int} &= -\frac{g}{\sqrt{2}}j_\mu^- W^{+\mu}+h.c.\\ &\quad -\frac{g}{\cos\theta_W}\{j_\mu^3-\sin^2\theta_W j_\mu^{em}\}Z^\mu \qquad (7.33)\\ &\quad -e_0 j_\mu^{em}A^\mu\end{aligned}}$$

where $\underline{g=\frac{e_0}{\sin\theta_W}}$.

However, the Lagrangian does not conain mass terms for the fermions and gauge bosons yet. The theory must be modified in such a way that the particles obtain their masses without getting into conflict with the gauge symmetries underlying the theory.

7.5 Introduction of the W, Z Boson and Fermion Masses

Let us repeat. With the currents

$$\begin{aligned} j_\mu^\pm &= \bar{l}_L \gamma_\mu \tau^\pm l_L \qquad \text{where} \quad l_L = (\nu_e, e)_L \\ j_\mu^3 &= \bar{l}_L \gamma_\mu \frac{1}{2} \tau^3 l_L \qquad (7.34) \\ j_\mu^{em} &= -\bar{e}\gamma_\mu e \qquad (7.35) \end{aligned}$$

the interaction Lagrangian can be written as

$$\begin{aligned} \mathcal{L}_{int} &= -\frac{g}{\sqrt{2}} j_\mu^- W^{+\mu} + h.c. \\ & -\frac{g}{\cos\theta_W}\{j_\mu^3 - \sin^2\theta_W j_\mu^{em}\} Z^\mu \qquad (7.36) \\ & -e_0 j_\mu^{em} A^\mu \qquad (7.37) \end{aligned}$$

and the couplings fulfill the relations

$$\begin{aligned} \frac{g'}{g} &= \tan\theta_W \\ \frac{G_F}{\sqrt{2}} &= \frac{g^2}{8m_W^2} \qquad (7.38) \\ e_0 &= g\sin\theta_W \ . \end{aligned}$$

The generation of masses for the 3 vector fields, hence the absorption of 3 Goldstone bosons, is not possible with 3 scalar fields. The minimal solution is the introcduction of one complex doublet with 4 degrees of freedom,

$$\phi = \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix} \qquad \text{with} \qquad \begin{aligned} \phi_+ &= \tfrac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \\ \phi_0 &= \tfrac{1}{\sqrt{2}}(\phi_3 + i\phi_4) \end{aligned} \qquad (7.39)$$

The Lagrangian of the doublet field ϕ is given by

$$\underline{\underline{\mathcal{L}_\phi = \partial_\mu\phi^*\partial^\mu\phi - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2}} \qquad (7.40)$$

It is $SU(2)_L \times U(1)_Y$ invariant. The field ϕ transforms as

$$\phi \to e^{-\frac{i}{2}g\vec{\alpha}\vec{\tau}} e^{-\frac{i}{2}g'\beta}\, \phi \qquad (7.41)$$

After spontaneous symmetry breaking the vacuum expectation value of the scalar field is

$$<\phi>= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \qquad v^* = v \ . \qquad (7.42)$$

It breaks the $SU(2)_L \times U(1)_Y$ symmetry, but is invariant under the $U(1)_{em}$ symmetry, generated by the electric charge operator. Since each (would-be) Goldstone boson is associated with a generator that breaks the vacuum, we have $4-1=3$ Goldstone bosons. The quantum numbers of the fields ϕ are

$$\left.\begin{aligned} I_3(\phi_+) &= +\tfrac{1}{2} \quad Y(\phi_+) = +1 \\ I_3(\phi_0) &= -\tfrac{1}{2} \quad Y(\phi_0) = +1 \end{aligned}\right\} \begin{aligned} Q(\phi_+) &= 1 \\ Q(\phi_0) &= 0 \end{aligned} \qquad (7.43)$$

(The field ϕ transforms as an $SU(2)_L$ doublet and therefore has to have the hypercharge $Y_\phi = 1$.) The gauge fields are introduced through minimal coupling,

$$i\partial_\mu \to i\partial_\mu - \frac{g}{2}\vec{\tau}\vec{W}_\mu - \frac{g'}{2}B_\mu \ . \tag{7.44}$$

Expanding about the minimum of the Higgs potential

$$\begin{aligned} & \phi_+(x) \to 0 \\ \phi_0(x) \to \frac{1}{\sqrt{2}}[v+\chi(x)] \qquad & \chi* = \chi \end{aligned} \tag{7.45}$$

one obtains from the kinetic part of the Lagrangian of the scalar field

$$\begin{aligned} \mathcal{L}_m &= \left|\left[(i\frac{g}{2}\vec{\tau}\vec{W} + i\frac{g'}{2}B)\begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}\right]\right|^2 \\ &= \frac{1}{2}\frac{v^2}{4}\begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ B \end{pmatrix}\begin{pmatrix} g^2 & & & \\ & g^2 & & \\ & & g^2 & -gg' \\ & & -gg' & g'^2 \end{pmatrix}\begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ B \end{pmatrix} \end{aligned} \tag{7.46}$$

with the eigenvalues of the mass matrix given by

$$\begin{aligned} m_1^2 &= m_2^2 = \frac{g^2v^2}{4} \\ m_3^2 &= \frac{(g^2+g'^2)v^2}{4} \\ m_4^2 &= 0 \end{aligned} \tag{7.47}$$

Thereby the masses of the gauge bosons read

$$\boxed{\begin{aligned} m_\gamma^2 &= 0 && (7.48) \\ m_W^2 &= \frac{1}{4}g^2v^2 && (7.49) \\ m_Z^2 &= \frac{1}{4}(g^2+g'^2)v^2 && (7.50) \end{aligned}}$$

They fulfill the folowing <u>mass relations</u>:

(i) W boson mass: We have $e_0^2 = g^2\sin^2\theta_W = 4\sqrt{2}G_F\sin^2\theta_W m_W^2$, from which follows

$$\boxed{m_W^2 = \frac{\pi\alpha}{\sqrt{2}G_F}\frac{1}{\sin^2\theta_W} \qquad (7.51)}$$

with $\alpha \approx \alpha(m_Z^2)$ (effective radiative correction). With $\sin^2\theta_W \approx 1/4$ the W boson mass is $m_W \approx 80$ GeV.

(ii) Z boson mass: With

$$\frac{m_W^2}{m_Z^2} = \cos^2\theta_W \tag{7.52}$$

we obtain

$$\boxed{\sin^2\theta_W = 1 - \frac{m_W^2}{m_Z^2} \qquad (7.53)}$$

Finally one obtains with Eq. (7.49) for the Higgs vacuum expectation value

$$\frac{1}{v^2} = \frac{g^2}{4m_W^2} = \sqrt{2}G_F \tag{7.54}$$

and thereby

$$\boxed{v = \frac{1}{\sqrt{\sqrt{2}G_F}} \approx 246 \text{ GeV} \qquad (7.55)}$$

The vacuum expectation value v is the characteristic scale of electroweak symmetry breaking.

The Higgs mechanism for charged lepton masses: The fermions couple via the gauge-invariant Yukawa coupling to the Higgs field ϕ. The interaction Lagrangian reads

$$\mathcal{L}(ee\Phi) \;=\; -f_e\overline{\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L}\phi e_R + h.c. \tag{7.56}$$

It is invariant under $SU(2)_L \times U(1)_Y$. After expansion of the Higgs field around the VEV one obtains

$$\begin{aligned} \mathcal{L}(ee\Phi) &= -f_e\frac{v}{\sqrt{2}}[\bar{e}_L e_R + \bar{e}_R e_L] + ... \\ &= -f_e\frac{v}{\sqrt{2}}\bar{e}e + ... \\ &= -m_e\bar{e}e + ... \end{aligned} \tag{7.57}$$

The electron mass is given by

$$\boxed{m_e = \frac{f_e v}{\sqrt{2}} \qquad (7.58)}$$

7.6 Quarks in the Glashow-Salam-Weinberg Theory

In this chapter the hadronic sector is implemented in the SM of the weak and electromagnetic interactions. This is done in the context of the quark model. Since quarks and leptons ressemble each other, the construction for the quark sector is obvious, but not trivial.

We know from the previous chapters that the lepton currents are built from the multiplets given by

$$\boxed{\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \quad e_R^- \qquad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L \quad \mu_R^- \qquad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L \quad \tau_R^- \qquad (7.59)}$$

This can be generalized to the quark currents.

For the quark currents for u, d, s we have:

1) The electromagnetic current, after summation over all possible charges, is given by

$$j_\mu^{elm} = \sum_{Q_q} Q_q \bar{q}\gamma_\mu q = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s \tag{7.60}$$

2) From low-energy experiments (pion and Kaon decays) it followed that the left-handed weak current, the Cabibbo current, is given by[4]

$$\begin{aligned} j_\mu^- &= \cos\theta_c \bar{u}\gamma_\mu \frac{1}{2}(1-\gamma_5)d + \sin\theta_c \bar{u}\gamma_\mu \frac{1}{2}(1-\gamma_5)s \\ &= \bar{u}\gamma_\mu \frac{1}{2}(1-\gamma_5)[\cos\theta_c d + \sin\theta_c s] \end{aligned} \tag{7.61}$$

with $\sin^2\theta_c \approx 0.05$. We define the Cabibbo rotated quarks

$$\begin{aligned} d_c &= \cos\theta_c d + \sin\theta_c s \\ s_c &= -\sin\theta_c d + \cos\theta_c s \end{aligned} \tag{7.62}$$

Here,

d, s	are different directions in the (u, d, s) space of quarks, characterized by different masses, *i.e.* we are in the mass basis.
d_c, s_c	are directions in the quark space, characterized through the weak interacation, they represent the current basis.

The current $j_\mu^\pm$ can be expressed through $j_\mu^\mp = \bar{Q}_L\gamma_\mu\tau^\pm Q_L$ with the definitions of the multiplets given by

$$\begin{pmatrix} u \\ d_c \end{pmatrix}_L \quad s_{cL} \quad \begin{matrix} u_R \\ d_{cR} \end{matrix} \quad s_{cR} \tag{7.63}$$

3) The corresponding neutral isovector current is then given by

$$\begin{aligned} j_\mu^3 &= \sum_{doublets} \bar{Q}_L\gamma_\mu \frac{1}{2}\tau^3 Q_L \\ &\sim \bar{u}_L\gamma_\mu u_L - \bar{d}_{cL}\gamma_\mu d_{cL} \\ &= \bar{u}_L\gamma_\mu u_L - \cos^2\theta_c \bar{d}_L\gamma_\mu d_L - \sin^2\theta_c \bar{s}_L\gamma_\mu s_L \\ &\quad -\sin\theta_c\cos\theta_c[\bar{d}_L\gamma_\mu s_L + \bar{s}_L\gamma_\mu d_L] \end{aligned} \tag{7.64}$$

The first line is a diagonal neutral current. The second line is a strangeness changing neutral current with the strength $\sim \sin\theta_c$, like the strangeness changing charged current.

[4] Cabibbo's conjecture was that the quarks that participate in the weak interactions are a mixture of the quarks that participate in the strong interaction. The mixing was originally postulated by Cabibbo (1963) to explain certain decay patterns in the weak interactions and originally had only to do with the d and s quarks.

This is in striking contradiction with the experimental non-observation of strangeness changing neutral current reactions. There are strict experimental limits on the decay rates that are mediated by strangeness changing neutral currents like

$$
\begin{aligned}
1)\ \frac{\Gamma(K_L \to \mu^+\mu^-)}{\Gamma(K^+ \to \mu^+\nu_\mu)} &< \sim 4 \cdot 10^{-9}(\mathrm{exp}) \\
2)\ \frac{\Gamma(K^+ \to \pi\nu\bar{\nu})}{\Gamma(K^+ \to all)} &< 1.4 \cdot 10^{-7}(\mathrm{exp}) \qquad (7.65) \\
3)\ \frac{|m(K_L) - m(K_S)|}{m(K)} &< 7 \cdot 10^{-15} m_{K^0}(\mathrm{exp}) \qquad (7.66)
\end{aligned}
$$

1) The observed rate for the decay $K_L \to \mu^+\mu^-$ can be understood in terms of QED and the known $K_L \to \gamma\gamma$ transition rate and leaves little room for an elementary $\bar{s}d \to \mu^+\mu^-$ transition.
2) The decay $K^+ \to \pi\nu\bar{\nu}$ can be understood in terms of the elementary reaction $\bar{s} \to \bar{d}\nu\bar{\nu}$.
3) Similarly the smallness of observables linked to to $|\Delta S| = 2$ transition amplitudes, such as the $K_L - K_S$ mass difference leaves little room for strangeness changing neutral currents.

Thus, in the Weinberg-Salam model, or more generally in models that allow for neutral current reactions that are proportional to the third component of the weak isospin, it is important to prevent the appearance of strangeness changing neutral currents. An elegant solution to the problem of flavour-changing neutral currents was proposed by Glashow, Iliopoulos and Maiani.

We need a "natural mechanism", *i.e.* originating from a symmetry, stable against perturbations, that suppresses 8 orders of magnitude. This can be achieved through the introduction of a fourth quark, the <u>charm quark c</u>. [Glashow, Iliopoulos, Maiani, PRD2(70)1985]

The new multiplet structure is then given by

$$
\boxed{\begin{pmatrix} u \\ d_c \end{pmatrix}_L \quad \begin{pmatrix} c \\ s_c \end{pmatrix}_L \quad \begin{matrix} u_R \\ d_{cR} \end{matrix} \quad \begin{matrix} c_R \\ s_{cR} \end{matrix} \qquad (7.67)}
$$

(a) The isovector current now reads:

$$
j^3_\mu = \sum_{doublets} \bar{Q}_L \gamma_\mu \frac{1}{2}\tau^3 Q_L = \frac{1}{2}[\bar{u}_L\gamma_\mu u_L - \bar{d}_L\gamma_\mu d_L + \bar{c}_L\gamma_\mu c_L - \bar{s}_L\gamma_\mu s_L] \qquad (7.68)
$$

The addition of the charm quark c diagonalizes the neutral current (*GIM mechanism*) and eliminates $(\Delta S \neq 0, \mathrm{NC})$ reactions.
(b) The electromagnetic current is given by:

$$
j^{em}_\mu = \frac{2}{3}[\bar{u}\gamma_\mu u + \bar{c}\gamma_\mu c] - \frac{1}{3}[\bar{d}\gamma_\mu d + \bar{s}\gamma_\mu s] \qquad (7.69)
$$

(c) The charged current reads:

$$
j^-_\mu = \bar{u}\gamma_\mu\frac{1}{2}(1-\gamma_5)[\cos\theta_c d + \sin\theta_c s] + \bar{c}\gamma_\mu\frac{1}{2}(1-\gamma_5)[-\sin\theta_c d + \cos\theta_c s] \qquad (7.70)
$$

The first term is the Cabibbo current, the second the charm current with strong (c, s) coupling.

In 1973 (1 year before the discovery of the charm quark!) Kobayashi and Maskawa extended Cabibbo's idea to six quarks. We thereby obtain a 3×3 matrix that mixes the quarks. Only in this way the CP violation can be explained. (We come back to this point later.) We also need the 3rd quark family to obtain an anomaly-free theory. We call anomalies terms that violate the classical conservation laws. Thus it can happen that a (classical) local conservation law derived from gauge invariance with the help of Noether's theorem holds at tree level but is not respected by loop diagrams. The simplest example of a Feynman diagram leading to an anomaly is a fermion loop coupled to two vector currents and one axial current. Because the weak interaction contains both vector and axial vector currents there is a danger that such diagrams may arise in the Weinberg-Salam theory and destroy the renormalizability of the theory. The anomaly is canceled if for each lepton doublet we introduce three quark doublets corresponding to the three quark coulours. Since we have three lepton doublets we need to introduce a third quark doublet. This was also supported by the observation of a fifth quark (the b quark) in the Υ family.

7.7 The CKM Matrix

7.7.1 The Fermion Yang-Mills Lagrangian

If we take the down-type quarks in the current basis, then the matrix for the weak interaction of the fermions is diagonal (see also Eqs. (7.62) and (7.70)). With the definitions

$$
\begin{aligned}
U &= \begin{pmatrix} u \\ c \\ t \end{pmatrix} \qquad D' = \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} \\
E &= \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix} \qquad N_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} ,
\end{aligned} \tag{7.71}
$$

where $'$ denotes the fields in the current basis, we obtain for the Yang-Mills Lagrangian

$$
\begin{aligned}
\mathcal{L}_{YM-F} &= (\bar{U}_L, \bar{D'}_L) i\gamma^\mu (\partial_\mu + igW_\mu^a \frac{\tau^a}{2} + ig'Y_L B_\mu) \begin{pmatrix} U_L \\ D'_L \end{pmatrix} \\
&+ (\bar{N}_L, \bar{E}_L) i\gamma^\mu (\partial_\mu + igW_\mu^a \frac{\tau^a}{2} + ig'Y_L B_\mu) \begin{pmatrix} N_L \\ E_L \end{pmatrix} \\
&+ \sum_{\Psi_R = U_R, D'_R, E_R} \bar{\Psi}_R i\gamma^\mu (\partial_\mu + ig'Y_R B_\mu) \Psi_R \\
&= \bar{U} i\partial\!\!\!/ U + \bar{D}' i\partial\!\!\!/ D' + \bar{E} i\partial\!\!\!/ E + \bar{N}_L i\partial\!\!\!/ N_L + \mathcal{L}_{int} \; .
\end{aligned} \tag{7.72}
$$

The interaction Lagrangian reads

$$
\mathcal{L}_{int} = -eJ^\mu_{em} A_\mu - \frac{e}{\sin\theta_W \cos\theta_W} J^\mu_{NC} Z_\mu - \frac{e}{\sqrt{2}\sin\theta_W} (J^{-\mu} W^+_\mu + h.c.) \; . \tag{7.73}
$$

The electromagnetic current is given by

$$J^\mu_{em} = Q_u \bar{U}\gamma^\mu U + Q_d \bar{D}'\gamma^\mu D' + Q_e \bar{E}\gamma^\mu E \ , \tag{7.74}$$

the neutral weak current by

$$\begin{aligned} J^\mu_{NC} &= (\bar{U}_L, \bar{D}'_L)\gamma^\mu \frac{\tau_3}{2} \begin{pmatrix} U_L \\ D'_L \end{pmatrix} + (\bar{N}_L, \bar{E}_L)\gamma^\mu \frac{\tau_3}{2} \begin{pmatrix} N_L \\ E_L \end{pmatrix} - \sin^2\theta_W J^\mu_{em} \\ &= \frac{1}{2}\bar{U}_L\gamma^\mu U_L - \frac{1}{2}\bar{D}'_L\gamma^\mu D'_L + \frac{1}{2}\bar{N}_L\gamma^\mu N_L - \frac{1}{2}\bar{E}_L\gamma^\mu E_L - \sin^2\theta_W J^\mu_{em} \end{aligned} \tag{7.75}$$

and the charged weak current by

$$\begin{aligned} J^{-\mu} &= (\bar{U}_L, \bar{D}'_L)\gamma^\mu \frac{\tau_1 + i\tau_2}{2} \begin{pmatrix} U_L \\ D'_L \end{pmatrix} + (\bar{N}_L, \bar{E}_L)\gamma^\mu \frac{\tau_1 + i\tau_2}{2} \begin{pmatrix} N_L \\ E_L \end{pmatrix} \\ &= \bar{U}_L\gamma^\mu D'_L + \bar{N}_L\gamma^\mu E_L \ . \end{aligned} \tag{7.76}$$

(The latter is purely left-handed and diagonal in generation space.)

7.7.2 Mass Matrix and CKM Matrix

Remark: Be χ_1, χ_2 $SU(2)$ doublets. Then there are two possibilities to form an $SU(2)_L$ singlet:
1) $\chi_1^\dagger\chi_2$ and $\chi_2^\dagger\chi_1$
2) $\chi_1^T\epsilon\chi_2$ and $\chi_2^T\epsilon\chi_1$, where

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ .$$

Proof: Perform an $SU(2)_L$ transformation

$$\begin{aligned} \chi_1(x) &\to U(x)\chi_1(x) \qquad \chi_1^\dagger \to \chi_1^\dagger U^{-1} \\ \chi_2(x) &\to U(x)\chi_2(x) \qquad \chi_2^\dagger \to \chi_2^\dagger U^{-1} \ , \end{aligned} \tag{7.77}$$

where

$$U(x) = e^{i\omega_a(x)\tau^a/2} \ . \tag{7.78}$$

1) is invariant under this transformation.
2) Here we have

$$(U\chi_1)^T\epsilon U\chi_2 = \chi_1^T U^T\epsilon U\chi_2 = \chi_1^T\epsilon\chi_2 \tag{7.79}$$

because with

$$U = e^{iA} = \sum_0^\infty \frac{(iA)^n}{n!} \Rightarrow U^T = \sum_n \frac{(iA^T)^n}{n!} \ , \quad A = \omega_a(x)\frac{\tau^a}{2} \ . \tag{7.80}$$

And since $(\tau^a)^T\epsilon = -\epsilon\tau^a$, we obtain

$$U^T\epsilon U = \epsilon U^{-1}U = \epsilon \ , \tag{7.81}$$

so that also 2) is invariant.

The Yukawa Lagrangian: We write up the most general, renormalizable, $SU(2)_L \times U(1)_Y$ invariant hermitean fermion-fermion-boson Lagrangian. With the $SU(2)_L$ doublets

$$\begin{pmatrix} U_L \\ D'_L \end{pmatrix} , \begin{pmatrix} N_L \\ E_L \end{pmatrix} , \Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{7.82}$$

and the $SU(2)_L$ singlets

$$U_R \, , D'_R \, , E_R \tag{7.83}$$

we can construct 2 $SU(2)_L$ invariant interactions,

$$\Phi^\dagger \begin{pmatrix} \psi_{1L} \\ \psi_{2L} \end{pmatrix} = (\phi^+)^* \psi_{1L} + (\phi^0)^* \psi_{2L} \tag{7.84}$$

and

$$\Phi^T \epsilon \begin{pmatrix} \psi_{1L} \\ \psi_{2L} \end{pmatrix} = \phi^+ \psi_{2L} - \phi^0 \psi_{1L} \, , \tag{7.85}$$

so that for the Yukawa Lagrangian that conserves also the hypercharge we obtain:

$$\begin{aligned} \mathcal{L}_{Yuk} &= -(\bar{e}_R, \bar{\mu}_R, \bar{\tau}_R) C_E \begin{pmatrix} \Phi^\dagger \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} \\ \Phi^\dagger \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix} \\ \Phi^\dagger \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix} \end{pmatrix} + (\bar{u}_R, \bar{c}_R, \bar{t}_R) C_U \begin{pmatrix} \Phi^T \epsilon \begin{pmatrix} u_L \\ d'_L \end{pmatrix} \\ \Phi^T \epsilon \begin{pmatrix} c_L \\ s'_L \end{pmatrix} \\ \Phi^T \epsilon \begin{pmatrix} t_L \\ b'_L \end{pmatrix} \end{pmatrix} \\ & -(\bar{d'}_R, \bar{s'}_R, \bar{b'}_R) C_D \begin{pmatrix} \Phi^\dagger \begin{pmatrix} u_L \\ d'_L \end{pmatrix} \\ \Phi^\dagger \begin{pmatrix} c_L \\ s'_L \end{pmatrix} \\ \Phi^\dagger \begin{pmatrix} t_L \\ b'_L \end{pmatrix} \end{pmatrix} \quad + \; h.c. \, . \end{aligned} \tag{7.86}$$

The C_E, C_U, C_D are arbitrary complex matrices. We perform through the following unitary transformations a transition into an equivalent field basis (fields are no observables!)

$$\begin{array}{lllc} N_L(x) & \rightarrow & V_1 N_L(x) & U_L(x) \rightarrow V_2 U_L(x) \\ E_L(x) & \rightarrow & V_1 E_L(x) & D'_L(x) \rightarrow V_2 D'_L(x) \\ E_R(x) & \rightarrow & U_1 E_R(x) & U_R(x) \rightarrow U_2 U_R(x) \\ & & & D'_R(x) \rightarrow U_3 D'_R(x) \, , \end{array} \tag{7.87}$$

where U_1, U_2, U_3, V_1, V_2 are unitary 3×3 matrices. Since the lepton and quark doublets transform in the same way this does not change the Yang-Mills-, the Higgs- and the Yang-Mills fermion Lagrangian. Only the C matrices are changed:

$$C_E \rightarrow U_1^\dagger C_E V_1 \qquad C_U \rightarrow U_2^\dagger C_U V_2 \qquad C_D \rightarrow U_3^\dagger C_D V_2 \, . \tag{7.88}$$

By choosing the $U_1^\dagger$ and V_1 matrices appropriately we can diagonalize C_E,

$$U_1^\dagger C_E V_1 = \begin{pmatrix} h_e & & \\ & h_\mu & \\ & & h_\tau \end{pmatrix} \qquad \text{with} \quad h_e, h_\mu, h_\tau \geq 0 \, . \tag{7.89}$$

Similarly,

$$U_2^\dagger C_U V_2 = \begin{pmatrix} h_u & & \\ & h_c & \\ & & h_t \end{pmatrix} \qquad \text{with} \quad h_u, h_c, h_t \geq 0\ . \tag{7.90}$$

Equation (7.90) fixes the matrix V_2. By choosing U_3 appropriately we obtain

$$U_3^\dagger C_D V_2 = \begin{pmatrix} h_d & & \\ & h_s & \\ & & h_b \end{pmatrix} V^\dagger \qquad \text{with} \quad h_u, h_c, h_t \geq 0\ , \tag{7.91}$$

where $V^\dagger$ is a unitary matrix. We transform D'_R through $D'_R \to V^\dagger D'_R$ and obtain

$$C_D \to V \begin{pmatrix} h_d & & \\ & h_s & \\ & & h_b \end{pmatrix} V^\dagger\ . \tag{7.92}$$

We expand Φ around the vacuum expectation value

$$\Phi = \begin{pmatrix} 0 \\ \frac{v+H(x)}{\sqrt{2}} \end{pmatrix}\ , \tag{7.93}$$

where $H(x)$ is a real field, and obtain

$$\begin{aligned} &(\bar{d}'_R, \bar{s}'_R, \bar{b}'_R) V \begin{pmatrix} h_d & & \\ & h_s & \\ & & h_b \end{pmatrix} V^\dagger \begin{pmatrix} \Phi^\dagger \begin{pmatrix} u_L \\ d'_L \end{pmatrix} \\ \Phi^\dagger \begin{pmatrix} c_L \\ s'_L \end{pmatrix} \\ \Phi^\dagger \begin{pmatrix} t_L \\ b'_L \end{pmatrix} \end{pmatrix} \\ = \ &(\bar{d}'_R, \bar{s}'_R, \bar{b}'_R) V \begin{pmatrix} h_d & & \\ & h_s & \\ & & h_b \end{pmatrix} V^\dagger \begin{pmatrix} \frac{1}{\sqrt{2}}(v+H(x)) d'_L \\ \frac{1}{\sqrt{2}}(v+H(x)) s'_L \\ \frac{1}{\sqrt{2}}(v+H(x)) b'_L \end{pmatrix}\ . \end{aligned} \tag{7.94}$$

After a basis transformation

$$\begin{pmatrix} d \\ s \\ b \end{pmatrix} = V^\dagger \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} \tag{7.95}$$

we finally have

$$(\bar{d}_R, \bar{s}_R, \bar{b}_R) \begin{pmatrix} h_d & & \\ & h_s & \\ & & h_b \end{pmatrix} \frac{1}{\sqrt{2}}(v+H(x)) \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix}\ . \tag{7.96}$$

The Yang-Mills and the Higgs Lagrangian do not change under the transformation (7.95). But the Yang-Mills fermion Lagrangian becomes

$$\begin{aligned} \mathcal{L}_{YM-F} &= \bar{U} i \partial\!\!\!/ U + \bar{D} i \partial\!\!\!/ D + \bar{E} i \partial\!\!\!/ E + \bar{N}_L i \partial\!\!\!/ N_L - e J^\mu_{em} A_\mu \\ &\quad - \frac{e}{\sin\theta_W \cos\theta_W} J^\mu_{NC} Z_\mu - \frac{e}{\sqrt{2}\sin\theta_W} (J^{-\mu} W^+_\mu + \ h.c.)\ , \end{aligned} \tag{7.97}$$

with

$$J^{-\mu} = \bar{U}_L \gamma^\mu D'_L + \bar{N}_L \gamma^\mu E_L = \bar{U}_L \gamma^\mu V D_L + \bar{N}_L \gamma^\mu E_L \ . \tag{7.98}$$

The unitary 3×3 matrix V is called CKM (Cabibbo-Kobayashi-Maskawa) mixing matrix.

The matrix V is unitary, *i.e.* $V^\dagger V = VV^\dagger = 1$. We investigate the number of free parameters. For a complex $n \times n$ matrix we have $2n^2$ free parameters. Since the matrix is unitary, the number of free parameters is reduced by n^2 equations. Furthermore the phases can be absorbed by a redefinition of the fermion fields, so that the number of free parameters is reduced by further $(2n-1)$ conditions:

Parameters:	$n \times n$ complex matrix:	$2n^2$
	unitarity:	n^2
	free phase choice:	$2n-1$
		$(n-1)^2$ free parameters

In the Euler parametrisation we have

Rotation angle:	$\frac{1}{2}n(n-1)$
Phases:	$\frac{1}{2}(n-1)(n-2)$

Thus we find for $n = 2, 3$

n	angles	phases
2	1	0
3	3	1

We thereby find that in a

$2-$family theory	$\sim$	Cabibbo:	no $\mathcal{CP}$ violation with L currents
$3-$family theory	$\sim$	KM:	complex matrix $\to \mathcal{CP}$ violation
			"Prediction of a 3-family structure"'

Next we investigate how we can parametrise the matrix:

(i) Esthetic parametrisation:

$$V_{CKM} = R_{sb}(\theta_2) U(\delta) R_{sd}(\theta_1) R_{sb}(\theta_3) \tag{7.99}$$

with

$$\begin{aligned} 0 &\leq \theta_i \leq \pi/2 \\ -\pi &\leq \delta \leq +\pi \end{aligned} \tag{7.100}$$

and

$$R_{sb}(\theta_2) = \begin{pmatrix} 1 & 0 & \\ 0 & \cos\theta_2 & \sin\theta_2 \\ 0 & -\sin\theta_2 & \cos\theta_2 \end{pmatrix} \qquad \text{etc.} \qquad U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix} \tag{7.101}$$

(ii) Convenient parametrisation (Wolfenstein):

$$V = \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} \tag{7.102}$$

The parameters are determined through e.g.

(a)	Cabibbo theory:	$\lambda = 0.221 \pm 0.002$	
(b)	$b \to c$ decays:	$V_{cb} = A\lambda^2$	$\to A = 0.78 \pm 0.06$
(c)	$b \to u$ decays:	$\lvert V_{ub}/V_{cb} \rvert = 0.08 \pm 0.02$	$\to (\rho^2 + \eta^2)^{1/2} = 0.36 \pm 0.09$
(d)	t matrix elements through unitarity		

(e) $\mathcal{CP}$ violation:

The unitarity of the CKM matrix leads to the unitarity triangle

$$\begin{aligned} V_{ud}^* V_{td} + V_{us}^* V_{ts} + V_{ub}^* V_{tb} &= 0 \\ A\lambda^3(1-\rho-i\eta) - A\lambda^3 + A\lambda^3(\rho+i\eta) &= 0 \\ \Rightarrow (\rho+i\eta) + (1-\rho-i\eta) &= 1 \end{aligned} \tag{7.103}$$

We hence have the unitarity triangle with the edges $(0,0)$, $(\rho, \eta$ and $(1,0)$ in the complex plane, cf. Fig. 7.8.

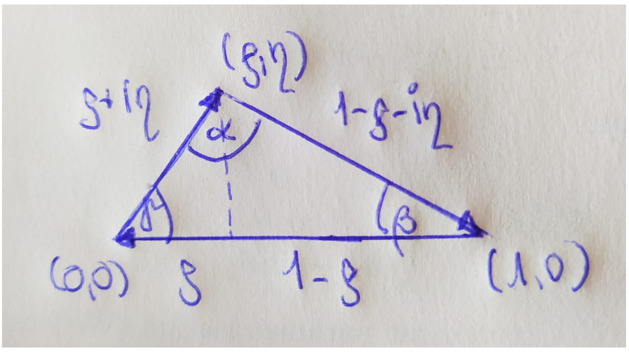


Figure 7.8: The unitary triangle.

For more information on the determination of the elements of the CKM matrix, cf. e.g. the pdg review article `https://pdg.lbl.gov/2020/reviews/rpp2020-rev-ckm-matrix.pdf`.

Chapter 8

Quantum Chromo Dynamics - QCD

8.1 Introduction of Color

QCD is the field-theoretical formulation of the strong interaction. Historically, the strong interaction was defined as

- the binding force of nucleons inside a nucleus
- the force in nucleon-nucleon scattering

Interaction distance:

$$d \sim 1\,\mathrm{fm} \to \sigma \sim \pi \frac{d^2}{4} \sim 10\,\mathrm{mb}\ . \tag{8.1}$$

Interaction strength:

$$\begin{aligned} V(R) &= \frac{g_s^2}{4\pi} e^{-\frac{R}{d}} \\ \frac{g_s^2}{4\pi} &\sim 10^2 \frac{g_{em}^2}{4\pi} \sim 1\ . \end{aligned} \tag{8.2}$$

Spin-statistics problem of the quark model:

$\Delta^{++}(s_z = \frac{3}{2}) = u(\uparrow)u(\uparrow)u(\uparrow)$ has a totally symmetrc spin wave function. Fermi-statistics, however, requires a spin wave function, which is totally antisymmetric.

(i) Ground state $\neq$ relativistic s-wave combination contrary to the naive expectation.
p-wave $\to$ knots $\to$ forbidden zones $\to$ higher energy because of the unertainty principle, contradicts naive expectation.

(ii) Magnetic moments of the nucleons
$\vec{\mu} = \frac{eQ}{2m}[\vec{l} + 2\vec{s}]$, s-waves $l = 0$: nucleon moments are built up additively from quark moments,

$$\mu_{\mathcal{N}} = < \mathcal{N}| \sum_{i=1}^{3} \mu(i)\sigma_3(i)|\mathcal{N} >\ , \tag{8.3}$$

where $\sigma_3(i)$ is the third component of the spin times two of the valence quarks in the nucleon. Due to the spin wave function: $[\mu_u = -2\mu_d]$ (the prefactors are the Clebsch-Gordan coefficients)

$$\begin{aligned}
\mu_p &= \frac{4}{3}\mu_u - \frac{1}{3}\mu_d = -(\frac{8}{3}+\frac{1}{3})\mu_d = -3\mu_d \qquad \text{for } m_u \approx m_d \\
\mu_n &= \frac{4}{3}\mu_d - \frac{1}{3}\mu_u = (\frac{4}{3}+\frac{2}{3})\mu_d = 2\mu_d
\end{aligned} \tag{8.4}$$

Ratio

$$\boxed{\frac{\mu_p}{\mu_n} = -\frac{3}{2} \qquad \text{exp.} = -1.46} \tag{8.5}$$

No $l \neq 0$ contribution required.
Effektive quark mass: $\mu_p = \frac{e}{2m_p}2.79 = -\frac{1}{3}\frac{e}{2m_d}(-3) = \frac{e}{2m_d} \Rightarrow$

$$\boxed{m_q^{eff} = \frac{m_p}{2.79} \approx 330 \text{ MeV}} \tag{8.6}$$

Solution: Quarks carry a 3-valued discriminator so that the symmetric quark model is possible.

> I. Color Hypothesis (Greenberg '64)
> Apart from flavor charges quarks also carry color charges; each quark appears in exactly 3 colors (red, blue, green = 1,2,3): $q = (q_1, q_2, q_3)$.

Color transformations: The maximal mixing group of the 3 color degrees of freedom is [$\neq$ common phase]

$$q \to q' = e^{-i\sum_{k=1}^{8}\alpha_k \frac{\lambda_k}{2}} q \tag{8.7}$$

$SU(3)_C$ transformations = unimodular, unitary 3×3 matrices [non-Abelian group].

Gell-Mann matrices: $\lambda_k,\ k = 1, ..., 8$. [3-dimensional extension of $\vec{\sigma}$ in $SU(2)$]

$$\begin{aligned}
\lambda_k^\dagger &= \lambda_k \Rightarrow e^{-i\alpha_k\frac{\lambda_k}{2}} \qquad \text{unitary: } U^\dagger U = \mathbf{1} \\
\mathrm{Tr}\lambda_k &= 0 \Rightarrow \text{unimodular: } \det U = +1\ .
\end{aligned} \tag{8.8}$$

Explicit representation:

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{aligned} \tag{8.9}$$

Properties:

$$
\begin{aligned}
\left[\frac{\lambda_i}{2}, \frac{\lambda_j}{2}\right] &= i f_{ijk} \frac{\lambda_k}{2} \qquad [A_2 \text{ algebra}] \\
\left\{\frac{\lambda_i}{2}, \frac{\lambda_j}{2}\right\} &= \frac{1}{3}\delta_{ij}\mathbf{1} + d_{ijk}\frac{\lambda_k}{2} \\
\mathrm{Tr}(\lambda_i\lambda_j) &= 2\delta_{ij} \qquad \mathrm{Tr}(\lambda_i) = 0 \ ,
\end{aligned}
\tag{8.10}
$$

with the f_{ijk} and d_{ijk} given in Fig. 8.1.

The Nonvanishing Values of f_{ijk} and d_{ijk}

(ijk)	f_{ijk}	(ijk)	d_{ijk}
123	1	118	$1/\sqrt{3}$
147	$\frac{1}{2}$	146	$\frac{1}{2}$
156	$-\frac{1}{2}$	157	$\frac{1}{2}$
246	$\frac{1}{2}$	228	$1/\sqrt{3}$
257	$\frac{1}{2}$	247	$-\frac{1}{2}$
345	$\frac{1}{2}$	256	$\frac{1}{2}$
367	$-\frac{1}{2}$	338	$1/\sqrt{3}$
458	$\sqrt{3}/2$	344	$\frac{1}{2}$
678	$\sqrt{3}/2$	355	$\frac{1}{2}$
		366	$-\frac{1}{2}$
		377	$-\frac{1}{2}$
		448	$-1/2\sqrt{3}$
		558	$-1/2\sqrt{3}$
		668	$-1/2\sqrt{3}$
		778	$-1/2\sqrt{3}$
		888	$-1/\sqrt{3}$

Figure 8.1: The values of f_{ijk} and d_{ijk}.

I'. Color Hypothesis (Gell-Mann '72)
The $SU(3)_C$ symmetry is exact. All physical (free) states, observables and interactions are $SU(3)_C$ singlets.

(a) Quarks, which are color triplets, do not exist as free particles.

(b) Color wave function

$$
\left.\begin{array}{lll}
\text{Baryons} & : & \frac{1}{\sqrt{6}}\epsilon_{ijk} \\
\text{Mesons} & : & \frac{1}{\sqrt{3}}\delta_{ij}
\end{array}\right\} \ \epsilon_{ijk}, \delta_{ij} SU(3)_C \text{ singlets} \ .
\tag{8.11}
$$

Example:

$$\begin{aligned}\Delta^{++}(s_z=+\frac{3}{2}) &= \frac{1}{\sqrt{6}}\epsilon_{ijk}u_i(\uparrow)u_j(\uparrow)u_k(\uparrow)\\ \Phi(s_z=+1) &= \frac{1}{\sqrt{3}}\delta_{ij}s_i(\uparrow)\bar{s}_j(\uparrow)\ .\end{aligned}$$

(c) Electromagnetic interaction:

$$\begin{aligned}\mathcal{L}_{em} &= -ej^\mu A_\mu\\ j_\mu &= \sum_{fl}\bar{q}\gamma_\mu Q_q q \equiv \sum_{fl}\sum_c \bar{q}_c\gamma_\mu Q_q q_c\ ,\end{aligned}\tag{8.12}$$

which is an $SU(3)$ singlet.

Tests of the color hypothesis:

1.) $\pi^0 \to \gamma\gamma$ decay

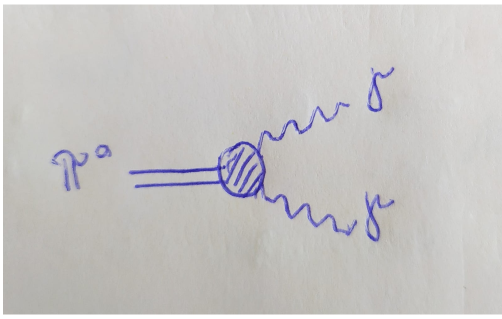


Figure 8.2: The pion decay into photons.

Decay width (cf. Fig. 8.2):

$$\boxed{\Gamma(\pi^0\to\gamma\gamma) = \frac{\alpha^2}{32\pi^3}\frac{m_\pi^3}{f_\pi^2}(Q_u^2-Q_d^2)^2N_C^2 \qquad (8.13)}$$

without color : $N_C=1$: $\Gamma=0.87$ eV
with color : $N_C=3$: $\Gamma=7.86$ eV
experimentally : $\Gamma_{exp}=7.48\pm0.33\pm0.31$ eV

2.) $e^+e^- \to$ hadrons

In the quark-parton model the production probability in $e^+e^-\to$ hadrons is determined by the one for $q\bar{q}$ pairs (cf. Fig. 8.3); final-state interactions are negligible for $\frac{d_{prod.\,q\bar{q}}}{d_{hadron}} \sim \frac{1\,\mathrm{GeV}}{E} \overset{(E\to\infty)}{\to} 0$.

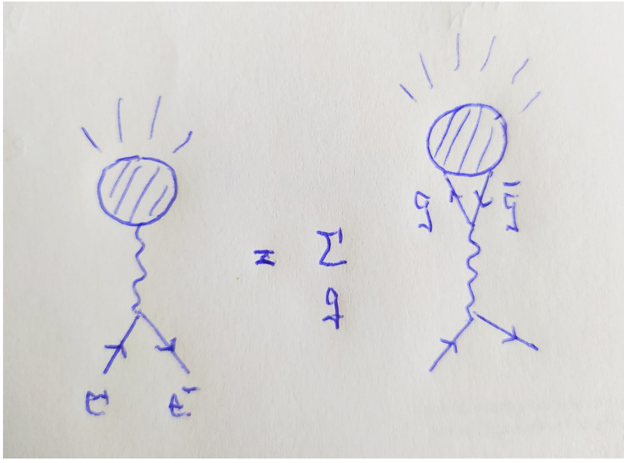


Figure 8.3: The process $e^+e^- \to$ hadrons.

$$\boxed{R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \sum_{fl,c} \frac{\sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_{fl} e_q^2 \qquad (8.14)}$$

q	e_q
u, c, t	$+\frac{2}{3}$
d, s, b	$-\frac{1}{3}$

energy	prod. quarks	R w/o color	R w/ color
< 3 GeV	u, d, s	$\frac{4}{9} + \frac{1}{9} + \frac{1}{9} = \frac{2}{3}$	2
> 5 GeV	$+c$	$\frac{6}{9} + \frac{4}{9} = \frac{10}{9}$	$\frac{10}{3}$
> 10 GeV	$+b$	$\frac{10}{9} + \frac{1}{9} = \frac{11}{9}$	$\frac{11}{3}$

The measurements (cf. Fig. 8.4) confirm the color hypothesis.

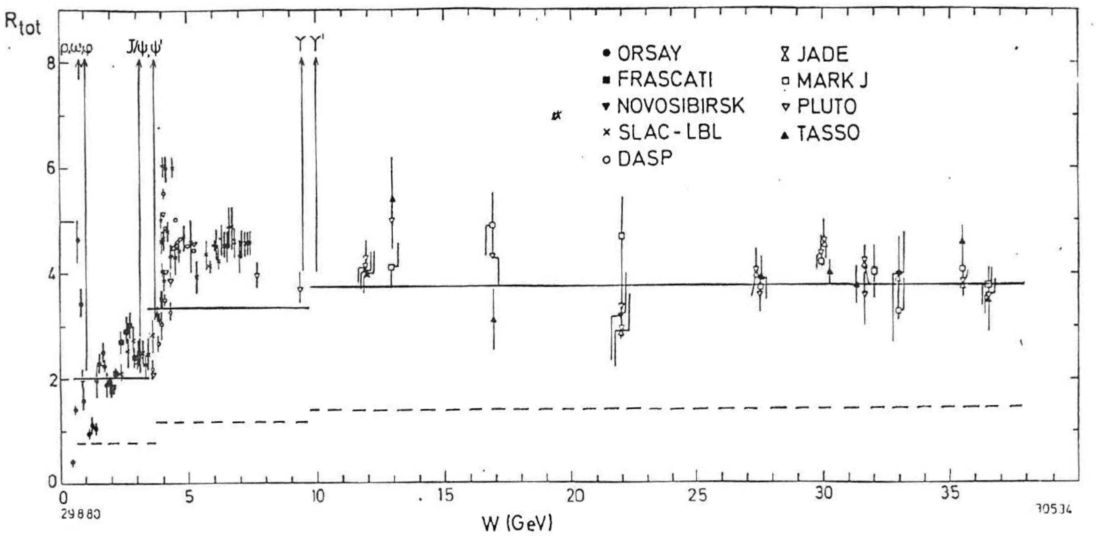


Figure 8.4: Results for R.

8.2 Gluon Gauge Fields

In anlogoy to QED:

> II.Color Hypothesis (Nambu '66, Fritzsch+Gell-Mann '72, Leutwyler '73)
> Color charges are sources of gauge fields ($\Rightarrow$ gluons) that build up the strong interaction between quarks.

Lagrangian for a color triplet:

$$\mathcal{L}_q = \bar{q}(x)(i\partial\!\!\!/ - m_q)q(x) \quad \text{with } q = (q_1, q_2, q_3),\ m_{q_1} = m_{q_2} = m_{q_3} \ \ SU(3)_C \text{ triplet} . \tag{8.15}$$

- $\mathcal{L}$ is invariant w.r.t. global, non-Abelian $SU(3)_C$ transformations

$$\left.\begin{array}{lcl} q(x) & \rightarrow & Sq(x) \\ \bar{q}(x) & \rightarrow & \bar{q}(x)S^{-1} \end{array}\right\} S = e^{-i\alpha_k T^k} \quad (T^k = \frac{\lambda_k}{2}) . \tag{8.16}$$

- $\mathcal{L}$ is not invariant w.r.t. local $SU(3)_C$ transformations: $\alpha_k = \alpha_k(x)$,

$$\mathcal{L}_q \rightarrow \mathcal{L}_q + \bar{q}(x)(S^{-1}i\partial\!\!\!/ S)q(x) . \tag{8.17}$$

$\mathcal{L}$ can be made locally gauge invariant by introducing 8 minimally coupled gluon fields $G^k_\mu(x)$ $(k = 1, ..., 8)$. (gluon matrix $G_\mu = G^k_\mu T^k$)

$$\begin{array}{lcl} i\partial_\mu & \rightarrow & i\partial_\mu - g_S G_\mu = iD_\mu \\ \mathcal{L}_q & = & \bar{q}(x)(iD\!\!\!/ - m_q)q(x) = \bar{q}(x)(i\partial\!\!\!/ - m_q - g_s\, G\!\!\!/(x))q(x) \end{array} \tag{8.18}$$

with

$$\begin{array}{lcll} q(x) & \rightarrow & S(x)q(x) & \qquad\qquad \alpha_k = \alpha_k(x) \\ \bar{q}(x) & \rightarrow & \bar{q}(x)S^{-1} & \\ G_\mu(x) & \rightarrow & SG_\mu S^{-1} - \frac{i}{g_S}S\partial_\mu S^{-1} . & \end{array} \tag{8.19}$$

The covariant derivative transforms as $[\partial_\mu(SS^{-1}) = 0]$

$$\begin{array}{rcl} iDq \rightarrow iD'q' & = & [i\partial - g_S SGS^{-1} - i(\partial S)S^{-1}]Sq \\ & = & S(i\partial - g_S G)q = SiDS^{-1}Sq . \end{array} \tag{8.20}$$

Thereby $\underline{\underline{D \rightarrow D' = SDS^{-1}}}$ (rotation).

Gluon Lagrangian We introduce

$$G_{\mu\nu} = D_\nu G_\mu - D_\mu G_\nu = \partial_\nu G_\mu - \partial_\mu G_\nu - ig_S[G_\mu, G_\nu] , \tag{8.21}$$

which transforms as: (with $G_{\mu\nu} = \frac{i}{g_S}[D_\mu, D_\nu]$)

$$G_{\mu\nu} \ \rightarrow \ G'_{\mu\nu} = SG_{\mu\nu}S^{-1} \qquad \text{pure rotation [no observable]} . \tag{8.22}$$

The Lagrangian reads

$$\mathcal{L}_g = -\frac{1}{2}\mathrm{Tr}G_{\mu\nu}^2 = -\frac{1}{4}(G_{\mu\nu}^k)^2 \; . \tag{8.23}$$

It is gauge invariant (no mass term: $+\frac{1}{2}m_g^2\mathrm{Tr}G_\mu^2$). The Lagrangian consists of

(a)	kinetic part	$= -\frac{1}{4}(\partial_\nu G_\mu^k - \partial_\mu G_\nu^k)^2$
(b)	trilinear coupling	$\sim g_S GGG$
(c)	quartic coupling	$\sim g_S^2 GGGG$.

- The gluon fields interact with themselves: color charged gluons are sources for gluons ($\neq \gamma$).
- g_S is universal, it is fixed in the gauge sector: color charges are quantized.

<u>Lagrangian I of the QCD:</u>

$$\begin{array}{rcll}
\mathcal{L} & = & \bar{q}(i\not{D} - m_q)q - \frac{1}{2}\mathrm{Tr}G_{\mu\nu}^2 & \\
& = & \bar{q}(i\not{\partial} - m_q)q - \frac{1}{2}\mathrm{Tr}(\partial_\nu G_\mu - \partial_\mu G_\nu)^2 & \text{kin. part} \\
& & -g_S\bar{q}\not{G}q & \text{quark-gluon coupling} \\
& & +ig_S\mathrm{Tr}(\partial_\nu G_\mu - \partial_\mu G_\nu)[G_\mu, G_\nu] & \text{3-Gluon Kopplung} \\
& & +\frac{g_S^2}{2}\mathrm{Tr}[G_\mu, G_\nu]^2 & \text{4-gluon coupling}
\end{array}$$

After gauge fixing and application of the Faddeev-Popov trick we obtain within the Feynman path integral formulation for the complete QCD Lagrangian the action functional

<u>Complete Lagrangian of the QCD</u>

$$\begin{array}{lcll}
W & \sim & \int \mathcal{D}\bar{q}\,\mathcal{D}q\,\mathcal{D}G\,\mathcal{D}c^*\,\mathcal{D}c\,\exp i\int d^4x\,\mathcal{L}_{eff} & \\
\mathcal{L}_{eff} & = & \mathcal{L}_{QCD} + \mathcal{L}_{g.f.} + \mathcal{L}_{FP} & \mathcal{L}_{QCD} = \text{ quark-gluon Lagrangian} \\
& & & \mathcal{L}_{g.f.} = \text{ gauge-fixing Lagrangian} \\
& & & \mathcal{L}_{FP} = \text{ ghost Lagrangian} \\
\text{Lorentz} & \text{gauge} & & \text{axial gauge:} \\
\mathcal{L}_{g.f.} & = & -\frac{1}{\xi}\mathrm{Tr}(\partial G)^2 & \mathcal{L}_{g.f.} = -\frac{1}{\xi}\mathrm{Tr}(nG)^2 \quad \text{for } \xi \to 0 \\
\mathcal{L}_{FP} & = & \partial c^*(\partial + g_S f G)c & \mathcal{L}_{FP} = 0
\end{array}$$

<u>Derivation of $\mathcal{L}_{FP}$ in the Lorentz gauge and in the axial gauge:</u>

The Faddeev-Popov Lagrangian is given by (cf. chapter **??**)

$$\mathcal{L}_{FP} = c_a^* M_{ab} c_b \; , \tag{8.24}$$

where c_a, c_b denote the ghost fields, a, b the color indices and M_{ab} the Faddeev-Popov determinant. The latter is given by

$$M_{ab}(x,y) = \left.\frac{\delta F^a[G_\alpha(x)]}{\delta \alpha^b(y)}\right|_{\vec{\alpha}=0} \; . \tag{8.25}$$

F^a denotes the gauge fixing condition. G_α is the gluon field, transformed under an infinitesimal gauge transformation with the gauge parameter α. The QCD Lagrangian is invariant under non-abelian $SU(3)$ gauge transformations, and we have ($G_\mu^a = 2\mathrm{Tr}T^a G_\mu$)

$$(G^\alpha)_\mu^a = G_\mu^a - f_{abc}G_\mu^b \alpha^c + \frac{1}{g_S}\partial_\mu \alpha^a + \mathcal{O}(\alpha^2) \; . \tag{8.26}$$

(i) Lorentz gauge: We have the gauge fixing condition $\partial G = f$, in detail

$$\partial^\mu (G^\alpha)^a_\mu - f^a = \underbrace{(\partial^\mu G^a_\mu - f^a)}_{=0} \quad \underbrace{-f_{abc}\partial^\mu G^b_\mu \alpha^c + \frac{1}{g_S}\partial^2 \delta_{ab}\alpha^b}_{\frac{1}{g_S}\int d^4y \{\partial^2\delta_{ab} + g_S f_{abc}\partial^\mu G^c_\mu\}\delta_4(x-y)\alpha^b(y)} \quad . \tag{8.27}$$

And we obtain for the Faddeev-Popov determinant

$$M_{ab}(x,y) = \frac{1}{g_S}[\partial^2\delta_{ab} + g_S f_{abc}\partial^\mu G^c_\mu]\delta_4(x-y) \; . \tag{8.28}$$

In non-abelian gauge theories the Faddeev-Popov determinant manifestly depends on the gauge field G. In abelian gauge theories the Faddeev-Popov determinant is independent of the gauge field ($f_{abc} \equiv 0$) and thereby ineffective so that it can be neglected in the effective Lagrangian.

Axial gauge: The gauge fixing condition is given by $nG = 0$, where n_μ is a four-vector with $n^2 = \pm 1, 0$. We then obtain for the gauge-transformed gluon field

$$\begin{aligned} n(G^\alpha)^a &= \underbrace{nG^a}_{=0} - f_{abc}\underbrace{nG^b}_{=0}\alpha^c + \frac{1}{g_S} n\partial\alpha^a \\ &= \frac{1}{g_S}\int d^4y \delta_{ab}\, n\partial\, \delta_4(x-y)\, \alpha^b(y) \; . \end{aligned} \tag{8.29}$$

The Faddeev-Popov determinant reads

$$M_{ab}(x,y) = \frac{1}{g_S}\, n_\mu \partial^\mu\, \delta_{ab}\, \delta_4(x-y) \; . \tag{8.30}$$

It is independent of the gauge field G and thereby ineffective so that it can be neglected in the effective Lagrangian. The Feynman rules have been given in Eq. (**??**)

8.3 Asymptotic Freedom

Through higher-order corrections the parameters of the theory become dependent on the energy scale. For QED, which is an abelian gauge theory, it is found that the electormagnetic coupling $\alpha = e^2/(4\pi)$ is given by

$$\alpha(Q^2) = \frac{\alpha(\mu^2)}{1 - \sum_f e_f^2 \frac{\alpha(\mu^2)}{3\pi} \ln \frac{Q^2}{\mu^2}} \; . \tag{8.31}$$

This means that the coupling strength decreases (increases) with increasing (decreasing) distance. We now want to determine the "running" of the strong coupling constant of QCD, which is a non-abelian gauge theory.

8.3.1 Determination of the Running Coupling

Detailed information on the following can also be found e.g. in the chapters 8.5, 8.6 and 8.8 in Pierre Ramond, "Field Theory: A Modern Primer", Frontiers in Physics.

For the derivation of the running coupling constant, we have to compute the loop-corrections to that part of the QCD Lagrangian which contains the terms relevant for the strong coupling constant. In the covariant gauge, this Lagrangian is at tree level given by the following bare Lagrangian

$$
\begin{aligned}
\mathcal{L}_{\text{bare}} &= \frac{1}{4}(\partial_\mu G^B_{\nu\,0} - \partial_\nu G^B_{\mu\,0})^2 - g_0' f^{ABC} G^B_{\mu\,0} G^C_{\nu\,0} \partial^\mu G_0^{\nu\,A} \\
&+\frac{1}{4}(g_0^{''})^2 f^{ABC} f^{ADE} G^B_{\mu\,0} G^C_{\nu\,0} G_0^{\mu\,D} G_0^{\nu\,E} \\
&+\frac{1}{2\xi_0}(\partial^\mu G^B_{\mu\,0})(\partial^\nu G^B_{\nu\,0}) + i(\partial^\mu \eta_0^{*B})(\partial_\mu \eta_0^B) - \frac{i}{2} g_0^{'''} f^{ABC} G^C_{\mu\,0} \eta_0^{*A} \overleftrightarrow{\partial}^\mu \eta_0^B \\
&-\frac{i}{2} g_0^{''''} f^{ABC} \eta_0^{*A} \eta_0^B (\partial_\mu G_0^{\mu\,C}) + \bar{\psi}_0 \partial\!\!\!/ \psi_0 + i g_0 G^B_{\mu\,0} \bar{\psi}_0 \gamma^\mu T^B \psi_0 + i m_0 \bar{\psi}_0 \psi_0 \;,
\end{aligned}
\tag{8.32}
$$

where $G^a_{\mu\,0}$ denote the bare gluon fields, η^a_0 denote the bare ghost fields, ψ_0 denote the bare fermion fields, m_0 denotes the bare fermion mass, and $g_0, g_0^{',\,'',\,''',\,''''}$ denote the bare strong QCD coupling constant, which, for the sake of generality, we take different in the various interaction vertices. The $A, B, C = 1, ..., 8$ denote the color indices, ξ_0 the bare gauge fixing parameter, and T^A the generators of the $SU(3)_C$. The bare fields, masses and coupling constants are related to their renormalised (finite) counterparts through the relations

$$\psi_0 = (1+\delta Z_2)^{1/2}\psi = Z_2^{1/2}\psi \tag{8.33}$$

$$G^B_{\mu\,0} = (1+\delta Z_3)^{1/2} G^B_\mu = Z_3^{1/2} G^B_\mu \tag{8.34}$$

$$\eta_0^B = (1+\delta Z_6)^{1/2}\eta^B = Z_6^{1/2}\eta^B \tag{8.35}$$

$$m_0 = m\frac{1+\delta Z_m}{1+\delta Z_2} = m\frac{Z_m}{Z_2} \tag{8.36}$$

$$\xi_0^{-1} = \xi^{-1}\frac{1+\delta Z_\xi}{1+\delta Z_3} = \xi^{-1}\frac{Z_\xi}{Z_3} \tag{8.37}$$

$$g_0 = g\mu^\epsilon \frac{1+\delta Z_1}{Z_2 Z_3^{1/2}} = g\mu^\epsilon \frac{Z_1}{Z_2 Z_3^{1/2}} \tag{8.38}$$

$$g_0^{'} = g\mu^\epsilon \frac{1+\delta Z_4}{Z_3^{3/2}} = g\mu^\epsilon \frac{Z_4}{Z_3^{3/2}} \tag{8.39}$$

$$g_0^{''} = g\mu^\epsilon \frac{(1+\delta Z_5)^{1/2}}{Z_3} = g\mu^\epsilon \frac{Z_5^{1/2}}{Z_3} \tag{8.40}$$

$$g_0^{'''} = g\mu^\epsilon \frac{1+\delta Z_7}{Z_3^{1/2} Z_6} = g\mu^\epsilon \frac{Z_7}{Z_3^{1/2} Z_6} \tag{8.41}$$

$$g_0^{''''} = g\mu^\epsilon \frac{1+\delta Z_8}{Z_3^{1/2} Z_6} = g\mu^\epsilon \frac{Z_8}{Z_3^{1/2} Z_6} \;. \tag{8.42}$$

With these relations we get for the counterterm Lagrangian

$$
\begin{aligned}
\mathcal{L}_{\text{ct}} &= \frac{1}{4}\delta Z_3(\partial_\mu G^B_\nu - \partial_\nu G^B_\mu)^2 - \delta Z_4 g\mu^\epsilon f^{ABC} G^B_\mu G^C_\nu \partial^\mu G^{\nu\,A} \\
&+\frac{1}{4}\delta Z_5 g^2 \mu^{2\epsilon} f^{ABC} f^{ADE} G^B_\mu G^C_\nu G^{\mu\,D} G^{\nu\,E} \\
&+\frac{1}{2\xi}\delta Z_\xi (\partial^\mu G^B_\mu)(\partial^\nu G^B_\nu) + i\delta Z_6(\partial^\mu \eta^{*B})(\partial_\mu \eta^B) - \frac{i}{2} g\mu^\epsilon \delta Z_7 f^{ABC} G^C_\mu \eta^{*A} \overleftrightarrow{\partial}^\mu \eta^B \\
&-\frac{i}{2} g\mu^\epsilon \delta Z_8 f^{ABC} \eta^{*A} \eta^B (\partial_\mu G^{\mu\,C}) + \delta Z_2 \bar{\psi} \partial\!\!\!/ \psi + i g\mu^\epsilon \delta Z_1 G^B_\mu \bar{\psi}\gamma^\mu T^B \psi + i m \delta Z_m \bar{\psi}\psi \;.
\end{aligned}
\tag{8.43}
$$

The five coupling constants $g_0, ..., g_0''''$ are identical due to the Slavnov-Taylor identities.[1] From this follows that

$$\frac{Z_1}{Z_2} = \frac{Z_4}{Z_3} = \sqrt{\frac{Z_5}{Z_3}} = \frac{Z_7}{Z_6} = \frac{Z_8}{Z_6} \ . \tag{8.44}$$

The one-loop corrections relevant for QCD are shown in Fig. 8.5.

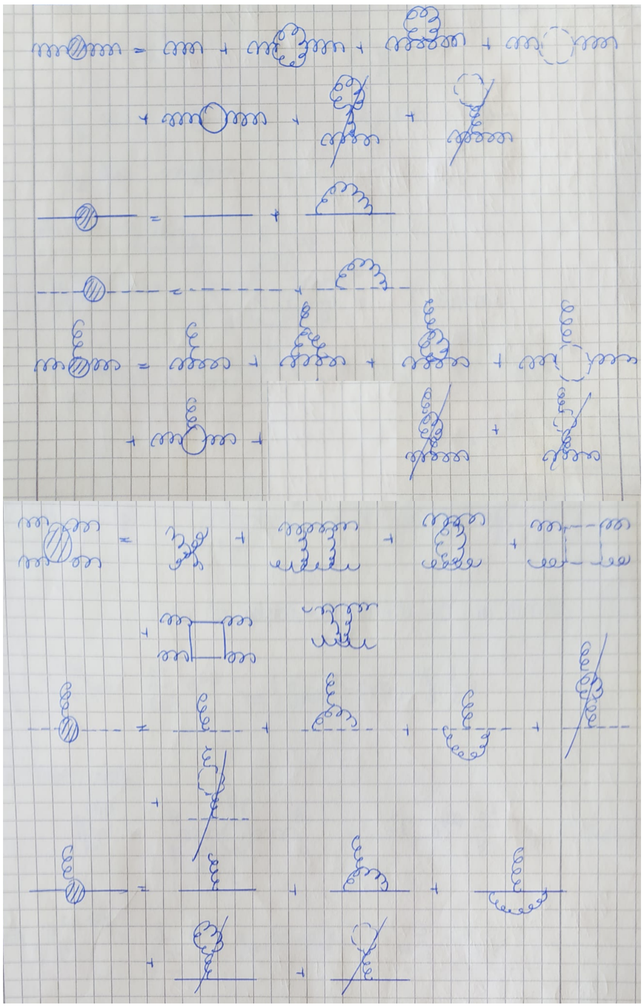
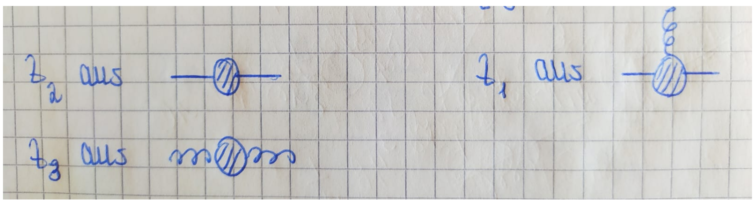


Figure 8.5: One-loop corrections to quark and gluon propagators and vertices.

For the determination of the 1-loop-corrected coupling we need Z_1, Z_2, and Z_3, as

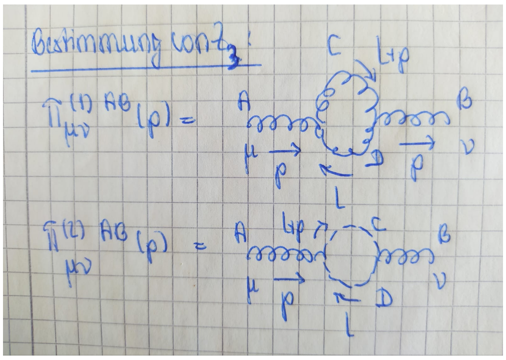
$$g_0 = g\mu^\epsilon \frac{Z_1}{Z_2 Z_3^{1/2}} \ . \tag{8.45}$$

We obtain them from



[1]They correspond to the Ward identities of the Yang Mills theory.

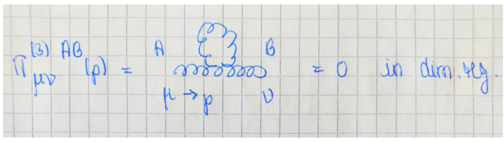
Determination of Z_3 We have the following contributions to the gluon self-energy: The sum of the two diagrams



results in

$$\begin{aligned}
&\Pi^{(1)\,AB}_{\mu\nu}(p) + \Pi^{(2)\,AB}_{\mu\nu}(p) \\
&= \frac{g^2}{32\pi^2} f^{ACD} f^{BCD} (g_{\mu\nu}p^2 - p_\mu p_\nu) \left\{ \frac{10}{3}\frac{1}{\epsilon} + \frac{62}{9} - \frac{10}{3}\gamma_E - \frac{10}{3}\ln\frac{p^2}{4\pi\mu^2} \right\} .
\end{aligned} \tag{8.46}$$

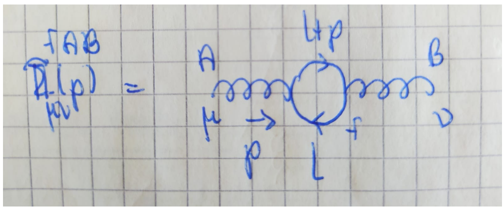
The following diagram is zero in dimensional regularisation:



Also the tadpole diagrams are zero in dimensional reguarlisation. Summing up, we have for the UV-divergent part

$$\Pi^{g\,AB}_{\mu\nu}(p) = \sum_{i=1}^{3} \Pi^{(i)\,AB}_{\mu\nu}(p) = \frac{g^2}{16\pi^2}\delta^{AB} C_{\mathrm{ad}} (g_{\mu\nu}p^2 - p_\mu p_\nu)\frac{5}{3}\frac{1}{\epsilon} + \ldots , \tag{8.47}$$

where $C_{\mathrm{ad}} = N = 3$. We furthermore have



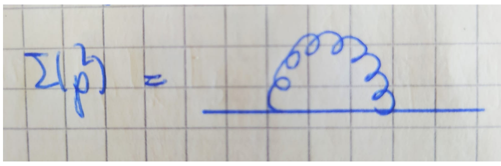
which results in

$$\Pi^{f\,AB}_{\mu\nu}(p) = -C_f \delta^{AB} \frac{g^2}{16\pi^2}(g_{\mu\nu}p^2 - p_\mu p_\nu)\frac{4}{3}\frac{1}{\epsilon} + \ldots , \tag{8.48}$$

for the divergent part, with $C_f = 1/2$. Altogether we hence have for the divergent part of the gluon self-energy at one-loop order in QCD

$$\Pi^{AB}_{\mu\nu} = \Pi^{g\,AB}_{\mu\nu}(p) + \Pi^{f\,AB}_{\mu\nu}(p) = \frac{g^2}{16\pi^2}\delta^{AB}(g_{\mu\nu}p^2 - p_\mu p_\nu)\left[C_{\mathrm{ad}}\frac{5}{3} - C_f\frac{4}{3}\right]\frac{1}{\epsilon} + \ldots \tag{8.49}$$

Determination of Z_2 We have one diagram that contributes to the quark self-energy:

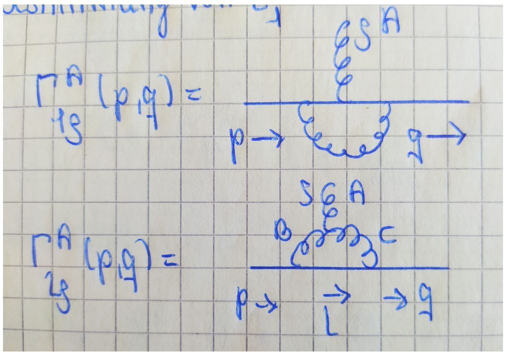


It can be expressed in terms of the QED self-energy after the appropriate coupling replacement,

$$\Sigma(p) = T^A T^A \Sigma^{\text{QED}}(p) = -i\frac{N}{d_f}C_f\frac{g^2}{16\pi^2}[\not{p} + 4m]\frac{1}{\epsilon} + \dots\,, \tag{8.50}$$

where $d_f = N = 3$.

Determination of Z_1 For the corrections to the quark-quark-gluon vertex we have two contributions as the tadpole contributions vanish:



For these diagrams we obtain

$$\Gamma^A_{1\rho}(p,q) = -ig\mu^\epsilon T^A\gamma_\rho\left(-\frac{1}{2}C_{\text{ad}} + C_f\frac{N}{d_f}\right)\frac{g^2}{16\pi^2}\frac{1}{\epsilon} + \dots \tag{8.51}$$

and

$$\Gamma^A_{2\rho}(p,q) = -3ig\mu^\epsilon\gamma_\rho T^A\frac{g^2}{32\pi^2}C_{\text{ad}}\frac{1}{\epsilon}\,. \tag{8.52}$$

We hence obtain

$$Z_1 = 1 - \frac{g^2}{16\pi^2}\left(\left[C_{\text{ad}} + C_f\frac{N}{d_f}\right]\frac{1}{\epsilon} + F_1\right)\dots \tag{8.53}$$

$$Z_2 = 1 - \frac{g^2}{16\pi^2}\left(C_f\frac{N}{d_f}\frac{1}{\epsilon} + F_2\right)\dots \tag{8.54}$$

$$Z_3 = 1 + \frac{g^2}{16\pi^2}\left(\left[\frac{5}{3}C_{\text{ad}} - \frac{4}{3}C_f\right]\frac{1}{\epsilon} + F_3\right)\dots;\,, \tag{8.55}$$

where F_1, F_2 and F_3 are the finite part of the counterterms.

Asymptotic Freedom With the computed Z_1, Z_2 and Z_3 we can now relate the bare coupling g_0 to the renormalised coupling g through

$$g_0 = g\mu^\epsilon\frac{Z_1}{Z_2 Z_3^{1/2}}\,. \tag{8.56}$$

We obtain

$$
\begin{aligned}
g_0 &= g\mu^\epsilon \left[1 - \frac{g^2}{16\pi^2}\left\{\left(C_{\rm ad} + C_f\frac{N_f}{d_f} - C_f\frac{N_f}{d_f} + \frac{5}{6}C_{\rm ad} - \frac{4}{6}C_f\right)\frac{1}{\epsilon} + F_1 - F_2 + \frac{F_3}{2}\right\}\right] \\
&= g\mu^\epsilon \left[1 - \frac{g^2}{16\pi^2}\left\{\left(\frac{11}{6}C_{\rm ad} - \frac{2}{3}C_f\right)\frac{1}{\epsilon} + F_1 - F_2 + \frac{F_3}{2}\right\}\right] . \qquad (8.57)
\end{aligned}
$$

Applying the minimal substraction renormalisation scheme, the finite terms are zero, $F_1 = F_2 = F_3 = 0$. And we get

$$
g_0 = g\mu^\epsilon \left[1 - \frac{g^2}{16\pi^2}\left(\frac{11}{6}C_{\rm ad} - \frac{2}{3}C_f\right)\frac{1}{\epsilon}\right] . \qquad (8.58)
$$

We now define the β-function, which describes the energy dependence of the coupling,

$$
\mu\frac{\partial g}{\partial \mu} = \beta(g) . \qquad (8.59)
$$

For this we take the partial derivative $\partial/\partial\mu$ on the left- and the right-handed side of Eq. (8.58) and get

$$
0 = \epsilon\mu^{\epsilon-1}\left\{g - \frac{g^3}{16\pi^2}\left(\frac{11}{6}C_{\rm ad} - \frac{2}{3}C_f\right)\frac{1}{\epsilon}\right\} + \mu^\epsilon\frac{\partial g}{\partial\mu}\frac{\partial}{\partial g}\left\{g - \frac{g^3}{16\pi^2}A\frac{1}{\epsilon}\right\} , \qquad (8.60)
$$

with

$$
A = \frac{11}{6}C_{\rm ad} - \frac{2}{3}C_f . \qquad (8.61)
$$

Multiplication with $\mu^{1-\epsilon}$ leads to

$$
\begin{aligned}
0 &= \epsilon g - \frac{g^3}{16\pi^2}A + \mu\frac{\partial g}{\partial\mu}\frac{\partial}{\partial g}\left\{g - \frac{g^3}{16\pi^2}A\frac{1}{\epsilon}\right\} \\
\Leftrightarrow 0 &= \epsilon g - \frac{g^3}{16\pi^2}A + \mu\frac{\partial g}{\partial\mu}\left\{1 - \frac{3g^2}{16\pi^2}A\frac{1}{\epsilon}\right\} \\
\Leftrightarrow \mu\frac{\partial g}{\partial\mu} &= \left\{\frac{g^3}{16\pi^2}A - \epsilon g\right\}\left\{1 - \frac{3g^2}{16\pi^2}A\frac{1}{\epsilon}\right\}^{-1} . \qquad (8.62)
\end{aligned}
$$

Expansion in ϵ leads to

$$
\mu\frac{\partial g}{\partial\mu} = \frac{g^3}{16\pi^2}A + \frac{3g^5}{(16\pi^2)^2}A^2\frac{1}{\epsilon} - \frac{3g^3}{16\pi^2}A + \dots . \qquad (8.63)
$$

Expansion up to strictly order g^3, we obtain

$$
\mu\frac{\partial g}{\partial\mu} = \left(g\frac{\partial}{\partial g} - 1\right)\left(-\frac{g^3}{16\pi^2}\right)\left(\frac{11}{6}C_{\rm ad} - \frac{2}{3}C_f\right) = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}C_{\rm ad} - \frac{4}{3}C_f\right) , \qquad (8.64)
$$

from where we can read off the β-function

$$
\beta(g) = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}C_{\rm ad} - \frac{4}{3}C_f\right) . \qquad (8.65)
$$

As long as

$$\frac{11}{2}C_{\text{ad}} - \frac{4}{3}C_f > 0 \tag{8.66}$$

the coupling constant decreases with μ. This is called **asymptotic freedom**. In QCD with the gauge group $SU(3)$ we have $C_{\text{ad}} = 3$ and $C_f = 1/2$ for each of the 6 quarks so that **QCD is asymptotically free**. The behavior of g as a function of the inverse energy scale μ^{-1} is shown in Fig. 8.6. As can be inferred from the figure, for small energy scales, respectively large distances, the coupling becomes non-perturbative so that we cannot trust the perturbative calculations any more. Only at short distances, where the particles become *asymptotically free*, we can apply our perturbative calculations.[2] As the QCD coupling rises with increasing distance, the perturbative states cannot leave the interaction region to become asymptotic states. The perturbative states could escape and form asymptotic states only by forming composite states which are neutral w.r.t. the long range force. We hence have the *confinement hypothesis*, which states that in an asymptotically free theory only singlets under the gauge force could be asymptotic states. QCD obviously is asymptotically free. The asymptotic states of the theory are not the quarks or the gluons, but composite states built up by the quarks, antiquarks and gluons. These are to be identified with the strongly interacting particles detected in the laboratory, such as protons, neutrons π-mesons etc.

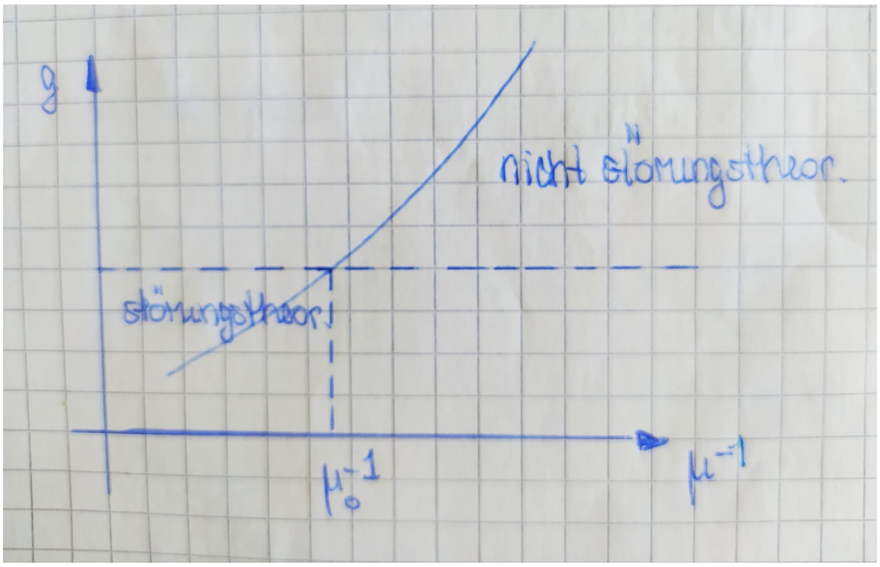


Figure 8.6: The strong coupling constant as a function of the inverse of the energy scale. For small energy scales, respectively large distances, the coupling exceeds the perturbative limit.

Let us finish this section by deriving the formula for the energy scale dependence of the

[2]The $SU(2)_L$ is also a non-abelian gauge theory such that the weak coupling in the SM also increases with decreasing energy, respectively, increasing distance. However, as we the gauge bosons of the weak interaction are massive, the interaction potential is exponentially supressed for large distances, so that we do not have confinement in the weak interaction.

strong coupling constaint α_s. Starting from

$$\mu\frac{\partial g}{\partial\mu} = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}C_{\mathrm{ad}} - \frac{4}{3}C_f\right) , \tag{8.67}$$

we make a variable transformation $\mu \to \mu^2$, $g \to g^2$ using

$$\partial\mu^2 = 2\mu\partial\mu \rightsquigarrow \frac{\partial}{\partial\mu} = 2\mu\frac{\partial}{\partial\mu^2} \tag{8.68}$$

and

$$\frac{\partial g^2}{\partial\mu} = 2g\frac{\partial g}{\partial\mu} \rightsquigarrow \frac{\partial g}{\partial\mu} = \frac{1}{2g}\frac{\partial g^2}{\partial\mu} \tag{8.69}$$

so that

$$\mu\frac{\partial g}{\partial\mu} = \frac{\mu^2}{g}\frac{\partial g^2}{\partial\mu^2} \tag{8.70}$$

and arrive at $(\alpha_s = g^2/(4\pi))$

$$\mu^2\frac{\partial\alpha_s(\mu)}{\partial\mu^2} = \underbrace{-\frac{\alpha_s^2}{\pi}\frac{1}{12}[33 - 2N_F]}_{\beta(\alpha_s)} . \tag{8.71}$$

Integration results in $(\beta_0 = 1/12(33 - 2N_F))$

$$\begin{aligned}
\int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2} \frac{d\alpha_s}{\beta(\alpha_s)} &= \int_{\mu^2}^{Q^2} \frac{d\mu'^2}{\mu'^2} = \ln\frac{Q^2}{\mu^2} \\
\Leftrightarrow \int_{\mu^2}^{Q^2} \frac{d\alpha_s}{\alpha_s^2(\mu)} &= -\frac{\beta_0}{\pi}\ln\frac{Q^2}{\mu^2} \\
\Leftrightarrow \frac{1}{\alpha_s(\mu^2)} + \frac{1}{\alpha_s(\mu^2)}\frac{\beta_0\alpha_s(\mu^2)}{\pi}\ln\frac{Q^2}{\mu^2} &= \frac{1}{\alpha_s(Q^2)} ,
\end{aligned} \tag{8.72}$$

so that we finally get

$$\boxed{\alpha_S(Q^2) = \frac{\alpha_S(\mu^2)}{1 + \frac{33-2N_F}{12}\frac{\alpha_S(\mu^2)}{\pi}\ln\frac{Q^2}{\mu^2}} \qquad (8.73)}$$

As can be inferred from this formula, with growing Q^2 the effective color charge vanishes. This asymptotic freedom is realized for the non-abelian $SU(3)$ for $N_F \leq 16$. It is a consequence of the non-abelian gauge boson loops, and is in contrast to the $U(1)$ where we do not have photon self-interactions.

The renormalisation group equation determines the asymptotic behaviour of the running coupling. Including higher orders we have

$$\begin{aligned}
\beta(\alpha_S) &= -\frac{\alpha_S^2}{\pi}\left[\beta_0 + \beta_1\frac{\alpha_S}{\pi} + \beta_2\left(\frac{\alpha_S}{\pi}\right)^2\right] \\
\beta_1 &= \frac{153 - 19N_F}{24} \qquad\qquad \beta_2 = \frac{1}{128}\left[2857 - \frac{5033}{9}N_F + \frac{325}{27}N_F^2\right] \\
\alpha_S(Q^2) &= \frac{\pi}{\beta_0\ln\frac{Q^2}{\Lambda^2}}\left\{1 - \frac{\beta_1}{\beta_0^2}\frac{\ln\ln\frac{Q^2}{\Lambda^2}}{\ln\frac{Q^2}{\Lambda^2}} + ...\right\} .
\end{aligned} \tag{8.74}$$

Scale parameter of QCD: Quantum theory introduces a scale into unscaled classical chromodynamics (for $m_q = 0$) via renormalisation. The coupling constant at a default distance is given by

$$\underbrace{\alpha_s}_{\text{experimentally determined}} = \alpha_s(\mu^2)$$

If hence the value of α_s is known at a certain scale through the comparison with experiment, one can deduce the scale from which on perturbation theory is not valid any more. A reformulation leads to

$$\frac{1}{\alpha_S(Q^2)} = \underbrace{\frac{1}{\alpha_S(\mu^2)} - \frac{33-2N_F}{12\pi}\ln\mu^2}_{\equiv \frac{33-2N_F}{12\pi}\ln\frac{1}{\Lambda^2}} + \frac{33-2N_F}{12\pi}\ln Q^2 \tag{8.75}$$

And thereby

$$\boxed{\alpha_S(Q^2) = \frac{12\pi}{(33-2N_F)\ln\frac{Q^2}{\Lambda^2}}} \tag{8.76}$$

With the confinement radius $\Lambda^{-1} \sim$ fm we have

$$\boxed{\Lambda \sim 100 - 300 \text{ MeV}} \tag{8.77}$$

And $\frac{\alpha_S(Q^2)}{\pi} \leq 10^{-1}$ for $Q^2 \geq 2$ GeV^2. This is the range in which perturbation theory can be applied.

Renormalisation schemes

We have for the inverse fermion propapator including higher-order corrections (i.e. the self-energy correction)

$$S^{-1} = \not{p}\left[1 - \tilde{\Sigma}(p)\right] , \tag{8.78}$$

where $\tilde{\Sigma}$ denotes the self-energy, which in dimensional regularisation ($n = 4 - 2\epsilon$) is given by

$$\tilde{\Sigma}(p) = \frac{4}{3}\frac{g_S^2}{(4\pi)^{2-\epsilon}}(\mu f)^{2\epsilon}\frac{\Gamma(\epsilon)}{(-p^2)^{\epsilon}}2(1-\epsilon)B(2-\epsilon, 1-\epsilon) . \tag{8.79}$$

In dimensional regularisation we replace the strong coupling constant as $g_S^2 \to g_S^2(\mu f)^{2\epsilon}$, where f is an arbitrary constant, in order to ensure that the action is dimensionless in $n = 4 - 2\epsilon$ dimensions. After expansion in ϵ we obtain

$$S^{-1}(p) = \not{p}\left\{1 - \frac{4}{3}\frac{g_S^2}{16\pi^2}\left[\frac{1}{\epsilon} - \ln\frac{-p^2}{(\mu f)^2} + 1 + \ln 4\pi - \gamma_E\right]\right\} . \tag{8.80}$$

Using multiplicative renormalisation we have the following relation between the bare and the renormalised propagator,

$$S^{-1}(p) = Z_\Psi^{-1} S_R^{-1}(p) . \tag{8.81}$$

We consider in the following different renormalisation schemes:

(i) Dyson renormalisation scheme

The scheme is characterised through the following condition,

$$\left.\begin{array}{rcl} f & = & 1 \\ S_R^{-1} & = & \not{p} \quad \text{for } \mu^2 = -p^2 \end{array}\right\} S^{-1}(p) = \not{p}\left[1 - \tilde{\Sigma}(\mu)\right]\left[1 - \tilde{\Sigma}(p) + \tilde{\Sigma}(\mu)\right] . \tag{8.82}$$

The solution is given by

$$\begin{aligned} Z_\Psi^{-1} &= 1 - \frac{4}{3}\frac{g_S^2}{16\pi^2}\left[\frac{1}{\epsilon} + \ln 4\pi - \gamma_E + 1\right] \\ S_R^{-1} &= \not{p}\left[1 + \frac{4}{3}\frac{g_{S,\mathrm{MOM}}^2}{16\pi^2}\ln\frac{-p^2}{\mu^2}\right] , \end{aligned} \tag{8.83}$$

where MOM stands for “momentum substraction” .

$$\boxed{\text{The coupling/charge depends on the renormalisation scheme.}}$$

(ii) ’t Hooft: Minimal Substraction (MS)

Here we demand

$$\begin{array}{rcl} f & = & 1 \\ Z_\Psi^{-1} & & \text{only takes off the } \frac{1}{\epsilon} \text{ pole. .} \end{array} \tag{8.84}$$

We demand for $S^{-1}(p)$ that

$$S^{-1}(p) = \not{p}\left[1 - \frac{4}{3}\frac{g_S^2}{16\pi^2}\frac{1}{\epsilon}\right]\left\{1 - \frac{4}{3}\frac{g_S^2}{16\pi^2}\left[-\ln\frac{-p^2}{\mu^2} + \ln 4\pi - \gamma_E + 1\right]\right\} . \tag{8.85}$$

The solution is

$$\begin{aligned} Z_\Psi^{-1} &= 1 - \frac{4}{3}\frac{g_S^2}{16\pi^2}\frac{1}{\epsilon} \\ S_R^{-1} &= \not{p}\left\{1 - \frac{4}{3}\frac{g_{S,MS}^2}{16\pi^2}\left[-\ln\frac{-p^2}{\mu^2} + \ln 4\pi - \gamma_E + 1\right]\right\} . \end{aligned} \tag{8.86}$$

(iii) Modified Minimal Substraction ($\overline{\mathrm{MS}}$)

We demand

$$f = \exp\left[-\frac{1}{2}\left(\ln 4\pi - \gamma_E\right)\right] . \tag{8.87}$$

The goal is to take off all trivial constants. This means that we demand

$$S^{-1}(p) = \not{p}\left[1 - \frac{4}{3}\frac{g_S^2}{16\pi^2}\frac{1}{\epsilon}\right]\left[1 - \frac{4}{3}\frac{g_S^2}{16\pi^2}\left(1 - \ln\frac{-p^2}{\mu^2}\right)\right] . \tag{8.88}$$

The solution is given by

$$Z_\Psi^{-1} = 1 - \frac{4}{3}\frac{g_S^2}{16\pi^2}\frac{1}{\epsilon} \tag{8.89}$$

$$S_R^{-1}(p) = \not{p}\left\{1 - \frac{4}{3}\frac{g_{S,\overline{\mathrm{MS}}}^2}{16\pi^2}\left[1 - \ln\frac{-p^2}{\mu^2}\right]\right\} . \tag{8.90}$$

The relation between the MS and the $\overline{\text{MS}}$ scheme is given by:

$$\underline{\text{MS} \leftrightarrow \overline{\text{MS}}:} \ \mu^2_{\text{MS}} \leftrightarrow \mu^2_{\overline{\text{MS}}} \exp\left[-\ln 4\pi + \gamma_E\right] \ . \tag{8.91}$$

And for the scale parameter (see side calculation)

$$\Lambda^2_{\text{MS}} = \mu^2 \exp\left\{-\frac{4\pi^2}{\beta_0 g^2_{S,\text{MS}}} + \frac{\beta_1}{\beta_0^2}\ln(4\pi^2/(\beta_0 g^2_{S,\text{MS}}))\right\} \tag{8.92}$$

$$\Lambda^2_{\overline{\text{MS}}} = \mu^2 \exp\left\{-\frac{4\pi^2}{\beta_0 g^2_{S,\overline{\text{MS}}}} + \frac{\beta_1}{\beta_0^2}\ln(4\pi^2/(\beta_0 g^2_{S,\overline{\text{MS}}}))\right\} \ . \tag{8.93}$$

$$\boxed{\Lambda_{\overline{\text{MS}}} = \Lambda_{\text{MS}} \exp\left\{\frac{\ln 4\pi - \gamma_E}{2}\right\} \qquad (8.94)}$$

β_0, β_1 are independent of the renormalisation scheme (not $\beta_{i\geq 2}$) and $\alpha_{S,\overline{MS}}(Q^2) > \alpha_{S,\text{MS}}(Q^2)$.

Quark masses

We now consider the quark self-energy which is given by

$$\Sigma(\not{p} = m) = m C_F \frac{\alpha_S}{\pi}\Gamma(1+\epsilon)\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon}\left(\frac{3}{4\epsilon} + 1\right) \ . \tag{8.95}$$

The pole mass is given by

$$m = m_0 + \Sigma(\not{p} = m) \tag{8.96}$$

and the $\overline{\text{MS}}$ mass by

$$\overline{m}(\mu^2) = m_0 + \delta\overline{m} \ , \tag{8.97}$$

where

$$\delta\overline{m} = m C_F \frac{\alpha_S}{\pi}\Gamma(1+\epsilon)(4\pi)^{\epsilon}\frac{3}{4\epsilon} \ . \tag{8.98}$$

The relation between the pole mass and the $\overline{\text{MS}}$ mass is

$$\begin{aligned}\overline{m}(\mu^2) &= m - [\Sigma(\not{p} = m) - \delta\overline{m}] = m\left[1 - C_F\frac{\alpha_S}{\pi}\left(\frac{3}{4}\ln\frac{\mu^2}{m^2} + 1\right)\right] \\ &= m\left[1 - C_F\frac{\alpha_S}{\pi}\right]\left[1 - \frac{3}{4}C_F\frac{\alpha_S}{\pi}\ln\frac{\mu^2}{m^2}\right] \ ,\end{aligned} \tag{8.99}$$

hence

$$\boxed{\begin{aligned}\overline{m}(m^2) &= m\left[1 - C_F\frac{\alpha_S(m^2)}{\pi}\right] \\ \overline{m}(\mu^2) &= \overline{m}(m^2)\left[1 - \frac{\alpha_S}{\pi}\ln\frac{\mu^2}{m^2}\right] \qquad (8.100)\end{aligned}}$$

The renormalisation group equation reads

$$\mu^2 \frac{\partial \overline{m}(\mu^2)}{\partial \mu^2} = -\gamma_m(\alpha_S(\mu^2))\overline{m}(\mu^2) \; , \tag{8.101}$$

where

$$\gamma_m(\alpha_S) = \frac{\alpha_S}{\pi} + \mathcal{O}(\alpha_S^2) \tag{8.102}$$

denotes the anomalous dimension. With

$$\alpha_S(\mu^2) = \frac{\pi}{\beta_0 \ln \frac{\mu^2}{\Lambda^2}} \tag{8.103}$$

the solution is given by

$$\overline{m}(\mu^2) = \overline{m}(m^2) \exp\left\{ \frac{-1}{\beta_0} \int_{m^2}^{\mu^2} \frac{dQ^2}{Q^2 \ln \frac{Q^2}{\Lambda^2}} \right\} = \overline{m}(m^2) \left[\frac{\alpha_S(\mu^2)}{\alpha_S(m^2)} \right]^{\frac{1}{\beta_0}} \; . \tag{8.104}$$

We hence obtain

$$\boxed{\begin{aligned} \overline{m}(\mu^2) &= \hat{m} \left[\alpha_S(\mu^2)\right]^{\frac{1}{\beta_0}} \\ \hat{m} &= \overline{m}(m^2) \left[\alpha_S(m^2)\right]^{-\frac{1}{\beta_0}} \qquad (8.105) \end{aligned}}$$

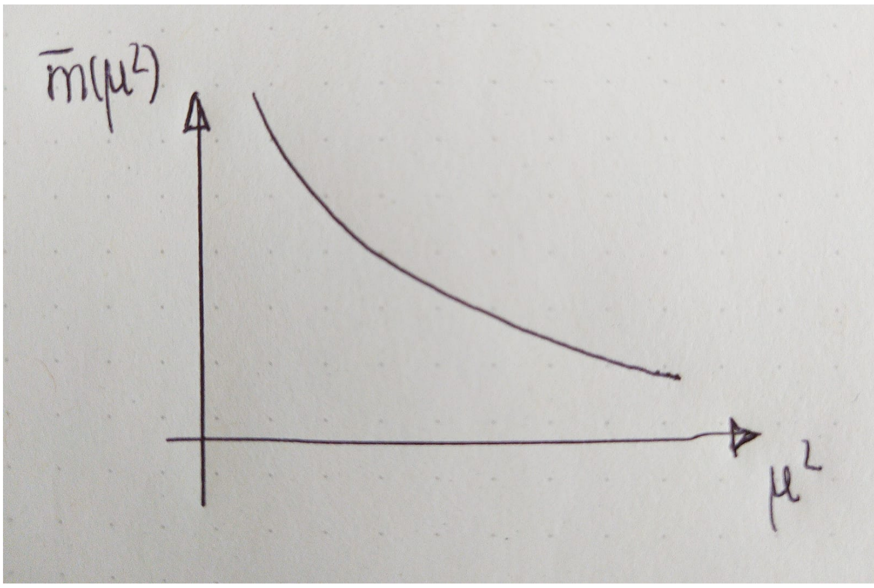


Figure 8.7: The $\overline{m}$ mass as function of the energy scale.

$\boxed{\text{With rising } \mu^2 (R \to 0) \text{ the effective quark mass vanishes, cf. Fig. 8.7.}}$

Examples:

Bottom Quark: $m_b = 4.8$ GeV.

$$\overline{m}(m_b^2) = 4.2 \text{ GeV} \qquad \overline{m}(M_Z^2) = 2.9 \text{ GeV} . \tag{8.106}$$

Charm Quark: $m_c = 1.6$ GeV.

$$\overline{m}(m_c^2) = 1.2 \text{ GeV} \qquad \overline{m}(M_Z^2) = 0.6 \text{ GeV} . \tag{8.107}$$

Light Quarks (QCD sum rules):

$$\begin{aligned} \overline{m}_u(1 \text{ GeV}^2) &\sim 5 \text{ MeV} \qquad \text{Gasser,Leutwyler} && (8.108)\\ \overline{m}_d(1 \text{ GeV}^2) &\sim 8 \text{ MeV} && (8.109)\\ \overline{m}_s(1 \text{ GeV}^2) &\sim 200 \text{ MeV} . && (8.110) \end{aligned}$$

The higher-order corrections are given by

$$\overline{m}(m^2) = \frac{m}{1 + C_F \frac{\alpha_S(m^2)}{\pi} + K \left(\frac{\alpha_S(m^2)}{\pi}\right)^2} \qquad \text{Gray,Broadhurst,Grafe,Schilcher} \tag{8.111}$$

where

$$K_t \sim 10.9 \qquad K_b \sim 12.4 \qquad K_c \sim 13.4 . \tag{8.112}$$

And we have

$$\overline{m}(\mu^2) = \overline{m}(m^2) \frac{c[\alpha_S(\mu^2)/\pi]}{c[\alpha_S(m^2)/\pi]} , \tag{8.113}$$

with

$$\begin{aligned} c(x) &= \left(\frac{9}{2}x\right)^{4/9} \left[1 + 0.895x + 1.371x^2 + 1.952x^3\right] \qquad m_s < \mu < m_c\\ c(x) &= \left(\frac{25}{6}x\right)^{12/25} \left[1 + 1.014x + 1.389x^2 + 1.091x^3\right] \qquad m_c < \mu < m_b\\ c(x) &= \left(\frac{23}{6}x\right)^{12/23} \left[1 + 1.175x + 1.501x^2 + 0.1725x^3\right] \qquad m_b < \mu < m_t\\ c(x) &= \left(\frac{7}{2}x\right)^{4/7} \left[1 + 1.389x + 1.793x^2 - 0.6834x^3\right] \qquad m_t < \mu && (8.114) \end{aligned}$$

Chetyrkin; Larin,van Ritbergen,Vermaseren .

The running of the coupling and of the bottom quark mass as determined by experiment are shown in Fig. 8.8 and Fig. 8.9, respectively.

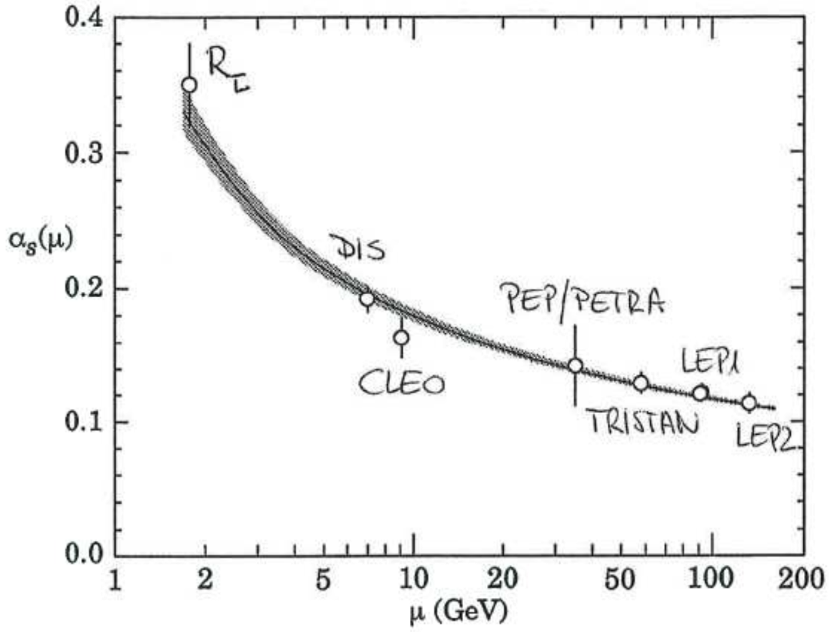


Figure 8.8: The running α_s.

8.4 Renormalisation Group

Parameters of a field theory (masses, couplings) are introduced for a certain μ^2; physical observables are independent of the particular choice of μ^2: The modification of μ^2 with the corresponding change of the parameters leads to the invariance which can be formulated as renormalisation group equations (RGEs), which are partial differenial equations.
Application: The μ^2-variation can be moved to a Q^2-variation by means of a dimensional analysis. In this way the Q^2 variation of Green's functions can be determined.

In the following, we will derive the renormalisation group equation. We start by considering the Green's function $G^{N_G N_\psi}(p)$ which depends on the number N_G of gauge fields and the number N_ψ of fermion fields and is given by the time-ordered product

$$G^{N_G N_\psi}(p) = \langle 0|T\{\psi(x_1)...\}|0\rangle_{\mathrm{FT}} \tag{8.115}$$

in field theory. The truncated Green's functions are obtained as

$$\Gamma^{N_G N_\psi}(p) = \frac{G^{N_G N_\psi}}{\Pi_G G^{2,0}(p_G)\Pi_\psi G^{0,2}(p_\psi)} \ . \tag{8.116}$$

For example we have

$$\Gamma^{0,2} = [G^{0,2}]^{-1}. \tag{8.117}$$

or

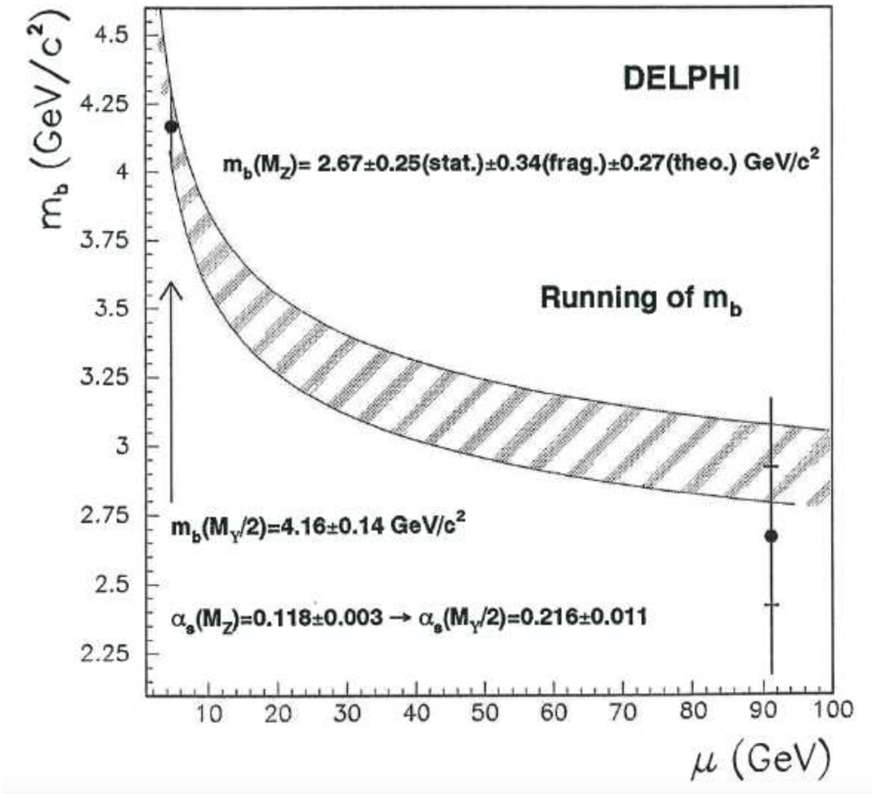
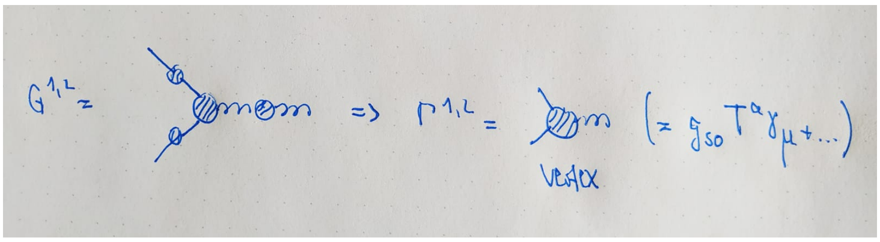


Figure 8.9: The running b-mass.



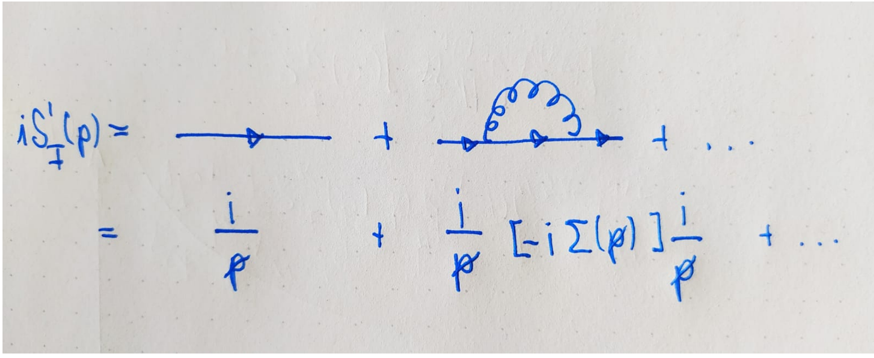
We have the following

Theorem of Multiplicative Renormalizability of Gauge Theories:

Divergent parts of Γ's can be separated as cut-off dependent factors; the remaining rest Γ_R is finite after the introduction of the renormalized coupling g and well-defined for the cut-off $\to \infty$;. The renormalisation constants depend only on the species of the external legs.

Examples

(i) Fermion propagator



The one-loop self-energy is given by

$$\Sigma(\not p, \epsilon) = -C_F \frac{\alpha_s}{4\pi}\Gamma(1+\epsilon)\left(\frac{4\pi\mu^2}{-p^2}\right)^{\epsilon}\left(\frac{1}{\epsilon}+1+\mathcal{O}(\epsilon)\right) . \tag{8.118}$$

We hence get for the propagator

$$\begin{aligned} \frac{i}{\not p} &\rightarrow \frac{i}{\not p}\left[1 - C_F\frac{\alpha_s}{4\pi}\Gamma(1+\epsilon)(4\pi)^{\epsilon}\left(\frac{1}{\epsilon}+1\right) - C_F\frac{\alpha_s}{4\pi}\log\frac{\mu^2}{-p^2}\right] + \mathcal{O}(\epsilon) \\ &= \frac{i}{\not p}\left[1 - C_F\frac{\alpha_s}{4\pi}\Gamma(1+\epsilon)(4\pi)^{\epsilon}\left(\frac{1}{\epsilon}+1\right)\right]\left[1 - C_F\frac{\alpha_s}{4\pi}\log\frac{\mu^2}{-p^2}\right] + \mathcal{O}(\epsilon,\alpha_s^2) . \end{aligned} \tag{8.119}$$

and thereby

$$S'_F(p) = \frac{Z_\psi(\alpha_s,\mu)}{\not p}\left[1 - C_F\frac{\alpha_s}{4\pi}\log\frac{\mu^2}{-p^2}\right] . \tag{8.120}$$

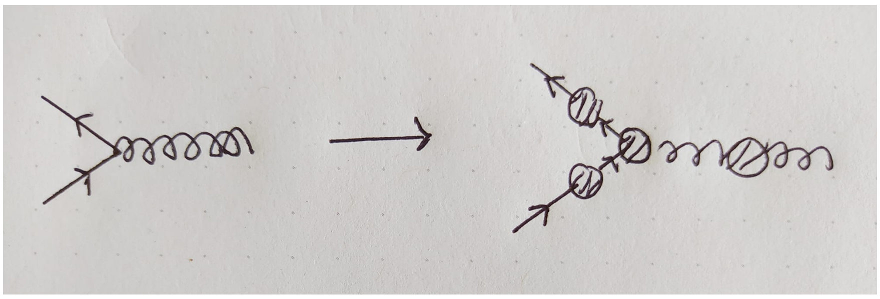
The renormalised propagator is

$$S^R_F(p) = \frac{1}{\not p} \quad \text{for} \quad \mu^2 = -p^2 . \tag{8.121}$$

And

$$\Gamma^{0,2} = Z_\psi^{-2/2}\Gamma_R^{0,2}(p) . \tag{8.122}$$

(ii) Vertex For the vertex



we have after inclusion of the one-loop corrections

$$S_F(p')g_{s0}T^a\gamma_\mu S_F(p)D_G^{\mu\nu}(k) \;\rightarrow\; S'_F(p')g_{s0}T^a\Gamma'_\mu S'_F(p)D'^{\mu\nu}_G(k) \tag{8.123}$$

We have

$$\begin{aligned}
& S'_F(p')g_{s0}T^a\Gamma'_\mu S'_F(p)D'^{\mu\nu}_G(k) \\
= \; & Z_\psi^{1/2}S_F^R(p')\left[g_{s0}\frac{Z_\psi Z_G^{1/2}}{Z_1}\right]T^a\Gamma^R_\mu S^R_F(p)Z_\psi^{1/2}D^{R\,\mu\nu}_G(k)Z_G^{1/2} \\
= \; & Z_\psi^{-1/2}S'_F(p')\underbrace{\left[g_{s0}\frac{Z_\psi Z_G^{1/2}}{Z_1}\right]}_{g_s}T^a\Gamma^R_\mu S'_F(p)Z_\psi^{-1/2}D'^{\mu\nu}_G(k)Z_G^{-1/2}\;,
\end{aligned} \tag{8.124}$$

so that we have

$$g_{s0}\Gamma'_\mu = Z_\psi^{-2/2}Z_G^{-1/2}g_s\Gamma^R_\mu\;. \tag{8.125}$$

Altogether we hence get

$$\Gamma^{N_G N_\psi}(p;g_{s0},\epsilon) = Z_G^{-N_G/2}(g_{s0},\mu)Z_\psi^{-N_\psi/2}(g_{s0},\mu)\Gamma_R^{N_G N_\psi}(p;g_s,\mu)\;. \tag{8.126}$$

In gauge theories, renormalisation constants and coupling constants (g_s for QCD) are theoretically fixed by 3 Green's functions (modulo gauge fixing parameters/ghosts):

$$\Gamma_R^{2,0}(p^2=-\mu^2) \;=\; Z_G(g_{s0},\mu)\Gamma^{2,0}(p^2=-\mu^2) = -g_{\mu\nu}p^2+p_\mu p_\nu \tag{8.127}$$

$$\Gamma_R^{0,2}(p^2=-\mu^2) \;=\; Z_\psi(g_{s0},\mu)\Gamma^{0,2}(p^2=-\mu^2) = \not{p} \tag{8.128}$$

$$\Gamma_R^{1,2}(p^2=-\mu^2) \;=\; \sqrt{Z}_G Z_\psi\Gamma^{1,2}(p^2=-\mu^2) = g_s\gamma_\mu\;. \tag{8.129}$$

As stated above, the synchronous variation of μ and $g_s(\mu)$ leaves the theory invariant and leads to the renormalisation group equation through

$$\mu\frac{d}{d\mu}\Gamma = 0\,, \tag{8.130}$$

resulting in

$$\left\{\mu\frac{\partial}{\partial\mu}+\mu\frac{\partial g_s}{\partial\mu}\frac{\partial}{\partial g_s}-\frac{N_G}{2}\mu\frac{\partial\ln Z_G}{\partial\mu}-\frac{N_\psi}{2}\mu\frac{\partial\ln Z_\psi}{\partial\mu}\right\}\Gamma_R^{N_G N_\psi}(p;g_s(\mu),\mu)=0\;. \tag{8.131}$$

Defining the β function as

$$\beta(g_s)=\mu\frac{\partial}{\partial\mu}g_s(g_{s0},\mu) \tag{8.132}$$

and the anomalous dimension as

$$\gamma(g_s)=\frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z(g_{s0},\mu) \tag{8.133}$$

we can re-write Eq. (8.131) as

$$\left\{\mu\frac{\partial}{\partial\mu}+\beta(g_s)\frac{\partial}{\partial g_s}-N_G\gamma_G(g_s)-N_\psi\gamma_\psi(g_s)\right\}\Gamma_R^{N_G N_\psi}(p;g_s(\mu),\mu)=0\;. \tag{8.134}$$

The variation of μ can be moved to a variation of p. Since the dimension of the Green's function in D dimensions is μ^D we can only have the following structure, $\Gamma_R = \mu^D f\left(\frac{p}{\mu}\right)$, where D is the physical dimension of Γ_R. Replacing $p \to e^t p$, we get

$$\left\{-\frac{\partial}{\partial t} + \beta(g_s)\frac{\partial}{\partial g_s} + D - N_G\gamma_G(g_s) - N_\psi\gamma_\psi(g_s)\right\}\Gamma_R^{N_G N_\psi}(g^t p; g_s(\mu), \mu) = 0 \;, \tag{8.135}$$

where $t = \ln Q/\mu$. In order to find the solution, we use that

$$\beta(g_s)\frac{\partial \bar{g}_s}{\partial g_s} = \beta(\bar{g}_s) = \frac{\partial \bar{g}_s}{\partial t} \;. \tag{8.136}$$

This follows from

$$\frac{\partial \bar{g}_s(g_s, t)}{\partial t} = \beta(\bar{g}_s(g_s, t)) \;, \quad \text{with} \quad \bar{g}_s(g_s, 0) = g_s \;, \tag{8.137}$$

from where we get

$$t = \int_{g_s}^{\bar{g}_s(g_s,t)} \frac{dg'}{\beta(g')} \;. \tag{8.138}$$

Differentiation of Eq. (8.138) w.r.t. t leads to

$$1 = \frac{1}{\beta(\bar{g}_s)}\frac{\partial \bar{g}_s}{\partial t} \;, \tag{8.139}$$

and differentiation of Eq. (8.138) w.r.t. g_s leads to

$$0 = -\frac{1}{\beta(g_s)} + \frac{1}{\beta(\bar{g}_s)}\frac{\partial \bar{g}_s}{\partial g_s} \;. \tag{8.140}$$

Combination of Eq. (8.139) and Eq. (8.140) leads to

$$\beta(g_s)\frac{\partial \bar{g}_s}{\partial g_s} = \beta(\bar{g}_s) = \frac{\partial \bar{g}_s}{\partial t} \;. \tag{8.141}$$

The most general solution is a function of $\bar{g}_s(g_s, t)$ modified by the special solution determined by the physical and anomalous dimensions:

$$\begin{aligned}\Gamma_R^{N_G,N_\psi}(e^t p, g_s) \;&=\; \Gamma_R^{N_G,N_\psi}(p, \bar{g}_s(g_s, t)) \\ &\quad \exp\left\{Dt - \int_0^t dt'[N_G\gamma_G(\bar{g}_s(g_s, t')) + N_\psi\gamma_\psi(\bar{g}_s(g_s, t'))]\right\} \;.\end{aligned} \tag{8.142}$$

Differentiation of Eq. (8.142) by using Eq. (8.141) shows that it fulfills Eq. (8.135).

We have (in the Landau gauge) the following anomalous dimensions and β functions,

$$\begin{aligned}\gamma_G(g_s) &= \left(-\frac{13}{2} + \frac{2}{3}N_F\right)\frac{\alpha_s}{4\pi} + ... \\ \gamma_\psi(g_s) &= 0 + ... \\ \frac{\beta(g_s)}{g_s} &= -\beta_0\frac{\alpha_s}{4\pi} - \beta_1\left(\frac{\alpha_s}{4\pi}\right)^2 + ... \quad \text{with} \\ \beta_0 &= 11 - \frac{2}{3}N_F \\ \beta_1 &= 102 - \frac{38}{3}N_F \;.\end{aligned} \tag{8.143}$$

The β_0 and β_1 are independent of the renormalisation scheme. The higher orders β_2, β_3, ... depend on the renormalisation scheme.

Scale variation of Green's functions

We have to lowest order

$$t = -\int_{g_s}^{\bar{g}_s} \frac{dg'}{bg'^3} = \frac{1}{2b}\left[\frac{1}{\bar{g}_s^2} - \frac{1}{g_s^2}\right] \quad \Rightarrow \quad \bar{g}_s^2(g_s,t) = \frac{g_s^2}{1+(11-\frac{2}{3}N_F)\frac{g_s^2}{8\pi^2}t}\ , \tag{8.144}$$

where $g_s^2 = g_s^2(\mu^2)$ and $t = 1/2\ln Q^2/\mu^2$. In order to derive the variation of the Green's functions with Q^2 we need to first look at the Q^2 variations of γ_G and g_s. We have

$$\begin{aligned} \gamma_G(\bar{g}_s^2) &= -d\bar{g}_s^2 + ...\ , \quad \text{with} \quad d = \frac{1}{16\pi^2}\left(\frac{13}{2} - \frac{2}{3}N_F\right) \\ \bar{g}_s^2(t) &= \frac{g_s^2}{1+2bg_s^2 t}\ , \quad \text{with} \quad b = \frac{1}{16\pi^2}\left(11 - \frac{2}{3}N_F\right)\ . \end{aligned} \tag{8.145}$$

From this we get

$$\int_0^t dt'\gamma_G(\bar{g}_s(g_s,t')) = \int_0^t dt'(-d)\frac{g_s^2}{1+2bg_s^2t'} = -\frac{d}{2b}\ln(1+2bg_s^2t) = -\ln(1+2bg_s^2t)^{\frac{d}{2b}} \tag{8.146}$$

We hence find

$$\Gamma_R \propto e^{Dt}e^{\ln(1+2bg_s^2t)^{\frac{d}{2b}}} \overset{t\to\infty}{\to} e^{Dt}t^{\frac{d}{2b}}\ . \tag{8.147}$$

The Green's function hence has the following scale dependence,

$$\Gamma_R \propto Q^D(\ln Q)^{\frac{d}{2b}}\ . \tag{8.148}$$

We see that the naive power law ($\propto Q^D$) is broken logarithmically: Green's functions vary logarithmically with Q^2 in asymptotic free theories. (This is in contrast to fixed point theories, where $g = g^\star \neq 0 \ \Rightarrow\ \Gamma_R \propto Q^D Q^{c^\star}$.)

8.5 QCD at Short Distances

8.5.1 Structure Functions of the Nucleon

Due to the asymptotic freedom we have:

(i) For small α_s to 0^{th} approximation the particles are approximately free at short distances, leading to the parton model.

(ii) We have a $\ln Q^2$ dependence of the Green's on the energy scale through the higher-order corrections (w.l.o.g. we consider the analogue of the electromagnetic structure functions).

We look at the scattering of an electron and a nucleon, cf. Fig. 8.10.

The matrix element is given by

$$\mathcal{M}(X) = ie^2\bar{u}'\gamma^\mu u\frac{1}{q^2}\langle X|j_\mu|\mathcal{N}_p\rangle \tag{8.149}$$

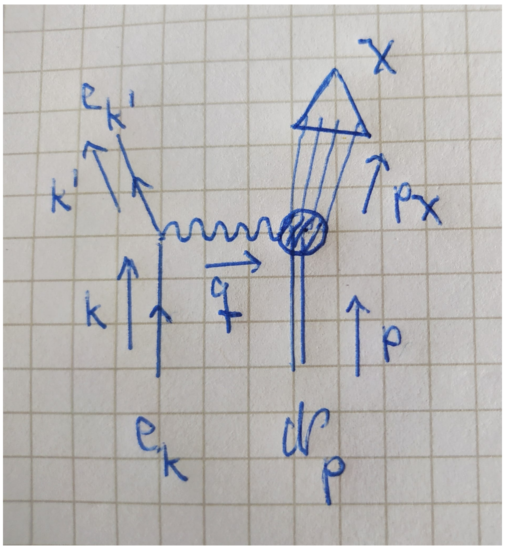


Figure 8.10: Electron-nucleon scattering.

and the cross section by

$$d\sigma = \frac{1}{4ME}\frac{d^3k'}{(2\pi)^3 2E'}\frac{1}{4}\sum_X (2\pi)^4\delta^{(4)}(p+q-p_X)|\mathcal{M}_X|^2 \ , \tag{8.150}$$

where E is the energy of the laboratory and

$$q = k - k' \ , \quad \text{with} \quad q^2 = -Q^2 < 0 \ . \tag{8.151}$$

We get for

$$\begin{aligned} &\frac{1}{4}\sum_X (2\pi)^4\delta^{(4)}(p+q-p_X)|\mathcal{M}_X|^2 \\ &= \left(\frac{e^2}{Q^2}\right)^2 \underbrace{\frac{1}{4}\sum_{\text{spins}}[\bar{u}'\gamma^\nu u][\bar{u}\gamma^\mu u']}_{=\mathcal{L}^{\mu\nu}\text{ lepton tensor}} \underbrace{\sum_X \langle\mathcal{N}|j_\mu|X\rangle\langle X|j_\nu|\mathcal{N}\rangle(2\pi)^4\delta^{(4)}(p+q-p_X)}_{=8\pi W_{\mu\nu}\text{ hadron tensor}} \ . \end{aligned} \tag{8.152}$$

The lepton tensor is given by

$$\mathcal{L}_{\mu\nu} = k_\mu k'_\nu + k_\nu k'_\mu - (k\cdot k')g_{\mu\nu} \ . \tag{8.153}$$

It is symmetric in μ, ν, k and k'. The hadron tensor is given by

$$\begin{aligned} W_{\mu\nu} &= \frac{1}{8\pi}\sum_{\text{spins}}\sum_X (2\pi)^4\delta^{(4)}(p+q-p_X)\langle\mathcal{N}_p|j_\mu^{\text{elm}}|X\rangle\langle X|j_\nu^{\text{elm}}|\mathcal{N}_p\rangle \\ &= \frac{1}{8\pi}\sum_{\text{spins}}\int d^4x\, e^{-iqx}\langle\mathcal{N}_p|\left[j_\mu^{\text{elm}}(0), j_\nu^{\text{elm}}(x)\right]|\mathcal{N}_p\rangle \ . \end{aligned} \tag{8.154}$$

The properties of the hadron tensor are

(i) It is symmetric in p_μ, q_μ, $g_{\mu\nu}$.

(ii) We have current conservation: $q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0$. $(\partial^\mu j_\mu^{\mathrm{elm}} = 0)$

(iii) The tensor is real ($\leftarrow$ hermiticity of the electromagnetic current).

We can decompose the hadronic tensor in invariants based on the general basis (taking into acount (i)-(iii)),

$$\underbrace{g_{\mu\nu} \quad q_\mu q_\nu}_{-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}} \quad \underbrace{p_\mu p_\nu \quad p_\mu q_\nu + p_\nu q_\mu \quad q_\mu q_\nu}_{\left[p_\mu - q_\mu \frac{pq}{q^2}\right]\left[p_\nu - q_\nu \frac{pq}{q^2}\right]} \tag{8.155}$$

With this we get the general form of the hadron tensor given by

$$\begin{aligned} W_{\mu\nu} &= W_1 \left[-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right] + W_2 \left[p_\mu - q_\mu \frac{pq}{q^2}\right]\left[p_\nu - q_\nu \frac{pq}{q^2}\right] &(8.156)\\ W_i &= \text{Lorentz scalar structure functions} &(8.157)\end{aligned}$$

The Lorentz structure functions are given in the variables relevant for the process. These are

(i) The electron state is characterised by the energy and the scattering angle.

(ii) The invariants are

$$Q^2 = -(k-k')^2 = -q^2 = 4EE'\sin^2\frac{\theta}{2}\ , \tag{8.158}$$

where we used $m_e^2 = 0 \rightsquigarrow E^{(\prime)} = |\vec{k}^{(\prime)}|$ and θ denotes the scattering angle. The energy loss in the electron sector is given by (in the laboratory system the proton is at rest, i.e. $\vec{p} = 0$)

$$\nu = pq = M(E - E') \tag{8.159}$$

Since $Q^2 \geq 0$ and $\nu \geq 0$ we have

$$\begin{aligned} (p+q)^2 &= W^2 \geq M^2 \quad \text{As we at least have one } \mathcal{N} \text{ in the final state}\\ \Leftrightarrow M^2 + 2pq + q^2 &\geq M^2 \Rightarrow 2\nu \geq Q^2\ . &(8.160)\end{aligned}$$

We hence have elastic scattering.

(iii) As scaling variables we choose

$$\begin{aligned} &\text{Bjorken variable} \qquad x = \frac{Q^2}{2\nu} \quad 0 \leq x \leq 1\\ &\text{relative energy loss} \qquad y = \frac{pq}{pk} \quad 0 \leq y \leq 1\ . &(8.161)\end{aligned}$$

The structure functions F_1, F_2 are given in terms of these variables and defined as

$$F_1(x,Q^2) = W_1(\nu,Q^2) \tag{8.162}$$
$$F_2(x,Q^2) = \nu W_2(\nu,Q^2)\,. \tag{8.163}$$

The cross section in the high-energy limit is given by

$$\frac{d^2\sigma}{dxdy} = \frac{4\pi\alpha^2}{Q^4} s_{e\mathcal{N}} \left[(1-y)F_2(x,Q^2) + y^2 x F_1(x,Q^2)\right]\,. \tag{8.164}$$

Interpretation of the structure functions:
The essence of the $e\mathcal{N} \to e' +$ everything scattering is the scattering $\gamma^\star + \mathcal{N} \to$ everything (the total absorption cross section of virtual photons). For virtual space-like photons we have

$$q_\mu = \left(\frac{\nu}{M}, 0, 0, \sqrt{Q^2 + \frac{\nu^2}{M^2}}\right)^T \tag{8.165}$$

in the laboratory frame. The transversal polarisation vectors $\epsilon_\mu(\pm)$ and the longitudinal polarisation vector $\epsilon_\mu(L)$ are given by

$$\begin{aligned}
\epsilon_\mu(\pm) &= \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(0, 1 \pm i, 0)^T \\
\epsilon_\mu(L) &= \frac{1}{\sqrt{Q^2}}\left(\sqrt{Q^2 + \frac{\nu^2}{M^2}}, 0, 0, \frac{\nu}{M}\right)^T\,.
\end{aligned} \tag{8.166}$$

with the normalisation

$$\epsilon_i^* \epsilon_j = \pm\delta_{ij}\,, \quad \epsilon_i q = 0\,, \quad \epsilon_\pm^* \epsilon_\pm = -1\,, \quad \epsilon_L^2 = +1\,. \tag{8.167}$$

The cross section for the scattering $\gamma^\star + \mathcal{N} \to$ everything is proportional to

$$\sigma(\gamma^\star \mathcal{N}) \propto \sum_X \epsilon^{\star\nu} \langle \mathcal{N}|j_\mu|X\rangle\langle X|j_\nu|\mathcal{N}\rangle \epsilon^\nu (2\pi)^4 \delta^{(4)}(p+q-p_X) \propto \epsilon^{\star\mu} W_{\mu\nu}\epsilon^\nu\,. \tag{8.168}$$

The cross section for the transversal polarisations is given by

$$\begin{aligned}
\sigma_\pm &= \epsilon_\pm^{\star\mu} W_{\mu\nu} \epsilon_\pm^\nu = W_1 = F_1 > 0 \\
&\quad [\mathcal{P}_{\rm elm} : \sigma_+ = \sigma_- = \frac{1}{2}\sigma_T]\,.
\end{aligned} \tag{8.169}$$

The cross section for longitudinal polarisation is given by

$$\begin{aligned}
\sigma_L = \epsilon_L^{\star\mu} W_{\mu\nu} \epsilon_L^\nu &= -W_1 + \left(\frac{\nu^2}{Q^2} + M^2\right) W_2 \geq 0 \\
&\stackrel{Q^2 \gg M^2}{\to} -F_1 + \frac{1}{2x} F_2\,.
\end{aligned} \tag{8.170}$$

We hence have for the ratio R of the two polarisations

$$R = \frac{\sigma_L}{\sigma_T} = \left(\frac{\nu^2}{Q^2} + M^2\right)\frac{W_2}{W_1} - 1 \to \frac{F_2 - 2xF_1}{2xF_1}\,. \tag{8.171}$$

Let us now discuss the experimental results:

1.) Bjorken scaling: Scattering off point-like scattering centers in the proton requires that at large Q^2 for fixed x the Q^2 dependence drops out:

In the Bjorken limit Q^2 large and x fixed we hence have

$$\nu W_2(\nu, Q^2) = F_2(x, Q^2) \rightsquigarrow F_2(x) \tag{8.172}$$

$$W_1(\nu, Q^2) = F_1(x, Q^2) \overset{Bj}{\rightsquigarrow} F_1(x) \,. \tag{8.173}$$

Indeed, this is experimentally found to some extent. In the experimental results a scaling with Q^2 is still visible, which is most pronounced for $x \sim 0.25$, cf. Fig. 8.11. We have

$$x \lesssim 0.25 : \quad F_2(x, Q^2) \text{ slightly increasing with } Q^2 \tag{8.174}$$

$$x \gtrsim 0.25 : \quad F_2(x, Q^2) \text{ slightly decreasing with } Q^2 \,. \tag{8.175}$$

The observed small logarithmic violation of the scaling is predicted by QCD.

2.) R ratio:

The R ratio in the Bjorken limit is given by

$$R(x, Q^2) \overset{Bj}{=} \frac{F_2(x) - 2xF_1(x)}{2xF_1(x)} \,. \tag{8.176}$$

For large Q^2 it is found that $R \to 0$ (cf. Fig: 8.12, i.e. the longitudinal absolute cross section vanishes and we obtain the

$$\underline{\text{Callan-Gross relation:}}\ F_2 = 2xF_1 \,. \tag{8.177}$$

3.) neutron/proton ratio:

The neutron/proton ratio $F_2^N(x)/F_2^P(x)$ decreases from the value 1 at $x = 0$ down to a value $\gtrsim 1/4$ for $x = 1$, cf. Fig. 8.13.

Classical quark-parton model For the scattering of an electron off a point-like object we have

$$e + \text{point-like} \to e + \text{point-like:}\ \frac{d\sigma^{\text{pt}}}{dQ^2} \sim \frac{1}{Q^4} \,. \tag{8.178}$$

For the electron scattering off a nucleon into the nucleon final state we have

$$e\mathcal{N} \to e\mathcal{N} : \frac{d\sigma^{\text{el}}}{dQ^2} \sim \frac{1}{Q^4}|F(Q^2)|^2 \sim \frac{d\sigma^{\text{pt}}}{dQ^2} \left(\frac{M^4}{Q^4}\right)^2 \,. \tag{8.179}$$

For the electron scattering off the nucleon into everything we have

$$e + \mathcal{N} \to e + \text{ everything:}\ \frac{d\sigma}{dQ^2} \sim \frac{1}{Q^4} F_2(x) \sim \frac{d\sigma^{\text{pt}}}{dQ^2} \,. \tag{8.180}$$

We have the scaling $F_2(x, Q^2) \approx F_2(x)$. For $Q^2 \to \infty$ the inclusive cross section hence behaves analogously to the point-like cross section (the Q^2 decrease is slower by 8 orders in Q compared to the elastic nucleon cross section (8.179)).

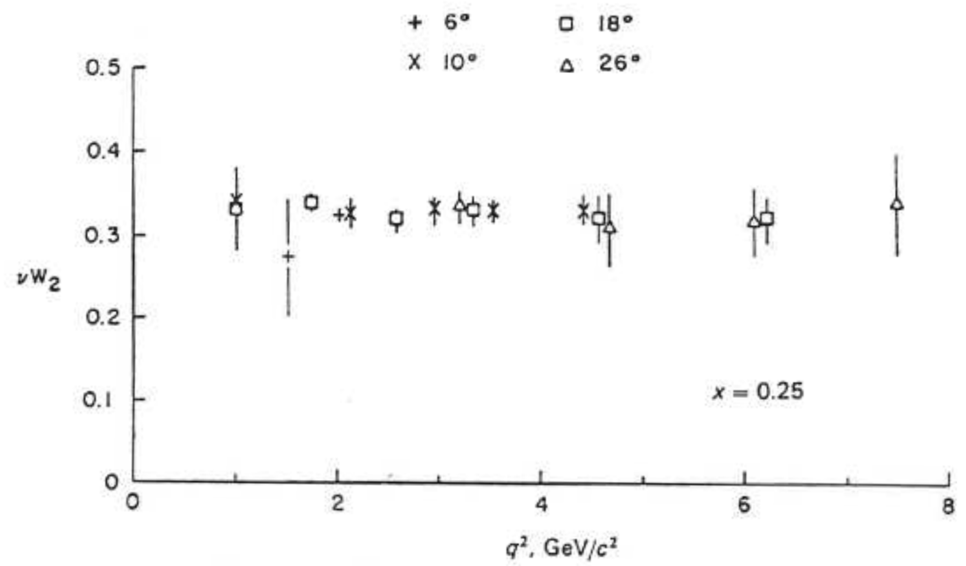


Figure 8.11: The νW_2 as function of q^2 for $x = 0.25$.

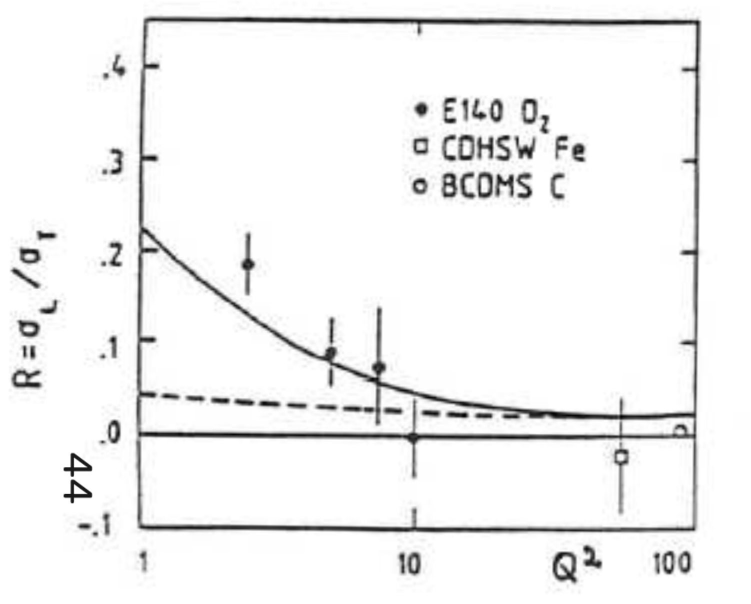


Figure 8.12: The ratio R as function of Q^2.

8.5.2 Scaling Violation: Altarelli-Parisi Equations (DGLAP)

The idea is that the parton-quarks are surrounded by a gluon cloud inside the nucleon. At sufficiently large Q^2 more and more quantum fluctuations are resolved. The momentum spectra of the quarks and gluons vary with Q^{-1} so that the microscopic parton distributions are Q^2-dependent.

Next, we discuss the splitting probability and look at the process (cf. Fig. 8.14)

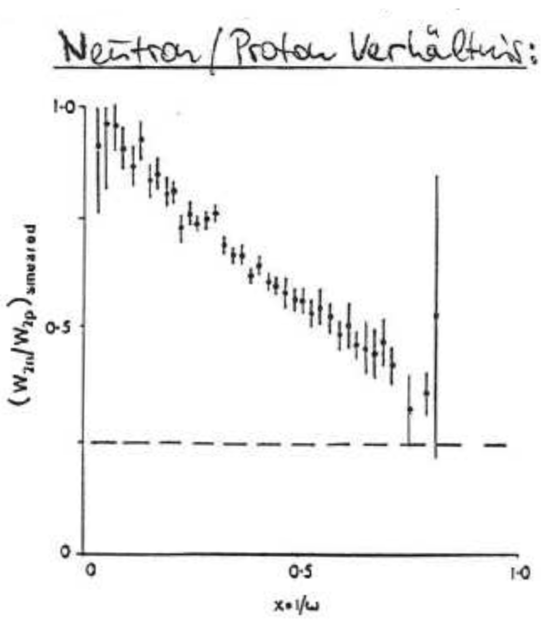


Figure 8.13: The ratio W_{2n}/W_{2p} as function of x.

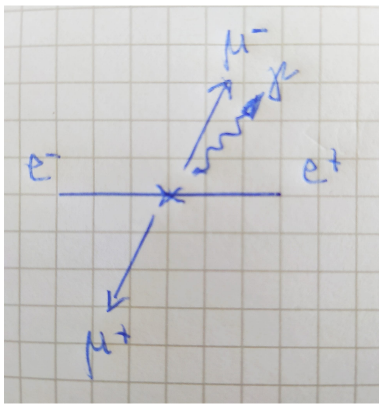


Figure 8.14: The process $e^+e^- \to \mu^+\mu^- + \gamma$.

$$e^+e^- \to \mu^+ + \underbrace{\mu^-\gamma}_{\theta_{\mu+\gamma}\ \text{small fixed}} \quad . \tag{8.181}$$

The contributing diagrams are shown in Fig. 8.15. Introducing the reduced electron/positron energies

$$x_{1,2} = \frac{E_\pm}{E} \tag{8.182}$$

we have

$$\frac{1}{\sigma_0}\frac{d^2\sigma}{dx_1 dx_2} = \frac{\alpha}{2\pi}\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \ . \tag{8.183}$$

Introducing the reduced photon energy

$$z = \frac{E_\gamma}{E} \tag{8.184}$$

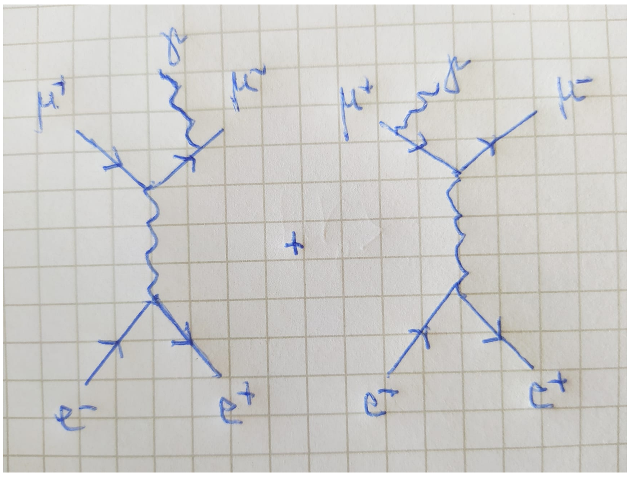


Figure 8.15: The diagrams contributing to $e^+e^- \to \mu^+\mu^- + \gamma$.

and

$$
\begin{aligned}
x_\perp &= \frac{2}{x_1}\sqrt{(1-x_1)(1-x_2)(1-z)} = \frac{p_\perp}{E} && (8.185)\\
\ln x_\perp^2 &\approx \ln(1-x_1) && (8.186)\\
d\ln p_\perp^2 &\approx \frac{dx_1}{1-x_1} && (8.187)
\end{aligned}
$$

$$x_1 + x_2 + z = 2 \tag{8.188}$$

We get for the fragmentation probability that μ^- splits into $\mu^- + \gamma$ (cf. Fig. 8.16), with $x_2 \approx 1 - z$,

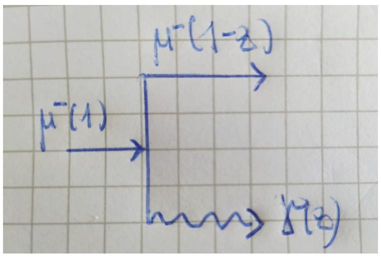


Figure 8.16: Fragmentation of μ^- into $\mu^- + \gamma$.

$$d\sigma = \sigma_0 \int^{Q^2} \frac{dp_\perp^2}{p_\perp^2}\frac{\alpha}{2\pi}\frac{1+(1-z)^2}{z}dz\ . \tag{8.189}$$

The cross section can be written as

$$\text{cxn} = \mu - \text{pair cxn } \times \text{particle flux } (\mu \to \mu\gamma)\ . \tag{8.190}$$

The increase of the particle flux at $Q^2 \to Q^2 + \delta Q^2$ is

$$\frac{\delta N(\mu \to \mu\gamma)}{\delta \ln Q^2} = \frac{\alpha}{2\pi}\frac{1+(1-z)^2}{z}dz\ . \tag{8.191}$$

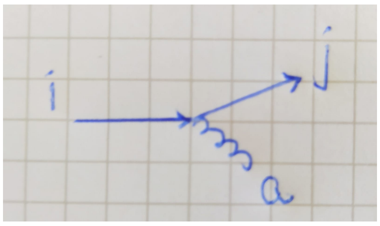


Figure 8.17: Fragmentation of q into $q+g$.

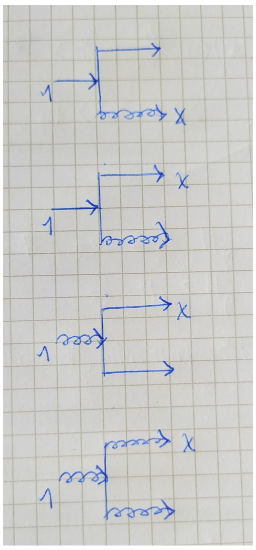
For the quark fragmentation (cf. Fig. 8.17) we have to include the color average and sum

$$\sum_{k,a} T^a_{ik} T^a_{jk} = \frac{4}{3}\delta_{ij}\ . \tag{8.192}$$

The QCD splitting probabilities

$$\frac{\delta N}{\delta \ln \frac{Q^2}{\Lambda^2}} = \frac{\alpha_s(Q^2)}{2\pi} P(x) dx \tag{8.193}$$

are given by (bs = bremstrahl-singularity)



$$q \to q + g(x) \qquad P_{gq} = \frac{4}{3}\frac{1+(1-x)^2}{x} \quad \text{bs for } x \to 0 \tag{8.194}$$

$$q \to q(x) + g \qquad P_{qq} = \frac{4}{3}\frac{1+x^2}{1-x} \quad \text{bs for } x \to 1 \tag{8.195}$$

$$g \to q(x) + \bar{q} \qquad P_{qg} = P_{\bar{q}g} = \frac{1}{2}[x^2 + (1-x)^2] \quad \text{finite} \tag{8.196}$$

$$g \to g(x) + g \qquad P_{gg} = 6\frac{[1-x+x^2]^2}{x(1-x)} \quad \text{bs for } x \to 0,1 \tag{8.197}$$

Altarelli-Parisi master equations for the parton densities We have

$$\begin{aligned} \frac{\partial q(x,Q^2)}{\partial \ln Q^2} &= \frac{\alpha_s(Q^2)}{2\pi}\int_0^1 dy \int_0^1 dz \delta(x-yz)\left[P_{qq}(y)\, q(z,Q^2) + P_{qg}(y)\, g(z,Q^2)\right] \\ &\quad -\frac{\alpha_s(Q^2)}{2\pi}\int_0^1 dy' P_{qq}(y') q(x,Q^2) && (8.198) \\ \int_0^1 dy\prime\, P_{qq}(y')\, q(x,Q^2) &= \int_0^1 dy \int_0^1 dz \delta(x-yz)\delta(y-1)\left[\int_0^1 dy' P_{qq}(y')\right] q(z,Q^2)\ . && (8.199) \end{aligned}$$

With this we obtain the Altarelli-Parisi equations:

$$
\begin{aligned}
\frac{\partial q(x,Q^2)}{\partial \ln Q^2} &= \frac{\alpha_s(Q^2)}{2\pi}\int_0^1 dy \int_0^1 dz\delta(x-yz)\left\{P^R_{qq}(y)\, q(z,Q^2) + P_{qg}(y)\, g(z,Q^2)\right\} \\
\frac{\partial g(x,Q^2)}{\partial \ln Q^2} &= \frac{\alpha_s(Q^2)}{2\pi}\int_0^1 dy \int_0^1 dz\delta(x-yz)\Bigg\{P_{gq}(y)\sum_{\text{fl}}[q(z,Q^2)+\bar{q}(z,Q^2)] \\
&\quad + P^R_{gg}(y) g(z,Q^2)\Bigg\} \\
P^R_{qq}(y) &= P_{qq}(y) - \delta(y-1)\int_0^1 dy' P_{qq}(y') \\
P^R_{gg}(y) &= P_{qq}(y) - \delta(y-1)\left[\frac{1}{2}\int_0^1 dy' P_{gg}(y') + N_F\int_0^1 dy' P_{qg}(y')\right] \\
\alpha_s(Q^2) &= \frac{12\pi}{(33-2N_F)\ln\frac{Q^2}{\Lambda^2}}
\end{aligned}
\tag{8.200}
$$

Partial disentanglement

$$\delta = q - q' \quad \text{non-singlet} \tag{8.201}$$

$$\left.\begin{array}{l} \Sigma = \sum_{\text{fl}}(q+\bar{q}) \\ g \end{array}\right\} \text{coupled singlet set} \tag{8.202}$$

The solutions are obtained by introducing the moments

$$q(N,Q^2) = \int_0^1 dx x^{N-1} q(x,Q^2) \tag{8.203}$$

This transforms the integro-differential system of equations into a system of usual differential equations. natural variable:

$$s = \ln\frac{\ln Q^2}{\ln Q_0^2}\ , \tag{8.204}$$

where Q_0 = reference momentum transfer. (For fixed coupling constant $t = \ln Q^2$ would be the natural variable.)

1.) Non-singlet density:

$$
\begin{aligned}
\frac{\partial}{\partial s}\delta(N,Q^2) &= \frac{6}{33-2N_F}\int_0^1 dy y^{N-1} P^R_{qq}(y)\delta(N,Q^2) \\
&= \underbrace{\frac{6}{33-2N_F}\frac{4}{3}\left[-\frac{1}{2}+\frac{1}{N(N+1)} - 2\sum_{j=2}^{N}\frac{1}{j}\right]}_{\equiv -d_{\text{NS}}(N)}\delta(N,Q^2) \\
\Leftrightarrow \quad \frac{\partial}{\partial s}\delta(N,Q^2) &= -d_{\text{NS}}(N)\delta(N,Q^2) \quad \Rightarrow \quad \delta = \delta_0 e^{-s d_{\text{NS}}}\ .
\end{aligned}
\tag{8.205}
$$

We have an ln violation of the Bjorken scaling:

$$\delta(N,Q^2) = \delta(N,Q_0^2)\left(\frac{\ln Q^2}{\ln Q_0^2}\right)^{-d_{\rm NS}} = \delta(N,Q_0^2)\left(\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right)^{d_{\rm NS}} \quad . \tag{8.206}$$

Interpretation:

(i)

$$\begin{aligned} \text{asymptotic freedom} \quad &\Rightarrow \left[\frac{\ln Q^2}{\ln Q_0^2}\right]^{-d} \\ \text{fixed coupling} \quad &\Rightarrow \left[\frac{Q^2}{Q_0^2}\right]^{-d} \end{aligned} \tag{8.207}$$

(ii)

$$d_{\rm NS}(N=1) \;=\; 0: \text{ net quark number unchanged} \tag{8.208}$$
$$d_{\rm NS}(N>1) \;>\; 0: \text{ moments decrease with increasing } Q^2 \tag{8.209}$$

(iii) moment comparison: test of anomalous dimensions Q^2-dependence of structure functions

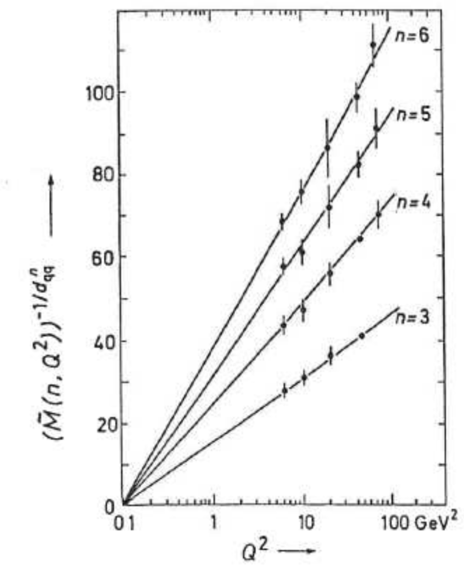


Figure 8.18: Moments of the structure functions.

Figure 8.18 shows the moments of the structure function, Fig. 8.19 displays the logarithms of the moments of the structure functions plotted against each other.

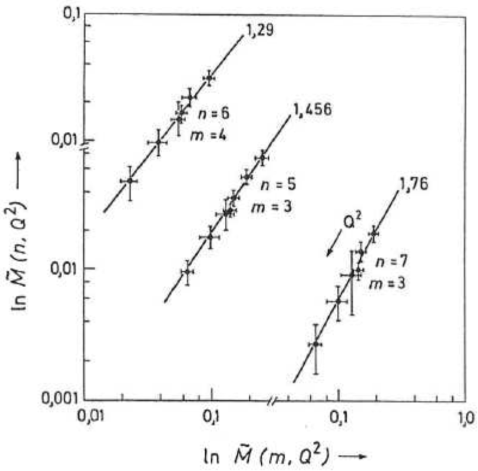


Figure 8.19: Logarithms of the moments of the structure functions plotted against each other. The QCD predictions are straight lines with calculable slope.

2.) Quark singlet and gluon densities:

$$\frac{\partial}{\partial s}\begin{pmatrix}\Sigma \\ G\end{pmatrix} = -\begin{pmatrix} d_{QQ} & d_{QG} \\ d_{GQ} & d_{GG}\end{pmatrix}\begin{pmatrix}\Sigma \\ G\end{pmatrix}, \quad \text{with } \Sigma = \Sigma(N, Q^2)\,. \tag{8.210}$$

We have

$$\begin{aligned} d_{QQ}(N) &= -\frac{6}{33-2N_F}\int_0^1 dy y^{N-1} P^R_{qq}(y) = \frac{4}{33-2N_F}\left[1 - \frac{2}{N(N+1)} + 4\sum_{j=2}^{N}\frac{1}{j}\right] \\ &\equiv d_{NS}(N) \end{aligned} \tag{8.211}$$

$$d_{QG}(N) = -\frac{6}{33-2N_F}\int_0^1 dy y^{N-1} 2N_F P_{qg}(y) = -\frac{6N_F}{33-2N_F}\frac{N^2+N+2}{N(N+1)(N+2)} \tag{8.212}$$

$$d_{GQ}(N) = -\frac{6}{33-2N_F}\int_0^1 dy y^{N-1} P_{gq}(y) = -\frac{8}{33-2N_F}\frac{N^2+N+2}{(N-1)N(N+1)} \tag{8.213}$$

$$\begin{aligned} d_{GG}(N) &= -\frac{6}{33-2N_F}\int_0^1 dy y^{N-1} P^R_{gg}(y) = \frac{9}{33-2N_F}\left[\frac{1}{3} - \frac{4}{N(N-1)}\right. \\ &\left. - \frac{4}{(N+1)(N+2)} + 4\sum_{j=2}^{N}\frac{1}{j} + \frac{2N_F}{9}\right] \end{aligned} \tag{8.214}$$

The solution is given by an exponential ansatz:

$$\Sigma = \frac{1}{\mu_+ - \mu_-}\left[(-\mu_-\Sigma_0 + G_0)e^{-d_+ s} + (\mu_+\Sigma_0 - G_0)e^{-d_- s}\right] \tag{8.215}$$

$$G = \frac{1}{\mu_+ - \mu_-}\left[\mu_+(-\mu_-\Sigma_0 + G_0)e^{-d_+ s} + \mu_-(\mu_+\Sigma_0 - G_0)e^{-d_- s}\right]\,, \tag{8.216}$$

with the eigenvalues

$$d_\pm(N) = \frac{1}{2}\left[(d_{GG}+d_{QQ}) \pm \sqrt{(d_{GG}-d_{QQ})^2 + 4d_{QG}d_{GQ}}\right] \tag{8.217}$$

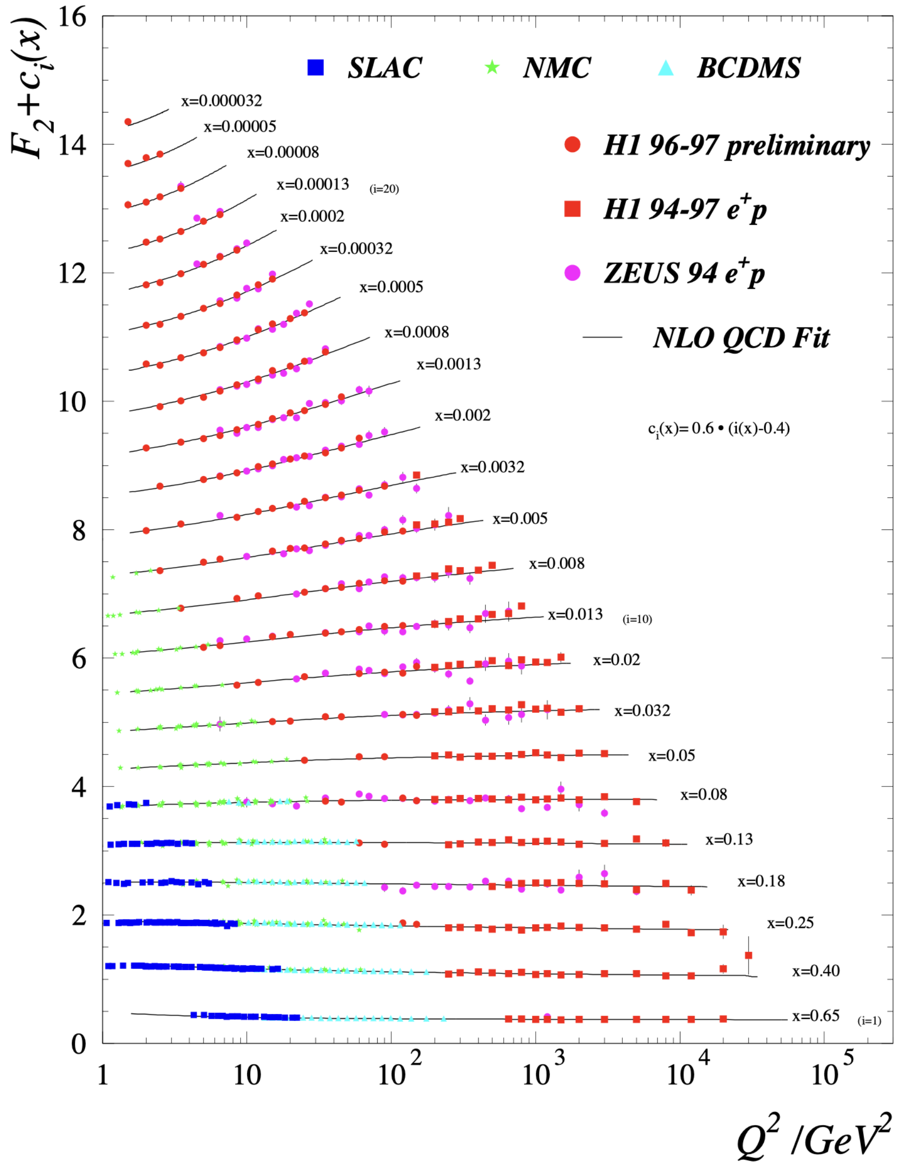


Figure 8.20: F_2 as function of Q^2 for different values of x.

and the eigenvectors

$$\mu_\pm(N) = \frac{d_\pm - d_{QQ}}{d_{QG}} = \frac{1}{2}\frac{d_{GG} - d_{QQ} \pm \sqrt{(d_{GG} - d_{QQ})^2 + 4d_{QG}d_{GQ}}}{d_{QG}} \; . \tag{8.218}$$

Physical consequences:

$$\frac{1}{A_N(s)} F_2^S(N-1, Q^2) = F_2^S(N-1, Q_0^2) + \frac{B_N(s)}{A_N(s)} G(N, Q_0^2) \; . \tag{8.219}$$

The left side is determined as straight line in $B_N(s)/A_N(s)$ with the gluon G density as slope, cf. Fig. 8.20.

8.6 Factorisation Theorem of QCD

At hadron colliders, we have the problem to relate the incoming quarks and gluons with the colliding protons and the outgoing particles with the observed hadronic jets. Scattering processes at high-energy hadron colliders hence consist of hard and soft processes. The hard processes like e.g. Higgs boson production from gluon fusion can be described through perturbation theory. The soft processes like e.g. the total hadronic cross section starting from the initial state protons and ending in final state hadrons, involve non-perturbative QCD effects which cannot be described by perturbation theory. The factorisation theorem of QCD states the following:

Factorisation theorem of QCD: Partonic cross sections have collinear divergences in the hadronic initial state, which factorise universally (i.e. independent of the process) from the hard scattering process and can be absorbed in the renormalised parton densities of the initial states. These renormalised parton densities are the DGLAP (Dokshitzer, Gribow, Lipatow, Altarelli and Parisi) equations.

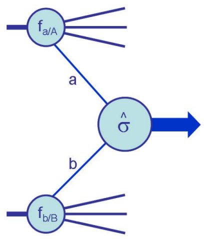


Figure 8.21: The process $pp \to X$.

The formula for the computation of the process $pp \to X$, cf. Fig. 8.21, applying the factorisation theorem, is given by

$$\sigma_{AB} = \sigma_{ab} \int dx_1 dx_2 \, f_{a/A}(x_1, \mu_F^2) f_{b/B}(x_2, \mu_F^2) \hat{\sigma}_{ab \to X} \;, \tag{8.220}$$

where A, B denote the incoming protons, $A, B = p$, and a, b sum over the quarks and gluons. The partonic cross section is denoted by $\hat{\sigma}$ and describes the reactions of the partons from the incoming hadrons, which interact at short distance. The functions $f_{a/A}(x_1, \mu_F^2)$ and $f_{b/B}(x_2, \mu_F^2)$ denote the parton distributions functions (pdf's) that quantify the probabliity of finding the parton a/b inside the hadron A/B carrying the momentum fraction $x_{1/2} = 2E_{a/b}/\sqrt{S}$ at the factorisation scale μ_F, which separates the short- and long-distance physics. The $\sqrt{S}$ denotes the hadronic (here proton) c.m. mass energy. The pdf's which involve non-perturbative effects have to be extracted from experiment, e.g. from deep inelastic scattering.

The full process including the final state hadrons X is calculated as

$$\sigma_{pp \to X} = \sum_{a,b,k} f_{a/p}(\mu_F^2) \otimes f_{b/p}(\mu_F^2) \otimes \hat{\sigma}_{ab \to k}(\alpha_s(\mu_R^2), \mu_R^2) \otimes D_{k \to X}(\mu_F^2) \;. \tag{8.221}$$

The partonic cross section $\hat{\sigma}_{ab \to k}$ is calculable within perturbation theory in powers of α_s,

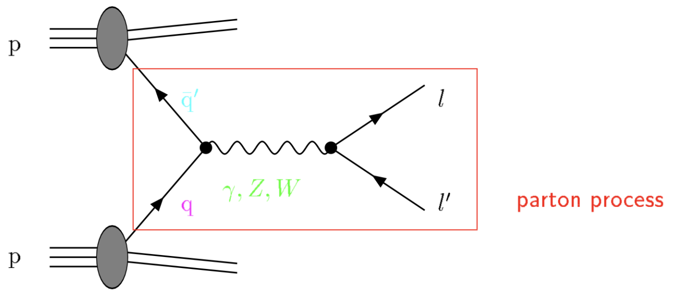
$$\hat{\sigma}_{ab \to k} = [\hat{\sigma}_0 + \alpha_s(\mu_R^2)\hat{\sigma}_1 + \alpha_s^2(\mu_R^2)\hat{\sigma}_2 + \ldots]_{ab \to k} \;. \tag{8.222}$$

The $f_{a/p}(\mu_F^2) \otimes f_{b/p}(\mu_F^2)$ relates to the luminosity of the collider and have to be determined experimentally. The transition to the final state X given by mesons, hadrons, jets ... is given by the fragmentation function $D_{k\to X}(\mu_F^2)$, the jet algorithms, and/or through Monte Carlo shower algorithms.

8.7 Example: Drell-Yan Process

Example Drell-Yan Process

- **Drell-Yan Process**



- **Cross section:** $\sigma(pp \to l^+l^-) = \sum_q \int dx_1 dx_2 \; f_q(x_1)\, f_{\bar{q}}(x_2)\, \hat{\sigma}(q\bar{q} \to l^+l^-)$

  ▷ $f_{q/\bar{q}}(x)dx$: probability to find (anti)quark with momenum fraction x

  process independent, measured in DIS

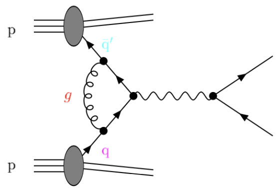
  ▷ $\hat{\sigma}(q\bar{q} \to l^+l^-)$: hard scattering cross section

  calculable in perturbation theory

Example Drell-Yan Process

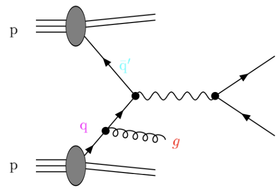
- **Factorisation not trivial beyond leading order**

virtual corrections



▷ UV divergences
▷ IR divergences

real corrections



▷ IR divergences
▷ collinear divergences

UV divergences $\longrightarrow$ renormalization $\alpha_S(\mu_R)$etc.
IR divergences $\longrightarrow$ cancel between virtual and real correction (Kinoshita-Lee-Nauenberg theorem)
collinear initial state divergences $\longrightarrow$ absorbed in pdf's

Example Drell-Yan Process

- **Scale dependence**

$$\begin{aligned}\sigma &= \int dx_1\, f_i(x_1,\mu_F) \int dx_2\, f_j(x_2,\mu_F) \\ &\quad \times \textstyle\sum_n \alpha_S^n(\mu_R) C_n(\mu_R,\mu_F)\end{aligned}$$

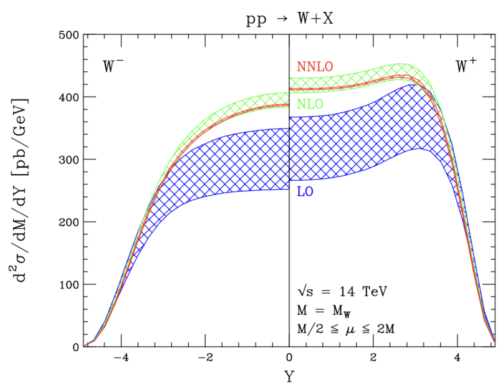
finite order in perturbation theory
$\rightsquigarrow$ artificial μ-dependence

$$\begin{aligned}\frac{d\sigma}{d\ln\mu_R^2} &= \textstyle\sum_{n=0}^N \alpha_S^n(\mu_R) C_n(\mu_R,\mu_F) \\ &= \mathcal{O}(\alpha_S(\mu_R)^{N+1})\end{aligned}$$

$\Rightarrow$ scale dependence $\sim$ theoretical uncertainty due to HO corrections

Rapidity distribution in $pp \longrightarrow W + X$

Anastasiou eal



$\Rightarrow$ significant reduction of μ dependence at (N)NLO

Parton luminosities

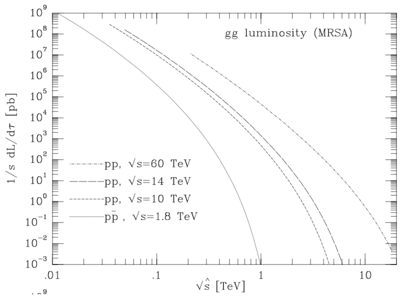
- **Assuming** that the total parton cross section $\hat{\sigma}$ depends only on $\hat{s}$ $\rightsquigarrow$ the cross section can be written as

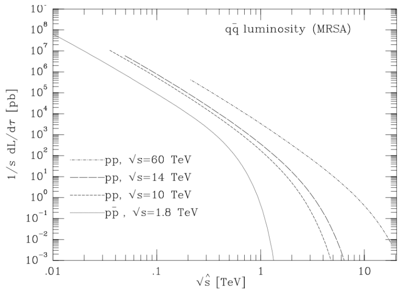
$$\sigma(s) = \sum_{\{ij\}} \int_{\tau_0}^{1} \frac{d\tau}{\tau} \left[\frac{1}{s} \frac{d\mathcal{L}_{ij}}{d\tau} \right] \hat{s}\, \hat{\sigma}_{ij}$$

▷ Sum over all relevant parton pairs $\{ij\}$ ▷ $\tau = x_1 x_2$

- **Differential luminosity** defined as

$$\tau \frac{d\mathcal{L}}{d\tau} = \int_0^1 dx_1 dx_2 [x_1 f_i(x_1, \mu_F)\, x_2 f_j(x_2, \mu_F) + (1 \leftrightarrow 2)]\, \delta(\tau - x_1 x_2)$$





Parton luminosity at the LHC

- **HERA precision data for F_2** cover most of the LHC x-range
- **Scale evolution of the PDFs** in Q over 2 to 3 orders
- **Sensitivity at the LHC**
  - ▷ 100 GeV physics: small x, sea partons
  - ▷ TeV scales: large x
  - ▷ rapidity distributions probe extreme x values