TTP2

Lecture 1

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1 Renormalization and regularization

We have seen in the previous semester that one-loop computations in the context of QED have led to ill-defined results quite often. We also saw that we could still make use of the perturbative computations if we accepted that quantities that appear in the Lagrangian do not need to be *physical* quantities, and that the relations between Lagrangian parameters and physical quantities need to be *established* to make theoretical predictions unambiguous. In this lecture we will discuss a few concepts and tools that will allow us to streamline this procedure and make it work in any Quantum Field Theory.

To define an interacting quantum theory fully, we need the concept of the *regularization*. To understand what this means we should remind ourselves about our experiences with one-loop QED computations. We saw that if we start with a standard QED action

$$S = \int \mathrm{d}^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\hat{\partial} - m)\psi - e\bar{\psi}\gamma^\mu \psi A_\mu \right), \tag{1.1}$$

and try to compute, say, a one-loop contribution to the matrix element of the electromagnetic current $\langle e(p_2)|J^\mu(0)|e(p_1)\rangle$, we will not be able to perform the calculation because the result *diverges* at large values of the loop momentum.[1] Specifically, we will face integrals of the form

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k_\alpha k_\beta}{((k+p_1)^2 - m^2)((k+p_2)^2 - m^2)k^2}, \tag{1.2}$$

that cannot be calculated as written. We addressed this problem by introducing a "heavy photon" into the theory and *subtracting* its contribution from matrix elements that contain contributions of an actual "physical" photon. If we do this, the above integral gets the modified as

$$\lim_{M\to\infty} \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k_\alpha k_\beta}{((k+p_1)^2 - m^2)((k+p_2)^2 - m^2)} \left(\frac{1}{k^2} - \frac{1}{k^2 - M^2} \right). \tag{1.3}$$

This is a convergent integral but, as a reminder that the limit $M \to \infty$ cannot be taken before the integration over k is performed, it depends on the logarithm of M.

[1] We have seen that there are also infra-red divergences, but we will ignore them for now.

The procedure that we just described is called the *Pauli-Villars regularization*. Another possible regularization is the *cut-off regularization* where, after the Wick rotation, all integrals are restricted to a finite volume of a four-dimensional Eucledian space-time.

It is important to understand that we introduce a regularization as an *intermediate* step to define the theory and to expose the nature of divergences that are present in the theory. Eventually, we will have to get rid of the regularization by expressing predictions through physical quantities and we will show that in certain theories that are called *renormalizable*, doing so allows to remove the dependence on the regularization procedure *completely*.

Since the regularization plays an auxiliary, technical role, there is a freedom to choose how it is done. A good regularization should not lead to unnecessary technical complications and, ideally, should not violate symmetries of a particular quantum field theory that we try to regularize. One of the most commonly used regularization procedure is the so-called *dimensional regularization* which is defined as a formal extension of a quantum field theory to a d-dimensional space-time.

Note that d is usually written as $d = 4 - 2\epsilon$ and that ϵ is a *complex parameter*. The dependence of the results of perturbative computations on d is supposed to be provided by *analytic* functions of d. Such functions can be *defined* for some values of d, where a calculation makes sense, and then analytically continued to the region around $d = 4$ which is of interest to us. This practice is standard in complex analysis. As we will see, results of perturbative computations already depend on d in the right way, so that the analytic continuation becomes a mere recognition of the properties of the functions that are used to express the results.

To see how this works, consider the following integral

$$I(n,d) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2 + i0)^n}. \tag{1.4}$$

Performing the Wick rotation, which formally amounts to writing $k_0 \to i k_0$, we find

$$I(n,d) = i(-1)^n \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \frac{1}{(k_E^2 + m^2)^n}, \tag{1.5}$$

where k_E is the Eucledian momentum and $k_E^2 = k_0^2 + k_1^2 + ... k_{d-1}^2$.[2] Introducing

[2]Note that this sum already requires a definition for a complex-valued d.

a radial coordinate, we write

$$I(n,d) = \frac{i(-1)^n \Omega_d}{(2\pi)^d} \int \frac{k_E^{d-1} \mathrm{d}k_E}{(k_E^2+m^2)^n} = \frac{i(-1)^n \Omega_d}{2(2\pi)^d} \int_0^\infty \frac{(k_E^2)^{\frac{(d-2)}{2}} \mathrm{d}k_E^2}{(k_E^2+m^2)^n}. \quad (1.6)$$

where $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the d-dimensional solide angle. To compute the last integral, we write

$$k_E^2 = m^2 \left(\frac{u}{1-u}\right), \quad 0 < u < 1, \quad (1.7)$$

and find

$$\begin{aligned} &\int \frac{(k_E^2)^{\frac{(d-2)}{2}} \mathrm{d}k_E^2}{(k_E^2+m^2)^n}. = (m^2)^{\frac{d-2}{2}+1-n} \int_0^1 (1-u)^{n-2-\frac{d-2}{2}} u^{\frac{(d-2)}{2}} \, \mathrm{d}u \\ &= (m^2)^{d/2-n} \frac{\Gamma(n-d/2)\Gamma(d/2)}{\Gamma(n)}. \end{aligned} \quad (1.8)$$

Using this result in Eq. (1.6), we find

$$\begin{aligned} I(n,d) &= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2-m^2+i0)^n} \\ &= \frac{i(-1)^n (m^2)^{\frac{d}{2}-n}}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)}. \end{aligned} \quad (1.9)$$

Let us first check that Eq. (1.9) gives what we want. The dependence of $I(n,d)$ on d appears in two places: first as power in m^d and $(4\pi)^d$ and also it appears in the argument of a Γ-function. Both, the Γ-function and an algebraic power function can be defined to be analytic functions in a complex plane. The Γ-function is known to have simple poles when its argument vanishes or is a negative integer. Hence, the above formula for I_n defines an analytic function of d that can be calculated for any value of d in the entire complex plane except for a countable set of values.

We are interested in $d = 4$. However, it is easy to see that there are many values of n for which choosing $d = 4$ leads to Γ-functions with negative integer arguments. Consider $n = 2$ as an example. Then $\Gamma(2-4/2) = \Gamma(0) = \infty$.

Hence, instead of computing $I(n=2,d=4)$ we compute $I(2,4-2\epsilon)$ where ϵ is assumed to be small. This computation is easy to do. To this end, we use the following property of the Γ-function

$$z\Gamma(z) = \Gamma(1+z), \tag{1.10}$$

and the fact that

$$\lim_{z\to 0} \Gamma(1+z) = 1 - \gamma_E \epsilon + \mathcal{O}(\epsilon^2), \tag{1.11}$$

where γ_E is the so-called Euler constant.

For $n=2$ we find

$$\Gamma\left(n-\frac{d}{2}\right) = \Gamma(2-2+\epsilon) = \Gamma(\epsilon) = \frac{\Gamma(1+\epsilon)}{\epsilon}, \tag{1.12}$$

so that

$$I(2,d) = \frac{i\Gamma(1+\epsilon)}{(4\pi)^{d/2}\epsilon}\,(m^2)^{-\epsilon}. \tag{1.13}$$

The factor $\Gamma(1+\epsilon)/(4\pi)^{d/2}$ in the above formula will appear quite often in calculations that use the dimensional regularization. For this reason it is convenient to keep this factor unexpanded in ϵ.

As another example, consider $n=1$. We write

$$\Gamma(n-d/2) \to \Gamma(-1+\epsilon) = \frac{\Gamma(\epsilon)}{\epsilon-1} = \frac{\Gamma(1+\epsilon)}{\epsilon(\epsilon-1)}, \tag{1.14}$$

and find

$$I(1,d) = \frac{i\Gamma(1+\epsilon)}{(4\pi)^{d/2}\epsilon}\,\frac{(m^2)^{1-\epsilon}}{(1-\epsilon)}. \tag{1.15}$$

We note that the result in Eq. (1.9) implies that we can interchange integration and the differentiation with respect to m. Indeed, it is easy to check that one can either differentiate the right hand side of Eq. (1.9) or the integral representation on the left hand side of this equation and get the same result. In general, changing the order of integration and differentiation with respect to a parameter works only for absolutely convergent integrals. However, formulas that we use define any integral to an absolutely convergent integral since they are obtained by an analytic continuation.

It is important to stress that sometimes this leads to "strange" results for integrals provided that we think about these integrals as Riemann integrals. For example, the result in Eq. (1.9) implies that $I(n,d)$ is identically zero for $n \leq 0$. This follows from the fact that $I(n,d) \sim 1/\Gamma(n)$ and $\Gamma(n) = \infty$ for $n = 0$ or n being negative integer. Therefore, all integrals of the form

$$\int \mathrm{d}^d k \, (k^2)^n, \quad n > 0, \tag{1.16}$$

are set to zero in the context of a consistent application of rules of dimensional regularization.

In fact, allowed manipulations of dimensionally-regulated integrals are the same as for absolutely convergent integrals. They include

- all changes of variables including momenta re-scaling and momenta shifts are allowed;
- the summation and integration can be interchanged;
- parameter differentiation and integration can be interchanged.

To see implications of these rules, consider the following integral

$$I_\alpha = \int \mathrm{d}^d k \, (k^2)^\alpha, \tag{1.17}$$

where α is an arbitrary (not necessarily integer) parameter. Changing the loop momentum $k = \lambda \, k_1$, we find

$$I_\alpha = \lambda^{2\alpha+d} I_\alpha. \tag{1.18}$$

Since this equation should hold for arbitrary λ, we conclude that $I_\alpha = 0$.

The above rules allow us to calculate any loop integral using dimensional regularization. To see this, consider a generic one-loop integral. We write

$$I_N = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \prod_{i=1}^{N} \frac{1}{(k+p_i)^2 - m_i^2}. \tag{1.19}$$

We use Feynman parameters to write

$$\prod_{i=1}^{N} \frac{1}{(k+p_i)^2 - m_i^2} = \Gamma(N) \int \frac{[\mathrm{d}x]_N}{((k+P_N)^2 - \Delta_N)^N}, \tag{1.20}$$

where $P_N = \sum_{i=1}^{N} p_i x_i$, $\Delta_N = P_N^2 - \sum_{i=1}^{N}(p_i^2 - m_i^2)x_i + i0$ and $[\mathrm{d}x]_N = \prod_{i=1}^{N} \mathrm{d}x_i\, \delta(1 - \sum_{i=1}^{N} x_i)$. To compute I_N, we interchange integration over Feynman parameters and the loop momentum, shift the loop momentum $k \to k - P_N$ and find

$$\begin{aligned} I_N &= \Gamma(N) \int [\mathrm{d}x]_N \int \frac{\mathrm{d}^d k}{(2\pi)^d} \, \frac{1}{(k^2 - \Delta_N)^N} \\ &= \frac{i(-1)^N \Gamma(N - \frac{d}{2})}{(4\pi)^{d/2}} \int \frac{[\mathrm{d}x]_N}{(\Delta_N)^{N-d/2}}. \end{aligned} \tag{1.21}$$

In general, the remaining integration over Feynman parameters is highly non-trivial but for *generic* momenta $p_{1,..,N}$ it should not lead to additional divergencies. However, we are often interested in computing Feynman integrals for *exceptional* momenta, e.g. for the on-shell $p_i^2 = m_i^2$ or the light-like $p_i^2 = 0$ ones, in which case infra-red divergences may appear. If this happens, integration over Feynman parameters *also* becomes singular in the $d \to 4$ limit. However, it can be treated in the same way as the loop integration namely as an analytic continuation from values of d where the integral over Feynman parameters converges to values of d which are of interest to us.

We have now discussed one of the most popular (if not *the most popular*) ways to regularize ultraviolet divergences of loop integrals in perturbation theory in a generic quantum field theory. We will now study which Green's functions need to be regularized.

To be concrete, we will consider Quantum Electrodynamics. We will also deal with Green's functions for generic values of the incoming momenta to avoid the need to deal with infra-red divegencies. We consider a typical contribution to an amputated one-particle-irreducible Green's function in QED. We can characterize it by

- the number of external electron and photon lines that we will refer to as N_l and N_γ;
- the number of internal (loop momentum dependent) lepton and photon propagators that we will denote as P_l and P_γ;
- the number of vertices which involve internal (loop momentum dependent) lines;

- the number of loops L.

We will be interested in understanding what happens to a diagram characterized by the above parameters if *all loop momenta become very large*. One can think about this in the following way – let us combine all loop momenta into a single vector in the $4L$-dimensional space and study what happens when the radial component of this vector becomes very large (infinite). We can also think about scaling all loop momenta by the same parameter λ and considering the limit $\lambda \to \infty$. The dependence of the result on λ tells us how the integrals diverges or converges in the integration region where all loop momenta become large.

For a particular Feynman integral I, we write

$$I(N_l, N_\gamma, P_l, P_\gamma, V, L) \sim \lambda^D, \tag{1.22}$$

where D is called *the superficial degree of divergence*. To compute it, we note that in QED vertices are momenta-independent, each photon propagator is $1/k^2 \sim 1/\lambda^2$ and each fermion propagator is $\hat{k}/k^2 \sim 1/\lambda$. We then find

$$D = 4L - P_l - 2P_\gamma. \tag{1.23}$$

We would like to express D through quantities that are easier to access than P_l and P_γ. To do so, we note that the number of loops is given by

$$L = P_l + P_\gamma - V + 1. \tag{1.24}$$

This formula is valid because the number of loops equals to the number of independent momenta that remain in the Green's function after we have accounted for the momentum conservation in each vertex. Then, we have $P_l + P_\gamma$ momenta to integrate over to start with and there are V momentum-conservation conditions to satisfy for internal momenta. However, one of these vertex δ-functions turns into a momentum conservation condition for the external momenta, this is the reason for $V \to V - 1$.

In QED, we can relate the number of vertices in a diagram to the number of internal and external lines. Indeed, each vertex has two electron lines and one photon line. Consider the photon lines. Since each vertex contains one photon line, it appears that the number of vertices equals to the number of

photon lines. However this is wrong because internal photon lines count twice since they appear in two vertices. Hence, the following equations are valid

$$V = 2P_\gamma + N_\gamma, \quad V = P_l + \frac{1}{2}N_l. \tag{1.25}$$

Note that it follows from the above equation that

$$2P_\gamma - P_l = \frac{1}{2}N_l - N_\gamma. \tag{1.26}$$

This equation will be important in the next step. We write

$$D = 4L - P_l - 2P_\gamma = 4(P_l + P_\gamma - V + 1) - P_l - 2P_\gamma. \tag{1.27}$$

We use the second equation in Eq. (1.25) to write V in terms of P_l and N_l and obtain

$$D = 2P_\gamma - P_l + 4 - 2N_l. \tag{1.28}$$

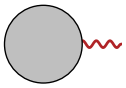
We can further simplify this equation using Eq. (1.26). We finally find

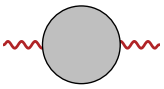
$$D = 4 - N_\gamma - \frac{3}{2}N_l. \tag{1.29}$$

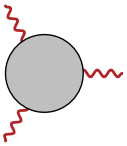
This formula has important implications. Indeed, the parameter D tells us the *overall* degree of divergence of a diagram that contributes to a particular Green's function. This is not the only divergence that may exists in a diagram since there can be sub-diagrams in a diagram that diverge. However, as we will see, when thinking about what to do with potential divergences, it is important to imagine that divergences can be removed iteratively, starting from sub-diagrams and continuing to an overall divergence that cannot be obtained by studying Green's functions with less loops. From this perspective, knowing the overal degree of divergence is very important.

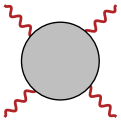
Eq. (1.29) tells us that whether or not a particular QED Green's function is divergent ($D \geq 0$) or convergent ($D < 0$), depends entirely on the number and type of external lines. The number of divergent Green's functions is obviously finite since $N_\gamma, N_l \geq 0$. Because external leptons can only appear in pairs, *seven* QED Green's functions are potentially divergent:

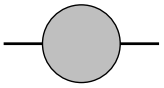
1.  $N_\gamma = N_l = 0$, with $D = 4$;

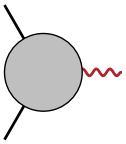
2.  $N_\gamma = 1$, with $D = 3$;

3.  $N_\gamma = 2$ with $D = 2$;

4.  $N_\gamma = 3$ with $D = 1$;

5.  $N_\gamma = 4$ with $D = 0$;

6.  $N_l = 2$ with $D = 1$;

7.  $N_l = 2$ and $N_\gamma = 1$, with $D = 0$.

This list can be further reduced. Indeed, the first Green's function has no external legs and, therefore, it is not needed for the construction of scattering amplitudes and cross sections. The second Green's function corresponds to the vacuum expectation value of the field A_μ. This quantity vanishes because of Lorentz invariance – there is no four-vector that this Green's function can depend upon. Furthermore, Green's functions with odd number of photons and no lepton fields vanish in QED because of Furry's theorem; this implies that also the fourth item on the list can be dropped.

Hence, if we are interested in a systematic understanding of ultraviolet divergences in QED, we need to find a way to make the following Green's functions finite

1. $\langle 0|T\psi(x)\bar{\psi}(0)|0\rangle, \quad D=1;$
2. $\langle 0|TA_\mu(x)A_\nu(0)|0\rangle, \quad D=2.$
3. $\langle 0|T\bar{\psi}(x_1)\psi(x_2)A_\mu(0)|0\rangle, \quad D=0;$
4. $\langle 0|A_\mu(x_1)A_\nu(x_2)A_\alpha(x_3)A_\rho(0)|0\rangle, \quad D=0.$

Before we discuss how this can be done, it is useful to point out that degrees of divergences in quantum field theories are affected by various factors. It is probably quite obvious that if everything else remains the same, the increase in space-time dimension d makes ultraviolet divergences worth. For example, in Eq. (1.23) $4L$ will turn into dL. We will not elaborate on this further because we are not planning to study quantum field theories in a space-time with dimensions that are different from 4.

Another important point is that the actual degree of divergence may depend on the symmetry properties of a particular theory. For example, in QED, gauge invariance and Lorentz symmetry *reduce the degree of divergence significantly*. Indeed, as we will see later, in the above list, the two-point electron's Green's function has $D=0$ rather than $D=1$, the two-point photon Green's function has $D=0$ instead of $D=2$, and the four-photon Green's function is actually *convergent*.

Furthermore, the degree of divergence depends on the structure of interactions that are present in a particular theory. In some cases this is quite obvious. For example, some interactions contain derivatives which turn into momenta which then contribute to the superficial degree of divegence. But there also cases that are more subtle. To see this, consider a theory defined by the following Lagrangian

$$L=\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi-\frac{1}{2}m^2\phi^2-\lambda\phi^n, \tag{1.30}$$

where ϕ is the scalar field.

We now consider an L-loop graph with P_ϕ internal lines, N_ϕ external lines and V vertices. To find the degree of divergence, we note that

$$\begin{aligned} D&=4L-2P_\phi,\\ L&=P_\phi-V+1, \end{aligned} \tag{1.31}$$

and since the total number of lines in case of a graph with V vertices is nV and since each internal line has to be counted twice, we have

$$nV = N_\phi + 2P_\phi. \tag{1.32}$$

Combining the three equations, we find

$$D = 4 - N_\phi + (n-4)V. \tag{1.33}$$

We see that, depending on n this equation leads to very different conclusions. If $n = 4$, the situation is very similar to that in QED since the degree of divergence only depends on the type of the Green's function (i.e. for $N_\phi > 4$, there are no overal divergencies anymore). On the contrary, if $n > 4$, every Green's function becomes divergent at *sufficiently high order in perturbation theory* because the degree of divegence increases with the increase in the number of vertices. Hence, if we are to define a prescription that makes Green's function convergent, in a theory with $n > 4$ we will have to cook up such a prescription for *all* Green's functions.[3] Such theories are called non-renormalizable.

Note also that $4 - n$ is the mass dimension of the coupling λ. Hence, non-renormalizable theories are theories where a coupling has *negative* mass dimension.

As we will see, in theories where the number of divergent Green's functions is finite, it is possible to absorb the divergences into a redefinition of various parameters of the theory. Such theories are called renormalizable and, as we have seen, the requirement of the renormalizability restricts the types of interactions that one can introduce into a Lagrangian. For a long time being renormalizable was considered to be an important criterion for a field theory to be "valid", but this perception has significantly changed in recent years.

[3] And there are infinitely many of them.