TTP2
Lecture 2

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2 Renormalized perturbative expansion in ϕ^4 theory

In this lecture, we will discuss the construction of the renormalized perturbation theory in a Quantum Field Theory of a scalar field ϕ with ϕ^4 interaction. The Lagrangian reads

$$L = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{2.1}$$

From the discussion in the previous lecture, we know that if we start calculating Green's functions in this theory, we will find ultraviolet divergencies. These divergencies affect Green's functions with zero, one, two, three and four external legs. Since the theory is invariant under $\phi \to -\phi$ symmetry, connnected Green's functions with odd number of ϕ-fields do not exist. Hence, we only need to understand what to do with Green's functions with two and four external legs.[1] Note that the Green's function with two external legs is the propagator of the field ϕ which depends on the mass parameter m. Furthermore, the amputated connected Green's function with four external legs is proportional to the coupling constant λ. Hence, we may suspect that some divergencies can be absorbed into a definition of the physical mass m and the physical coupling constant λ, both of which need to be operationally defined. Furthermore, as we have discussed in the previous semester, in the vicinity of the physical mass $m^2_{\rm phys}$, the two point function has the following behavior

$$\langle 0|T\phi(x)\phi(x)|0\rangle \sim \frac{Z}{p^2 - m^2_{\rm phys}}. \tag{2.2}$$

The construction of asymptotic states requires, however, that at $p^2 \sim m^2$, the propagator in the above equation reads $1/(p^2 - m^2)$. To accomplish this, we have to renormalize the field operators, as we have seen last semester.

The bottom line of this discussion is that all quantities in the Lagrangian, such as the field, the mass parameter and the coupling constant do not need to be "physical" (e.g. the physical mass of the particle). To make this more clear, we rewrite the Lagrangian in Eq. (2.1) as follows

$$L = \frac{1}{2}\partial_\mu\phi_0\,\partial^\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4. \tag{2.3}$$

[1] Green's functions with no external legs do not contribute to scattering amplitudes and cross sections.

We call the quantities with subscript 0 *bare* and distinguish them from *physical* parameters that can be determined through e.g. experimental measurements. To make this distinction explicit, we write

$$\phi_0 = Z^{1/2}\phi, \quad m_0^2 = Z_m m^2, \quad \lambda_0 = Z_\lambda \lambda. \tag{2.4}$$

Then, we rewrite Eq. (2.3) as

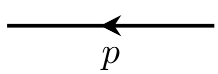
$$L = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + L_{\rm ct}, \tag{2.5}$$

where

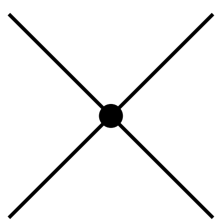
$$L_{\rm ct} = \frac{1}{2}(Z-1)\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2(Z_m Z - 1)\phi^2 - \frac{\lambda}{4!}(Z_\lambda Z^2 - 1)\phi^4, \tag{2.6}$$

is called the counter-term Lagrangian.

We can easily determine Feynman rules for the Lagrangian L treating the term $-\lambda\phi^4/4!$ and *all* terms in $L_{\rm ct}$ as perturbations. The Feynman rules read

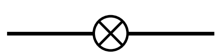


$$= \frac{i}{p^2 - m^2 + i0},$$

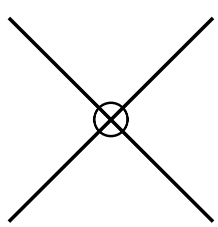


$$= -i\lambda,$$

(2.7)



$$= ip^2(Z-1) - im^2(Z_m Z - 1),$$



$$= -i\lambda(Z_\lambda Z^2 - 1).$$

At this point we *do not know* what the parameters Z, Z_m and Z_λ actually are. To fix them, we will require that m is the physical mass of a particle that corresponds to an excitation of the field ϕ, ϕ is the physical field and λ is the coupling constant that can be determine by measuring the scattering amplitude of the four ϕ-particles at rest.

Once we start computing the Green's functions in perturbation theory, we will find that *the above requirements are not satisfied automatically*. However, we will see that we will be able to *choose* all the Z-factors, order by order in the perturbative expansion, to ensure that they are fulfilled.

Let us now formulate the above conditions precisely. Consider first the two-point function. Denoting the self-energy as $\Sigma(p^2)$ (the relevant diagrams contribute to $-i\Sigma(p^2)$), we can write the two-point function as

$$\langle 0|T\phi(x)\phi(0)|0\rangle \to \frac{i}{p^2 - m^2 - \Sigma(p^2)}. \tag{2.8}$$

The condition that the two-point function must satisfy reads

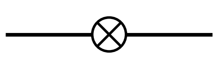
$$\langle 0|T\phi(x)\phi(0)|0\rangle \to \frac{i}{p^2 - m^2} + \mathrm{non-resonant\ terms}, \tag{2.9}$$

as $p^2 \to m^2$. This is only possible if

$$\Sigma(p^2)|_{p^2=m^2} = 0, \quad \frac{\mathrm{d}\Sigma(p^2)}{\mathrm{d}p^2}|_{p^2=m^2} = 0. \tag{2.10}$$

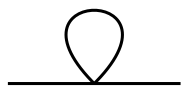
It is important to emphasize that when Σ is computed all relevant contributions, including the ones from the counter-term Lagrangian, are accounted for and adjusted accordingly to ensure that conditions in Eq. (2.10) are satisfied.

Let us compute $\Sigma(p^2)$ at one loop. We will start with the counter-term Lagrangian. Using the relevant Feynman rule from Eq. (2.7), we find



$$= -i\Sigma_{\mathrm{ct}}(p) = i(Z-1)p^2 - im^2(Z_m Z - 1). \tag{2.11}$$

Next, we consider the one-loop contribution to the self-energy. It reads



$$
\begin{aligned}
&= -i\Sigma^{(1)} = -i\frac{\lambda}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{i}{k^2-m^2} \\
&= \frac{i\lambda\Gamma(1+\epsilon)}{(4\pi)^{d/2}}\frac{(m^2)^{1-\epsilon}}{2\epsilon(1-\epsilon)},
\end{aligned}
\tag{2.12}
$$

and it is p^2-independent. We stress that this feature (p^2-independence) is the peculiarity of ϕ^4 theory.

The full self-energy Σ is the sum of $\Sigma_{\rm ct}$ and $\Sigma^{(1)}$ so that we require

$$
\begin{aligned}
&\Sigma_{\rm ct} + \Sigma^{(1)} = 0, \;\text{at } p^2 = m^2, \\
&\frac{\mathrm{d}\Sigma_{\rm ct}}{\mathrm{d}p^2} + \frac{\mathrm{d}\Sigma^{(1)}}{\mathrm{d}p^2} = 0, \;\text{at } p^2 = m^2.
\end{aligned}
\tag{2.13}
$$

Since $\Sigma^{(1)}$ is independent of p^2, the above equations simplify; from the second one we find

$$Z - 1 = 0, \tag{2.14}$$

which we rewrite as

$$Z = 1 + \mathcal{O}(\lambda^2). \tag{2.15}$$

since we have computed $\Sigma^{(1)}$ through $\mathcal{O}(\lambda)$ *only*.

Then, we use the first equation in Eq. (2.13) where we employ the (by now) known value of Z, and find

$$Z_m = 1 - \frac{\Sigma}{m^2} = 1 + \frac{\lambda\Gamma(1+\epsilon)}{(4\pi)^{d/2}}\frac{(m^2)^{-\epsilon}}{2\epsilon(1-\epsilon)} + \mathcal{O}(\lambda^2). \tag{2.16}$$

With this counter-term, the two-point function in ϕ^4 theory becomes *finite* and the limit $\epsilon \to 0$ can be taken. Note that in this case this limit is trivial since, with the above counter-term, the two-point function receives *no* one-loop corrections. This, however, is a feature that is particular to ϕ^4-theory at one loop.

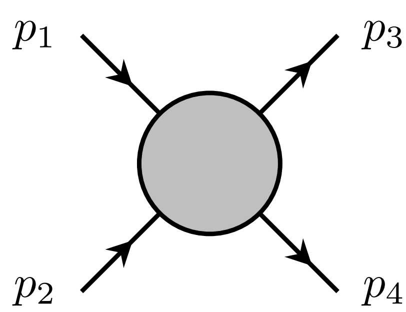


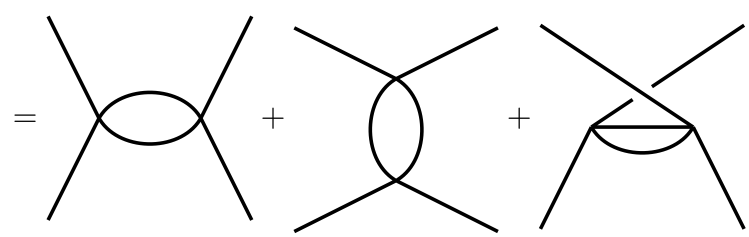
Figure 1: The four-point Green's function.

The next thing to discuss is the four-point amputated connected Green's function shown in Fig. 1. If we denote the momenta of the four particles in this Green's function as p_1, p_2, p_3, p_4 and consider $p_{1,2}$ to be incoming and $p_{3,4}$ – outgoing. At leading order we find

$$G(p_1, p_2, p_3, p_4) = -i\lambda. \tag{2.17}$$

The one-loop contribution can be written as

$$G^{(1)}(p_1, p_2, p_3, p_4) = \sum_{i=1}^{3} F(q_i^2)$$



(2.18)

where $q_1 = p_1 + p_2$, $q_2 = p_1 - p_3$ and $q_3 = p_1 - p_4$ and

$$F(q^2) = \frac{\lambda^2}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)((k + q_i)^2 - m^2)}. \tag{2.19}$$

It is easy to see that $F(q^2)$ is divergent and that its divergence is independent of q^2. We find

$$F(q^2) = \frac{i\lambda^2 \Gamma(1+\epsilon)}{(4\pi)^{d/2}} \left[\frac{1}{2\epsilon} + F_{\text{fin}}(q) \right]. \tag{2.20}$$

As we mentioned earlier, the definition of the physical coupling constant is that the scattering amplitude of four particles *at rest* is given by $-i\lambda$. This kinematic point corresponds to $p_1 = p_2 = p_3 = p_4 = (m, \vec{0})$, so that $q_1 = p_1 + p_2 = Q$, where $Q = (2m, \vec{0})$ and $q_3 = p_1 - p_3 = q_3 = p_2 - p_4 = 0$. Therefore,

$$G^{(1)}|_{\text{rest}} = \frac{i\lambda^2\Gamma(1+\epsilon)}{(4\pi)^{d/2}}\left[\frac{3}{2\epsilon} + F_{\text{fin}}(Q) + 2F_{\text{fin}}(0)\right]. \tag{2.21}$$

At the threshold the full amplitude is

$$-i\lambda\left[1 - \frac{\lambda\Gamma(1+\epsilon)}{(4\pi)^{d/2}}\left(\frac{3}{2\epsilon} + F_{\text{fin}}(Q) + 2F_{\text{fin}}(0)\right) + (Z_\lambda Z^2 - 1)\right]. \tag{2.22}$$

According to the condition that we imposed on the four-point Green's function, the expression in square brackets has to be equal to 1. Because we already found the the wave function renormalization constant Z is $1 + \mathcal{O}(\lambda^2)$, we derive

$$Z_\lambda = 1 + \frac{\lambda\Gamma(1+\epsilon)}{(4\pi)^{d/2}}\left[\frac{3}{2\epsilon} + F_{\text{fin}}(Q) + 2F_{\text{fin}}(0)\right]. \tag{2.23}$$

Using Z_λ, we obtain the following result for the four-point Green's function at an arbitrary kinematic point

$$G(\{p_i\}) = -i\lambda\left(1 - \frac{\lambda}{16\pi^2}\left[\sum_{i=1}^{3} F_{\text{fin}}(q_i^2) - F_{\text{fin}}(Q^2) - 2F_{\text{fin}}(0)\right]\right). \tag{2.24}$$

where we took the $\epsilon \to 0$ limit as appropriate.

Although we achieved our goals by removing divergences from the Green's functions and ensuring that conditions imposed on them are satisfied,[2] it is instructive to compute the Green's function explicitly. We find

$$F(q^2) = \frac{i\lambda^2\Gamma(1+\epsilon)}{2(4\pi)^{d/2}\epsilon}\int_0^1 \mathrm{d}x\, \left(m^2 - q^2x(1-x)\right)^{-\epsilon}. \tag{2.25}$$

Expanding in ϵ, we find

$$F_{\text{fin}}(q^2) = -\frac{1}{2}\int_0^1 \mathrm{d}x\, \ln(m^2 - x(1-x)q^2). \tag{2.26}$$

[2]Admittedly, only at one loop.

This integral can be computed explicitly in terms of logarithmic and rational functions. However, this computation requires a little bit of care because the argument of the logarithm is a quadratic form. To simplify it, I will consider the case when $s = (p_1 + p_2)^2 \gg m^2$, $|t| = |(p_3 - p_1)^2| \gg m^2$ and $|u| = |(p_1 - p_4)^2| \gg m^2$ and perform the calculation with the logarithmic accuracy.

If $|q^2| \gg m^2$, we easily find (with the logoarithmic accuracy)

$$F_{\rm fin}(q^2) \approx -\frac{1}{2}\ln(|q^2|). \tag{2.27}$$

On the contrary, if $q^2 = 0$ or $q^2 = 4m^2$, then

$$F_{\rm fin}(0) = -\frac{1}{2}\ln m^2, \tag{2.28}$$

or

$$F_{\rm fin}(4m^2) = -\frac{1}{2}\ln\frac{m^2}{2}, \tag{2.29}$$

respectively. Hence, the amputated four-point Green's function becomes (with the logarithmic accuracy)

$$G(\{p_i\}) = -i\lambda\Big(1 - \frac{\lambda}{16\pi^2}\Big[\ln\left(\frac{stu}{m^6}\right) + \cdots\Big]\Big) + \mathcal{O}(\lambda^3). \tag{2.30}$$

This formula shows an interesting feature which is worth pointing out. We have constructed a perturbative expansion of the scattering amplitude in the coupling constant λ. We see, however, that the expansion parameter can be very different since the size of the corrections is not determined by the coupling constant λ but rather by the quantity $\lambda \ln s/m^2$ which can be much larger than λ itself. Looking back at our calculation, it is easy to understand the origin of these logarithms. Indeed, terms $\ln s, \ln(-t), \ln(-u)$ come from computing the loop for a given set of momenta $p_{1,2,3,4}$. However, $\ln m^2$ terms come from the renormalization condition which is set at the threshold of the process $\phi(p_1) + \phi(p_2) \to \phi(p_3) + \phi(p_4)$ where energies of all particles are m.

This understanding and the possibility to change the renormalization condition, since the result of the calculation should not depend on it (provided of course that one changes the parameters accordingly), should allow us to account for certain types of "kinematically enhanced" corrections to all orders in perturbation theory. Methods to do so are known as the *renormalization group methods* and we will talk about them later.