TTP2
Lecture 3

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3 Renormalized perturbation theory for QED

In this lecture we will construct the renormalized perturbation theory for Quantum Electrodynamics (QED). We will consider a theory with a single lepton, the electron. The Lagrangian reads

$$L = -\frac{1}{4} F^{(0)}_{\mu\nu} F^{\mu\nu}_0 + \bar{\psi}_0 (i\hat{\partial} - m_0)\psi_0 - e_0 \bar{\psi}_0 \gamma_\mu \psi_0 A^\mu_0, \tag{3.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field-strength tensor. Similar to the case of the ϕ^4 theory, we interpret all the quantities in the above Lagrangian as *bare* quantities. These bare quantities are indicated by a subscript 0. The physical quantities are related to bare quantities by means of a multiplicative renormalization. We write

$$\psi_0 = Z_2^{1/2}\psi, \quad A^\mu_0 = Z_3^{1/2} A^\mu, \quad m_0 = Z_m m, \quad e_0 = Z_e e. \tag{3.2}$$

We now rewrite the Lagrangian as follows

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\hat{\partial} - m)\psi - e\bar{\psi}\gamma_\mu \psi A^\mu + L_{\rm ct}, \tag{3.3}$$

where the counter-term Lagrangian reads

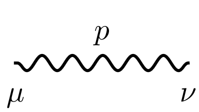
$$L_{\rm ct} = -\frac{\delta_3}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(\delta_2 i\hat{\partial} - m\delta_m)\psi - e\delta_e\, \bar{\psi}\gamma_\mu \psi A^\mu, \tag{3.4}$$

where

$$\delta_3 = Z_3 - 1, \quad \delta_2 = Z_2 - 1, \quad \delta_m = (Z_m Z_2 - 1), \quad \delta_e = (Z_e Z_2 Z_3^{1/2} - 1). \tag{3.5}$$

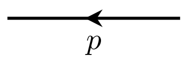
The renormalization constants are *unknown* at this point. As we already discussed in the previous lecture, we will determine them order by order in perturbation theory by insisting that divergent Green's functions satisfy (physical) conditions that we impose on them.

The Feynman rules[1] that follows from the QED Lagrangian in Eq. (3.4) are then

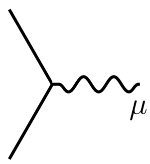
- 

 the photon propagator: $-ig_{\mu\nu}/(p^2 + i0)$;

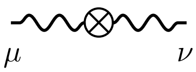
[1] The Feynman rules refer to Feynman gauge. If a general gauge is used, it may be necessary to also renormalize the gauge parameter.

- 

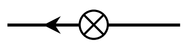
the electron propagator: $i/(\hat{p} - m + i0)$;

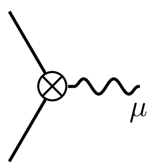
- 

an (amputated) electron-photon vertex: $-ie\gamma^\mu$;

- 

a counter-term contribution to the photon propagator: $-i(g^{\mu\nu}p^2 - p^\mu p^\nu)\delta_3$;

- 
a counter-term contribution to the electron propagator: $i(\hat{p}\delta_2 - \delta_m)$;

- 

a counter-term for an electron-photon vertex: $-ie\,\delta_e\gamma^\mu$
.

We now describe conditions that we impose on the relevant Green's functions. First, electron's self-energy $\hat{\Sigma}$ should fulfill the following equations

$$\hat{\Sigma}(\hat{p})|_{\hat{p}=m} = 0, \qquad \left.\frac{\mathrm{d}\hat{\Sigma}(\hat{p})}{\mathrm{d}\hat{p}}\right|_{\hat{p}=m} = 0. \tag{3.6}$$

These two conditions ensure that m is the physical mass of the electron and that the electron field ψ is properly normalized. We have talked about these conditions in the previous semester.

There is a similar condition that the photon self-energy (which in this case is called *vacuum polarization*) has to satisfy, however there is an important difference which concerns the *mass of the photon*. In QED, the mass of the photon is zero, as the consequence of gauge invariance. Since we regularize QED in such a way that gauge-invariance is not broken, we expect that the photon mass remains zero automatically and this condition does not need to be imposed. Technically, this follows from the fact that the photon vacuum polarization has the following form

$$i\Pi^{\mu\nu}(p) = i(g^{\mu\nu}p^2 - p^\mu p^\nu)\Pi(p^2), \tag{3.7}$$

where the function $\Pi(p^2)$ is not singular at $p^2 = 0$. To see what this form implies, we compute the photon propagator by summing up the vacuum polarization contributions. To this end, it is convenient to write

$$i\Pi^{\mu\nu}(p) = i\delta_T^{\mu\nu} p^2 \Pi(p^2), \tag{3.8}$$

where

$$\delta_T^{\mu\nu} = g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}. \tag{3.9}$$

Then, the photon propgator $D^{\mu\nu}$ becomes

$$D^{\mu\nu} = \frac{-ig^{\mu\nu}}{p^2} + \frac{-ig^\mu_\alpha}{p^2} i\delta_T^{\alpha\beta} p^2 \Pi(p^2) \frac{-ig^\nu_\beta}{p^2} + \ldots \tag{3.10}$$

We then replace the metric tensor with $\delta_T^{\mu\nu} + p^\mu p^\nu / p^2$ everywhere in the above formula. Since $p_\alpha \delta_T^{\alpha\beta} = 0$, when doing so the $p^\mu p^\nu$ terms can be simply dropped everywhere except in the first term in Eq. (3.10). Then, we find

$$D^{\mu\nu} = \frac{-ip^\mu p^\nu}{p^2} + \frac{-i\delta_T^{\mu\nu}}{(p^2 + i0)(1 - \Pi(p^2))}. \tag{3.11}$$

The first term is essentially a gauge term, which does not contribute to matrix elements because of the electromagnetic current conservation. The second term in the actual propagator, with a pole at $p^2 = 0$.

To have properly normalized asymptotic states, the residue at this pole has to be one. This implies that the vacuum polarization function $\Pi(p^2)$ has to satisfy the following condition

$$\Pi(p^2) = 0, \quad \text{at } p^2 = 0. \tag{3.12}$$

Finally, we would like e to be the physical charge of the electron.This parameter can be defined as the on-shell limit of the amputated Green's function which involves the electromagnetic field at zero momentum and two on-shell electrons at rest.

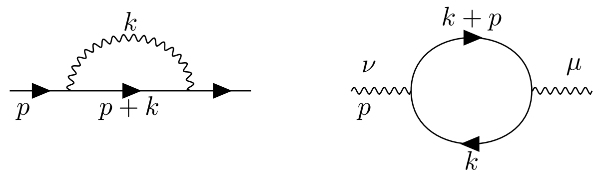


Figure 1: Electron self-self energy and photon vacuum polarization diagrams.

Among the various Green's functions in QED that have a non-negative superficious degree of divergence, the only one that is not part of this list is the four-photon Green's function. However, as we mentioned earlier, this Green's function is actually finite thanks to the gauge invariance. So, we will not need to discuss it.

We will now compute the renormalization constants explicitely. The electron self-energy was discussed last semester but we repeat this discussion here since we a) will employ the dimensional regularization and b) have to consider the counter-term contributions. We write

$$i\Sigma(\hat{p}) = i\Sigma_{1l}(\hat{p}) + i\Sigma_{\rm ct}, \tag{3.13}$$

where the first term on the right-hand side is the one-loop contribution and the second term is the counter-term contribution. The one-loop contribution reads (see Fig. 1)

$$i\Sigma_{1l}(\hat{p}) = (-ie)^2 \int \frac{{\rm d}^d k}{(2\pi)^d} \frac{\gamma^\alpha(\hat{p}+\hat{k}+m)\gamma_\alpha}{k^2((k+p)^2-m^2)}. \tag{3.14}$$

Since we use the dimensional regularization, we need to be careful with γ-matrices since their Lorentz indices should be considered to be d-dimensional. This has consequences; for example one can show that the following equation holds

$$\gamma^\alpha(\hat{p}+\hat{k}+m)\gamma_\alpha = (2-d)(\hat{p}+\hat{k}) + dm. \tag{3.15}$$

Then, we combine the propagators in Eq. (3.14) using Feynman parameters and integrate over the loop momentum k using results discussed in the first lecture. We find

$$i\Sigma_{1l}(\hat{p}) = -\frac{ie^2\Gamma(1+\epsilon)}{(4\pi)^{d/2}\epsilon}\int_0^1 \mathrm{d}x\,\frac{(2-d)\hat{p}(1-x)+dm}{(m^2x-p^2x(1-x))^\epsilon}. \tag{3.16}$$

We note that the divergent part of this quantity can easily be computed. We find

$$i\Sigma_{1l}(\hat{p})\Big|_{\mathrm{div}} = -\frac{i\Gamma(1+\epsilon)}{(4\pi)^{d/2}}\,\frac{1}{\epsilon}\left(-\hat{p}+4m\right). \tag{3.17}$$

To determine the constants Z_2 and Z_m we need to know $\Sigma(\hat{p})$ and its derivative at $\hat{p}=m$. We will now compute the corresponding contributions. First, taking $\hat{p}=m$ and $p^2=m^2$, and using $d=4-2\epsilon$, we find

$$\begin{aligned} i\Sigma_{1l}|_{\hat{p}=m} &= -\frac{ie^2\Gamma(1+\epsilon)}{(4\pi)^{d/2}\epsilon}\,m^{1-2\epsilon}\int_0^1 \mathrm{d}x\,\frac{2+2x(1-\epsilon)}{x^{2\epsilon}} \\ &= -\frac{ie^2\Gamma(1+\epsilon)}{(4\pi)^{d/2}\epsilon}\,m^{1-2\epsilon}\left(\frac{2}{1-2\epsilon}+1\right) = -\frac{ie^2\Gamma(1+\epsilon)}{(4\pi)^{d/2}\epsilon}\,m^{1-2\epsilon}\frac{3-2\epsilon}{1-2\epsilon}. \end{aligned} \tag{3.18}$$

Since we require that $\Sigma=0$ at $\hat{p}=m$, we derive the following equation

$$0 = -\frac{ie^2\Gamma(1+\epsilon)}{(4\pi)^{d/2}\epsilon}\,m^{1-2\epsilon}\frac{3-2\epsilon}{1-2\epsilon} + i(\hat{p}\,\delta_2-\delta_m m)|_{\hat{p}=m}, \tag{3.19}$$

where the last term is the counter-term contribution.

The second condition that needs to be satisfied is that the derivative of $\Sigma(\hat{p})$ w.r.t. $\hat{p}$ at $\hat{p}=m$ vanishes. We will start with computing the derivative of $\Sigma_{1l}(\hat{p})$. We will use the fact that

$$p^2 = \hat{p}\hat{p}, \tag{3.20}$$

so that

$$\frac{\partial}{\partial\hat{p}}p^2 = 2\hat{p}. \tag{3.21}$$

Since we use dimensional regularization, we can interchange integration and differentiation. Therefore, we find

$$
\begin{aligned}
\frac{\partial}{\partial \hat{p}} \int_0^1 \mathrm{d}x \, \frac{(2-d)\hat{p}(1-x)+dm}{(m^2x-p^2x(1-x))^{\epsilon}} &= \int_0^1 \mathrm{d}x \, \Big[\frac{(2-d)(1-x)}{(m^2x-p^2x(1-x))^{\epsilon}} \\
&+ \epsilon 2x(1-x)\hat{p}\frac{(2-d)\hat{p}(1-x)+dm}{(m^2x-p^2x(1-x))^{\epsilon+1}}\Big].
\end{aligned} \tag{3.22}
$$

We then set $\hat{p} \to m$ and find

$$
\frac{\partial}{\partial \hat{p}} i\Sigma_{1l}(\hat{p})|_{\hat{p}=m} = -\frac{ie^2\Gamma(1+\epsilon)m^{-2\epsilon}}{(4\pi)^{d/2}\epsilon} \int_0^1 \mathrm{d}x \, x^{-2\epsilon} f(x), \tag{3.23}
$$

where

$$
f(x) = \frac{4\epsilon}{x} - 2 + 2\epsilon - 4\epsilon^2 + (2 - 6\epsilon + 4\epsilon^2)x. \tag{3.24}
$$

Integrating over x, we find

$$
\frac{\partial}{\partial \hat{p}} i\Sigma_{1l}(\hat{p})|_{\hat{p}=m} = \frac{ie^2\Gamma(1+\epsilon)m^{-2\epsilon}}{(4\pi)^{d/2}\epsilon} \, \frac{3-2\epsilon}{1-2\epsilon}. \tag{3.25}
$$

We then compute the derivative of the full self-energy contribution, including the counter-term, and require that it vanishes at $\hat{p} = m$. We obtain the following equation

$$
0 = \frac{ie^2\Gamma(1+\epsilon)m^{-2\epsilon}}{(4\pi)^{d/2}\epsilon}\frac{3-2\epsilon}{1-2\epsilon} + i\delta_2, \tag{3.26}
$$

from where δ_2 and Z_2 can be determined. We find

$$
Z_2 = 1 - \frac{e^2\Gamma(1+\epsilon)\, m^{-2\epsilon}}{(4\pi)^{d/2}\, \epsilon} \, \frac{3-2\epsilon}{1-2\epsilon}. \tag{3.27}
$$

Then, from Eq. (3.19) we determine δ_m and Z_m, using teh result for Z_2 in Eq. (3.27). We find

$$
Z_m = 1 - \frac{e^2\Gamma(1+\epsilon)\, m^{-2\epsilon}}{(4\pi)^{d/2}\, \epsilon} \, \frac{3-2\epsilon}{1-2\epsilon}, \tag{3.28}
$$

so that $Z_m = Z_2$. The peculiar equality of these renormalization constants is an accident and does not have deep physical meaning.

Next, we consider the photon vacuum polarization contribution (see Fig. 1). We write

$$i\Pi^{\mu\nu}(p) = (-1)e^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{Tr}\left[\gamma^\mu(\hat{k}+\hat{p}+m)\gamma^\nu(\hat{k}+m)\right]}{(k^2-m^2)((k+p)^2-m^2)}. \tag{3.29}$$

where the (-1) pre-factor appears because of the closed fermion loop.

In general we can write

$$i\Pi^{\mu\nu} = i\Pi(p)(g^{\mu\nu}p^2 - p^\mu p^\nu) + p^\mu p^\nu i\Pi_L(p). \tag{3.30}$$

We will now prove that at one loop $\Pi_L(p) = 0$ (c.f. Eq. (3.7)).[2] To this end, we contract $i\Pi^{\mu\nu}$ with $p^\mu p^\nu$ and make use of the fact that $p_\mu p_\nu(g^{\mu\nu}p^2 - p^\mu p^\nu) = 0$. Hence, we find

$$i\Pi_L(p) = -\frac{e^2}{(p^2)^2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{Tr}\left[\hat{p}\,(\hat{k}+\hat{p}+m)\,\hat{p}\,(\hat{k}+m)\,\right]}{(k^2-m^2)((k+p)^2-m^2)}. \tag{3.31}$$

We then write

$$\begin{aligned} &(\hat{k}+m)\hat{p}(\hat{k}+\hat{p}+m) \\ &= (\hat{k}+m)\left((\hat{k}+\hat{p}-m)-(\hat{k}-m)\right)(\hat{k}+\hat{p}+m) \\ &= (\hat{k}+m)\left((k+p)^2-m^2\right)-(k^2-m^2)(\hat{k}+\hat{p}+m). \end{aligned} \tag{3.32}$$

We now use this expression in Eq. (3.31) and find

$$i\Pi_L(p) = \frac{e^2}{(p^2)^2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left[\frac{\mathrm{Tr}\left[\hat{p}\,(\hat{k}+\hat{p}+m)\right]}{(k+p)^2-m^2} - \frac{\mathrm{Tr}\left[\hat{p}\,(\hat{k}+m)\right]}{k^2-m^2} \right]. \tag{3.33}$$

Because integrals in the above equation are regularized, we can consider them separately. We then shift the loop momentum $k \to k-p$ in the first integral and observe that it becomes equal to the second one which, however, enters with the minus sign. Hence, we conclude that

$$\Pi_L(p^2) = 0. \tag{3.34}$$

[2] In fact, this statement is true at any number of loops.

We derived this result for the one-loop contribution to the photon vacuum polarization but it is valid for an arbitrary loop order; the reason for this is the fact that A^μ couples to the *conserved current*.

To determine the function $\Pi(p)$, we compute the trace and contract the result with the metric tensor. Computation of the trace is straightforward. By convention, even in d-dimensional space, we use

$$\mathrm{Tr}\,[1] = 4. \tag{3.35}$$

Hence, formulas for traces of γ-matrices do not change and we find

$$\begin{aligned}
&\mathrm{Tr}\left[\gamma^\mu(\hat{k}+\hat{p}+m)\gamma^\nu(\hat{k}+m)\right] \\
&= 4\left(k^\mu(k+p)^\nu + k^\nu(k+p)^\mu + g^{\mu\nu}\left(m^2 - k\cdot(k+p)\right)\right) \\
&= 4\left(2k^\mu k^\nu + k^\mu p^\nu + k^\nu p^\mu + g^{\mu\nu}\left(m^2 - k\cdot(k+p)\right)\right).
\end{aligned} \tag{3.36}$$

Contracting with $g_{\mu\nu}$, we obtain

$$i\Pi(p) = -\frac{4e^2}{(d-1)p^2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{2k\cdot(k+p) + d(m^2 - k\cdot(k+p))}{(k^2-m^2)((k+p)^2-m^2)}. \tag{3.37}$$

To simplify this expression, we rewrite the numerator in terms of the two propagators that appear in the denominator. Introducing the notation

$$d_1 = k^2 - m^2, \quad d_2 = (k+p)^2 - m^2, \tag{3.38}$$

we write

$$2k(k+p) + d(m^2 - k(k+p)) = 2m^2 + (1-\epsilon)p^2 - (1-\epsilon)(d_2+d_1), \tag{3.39}$$

so that

$$\begin{aligned}
i\Pi(p) = -\frac{4e^2}{(d-1)p^2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\Bigg(&\frac{2m^2+(1-\epsilon)p^2}{d_1 d_2} \\
&-(1-\epsilon)\left(\frac{1}{d_2}+\frac{1}{d_1}\right)\Bigg).
\end{aligned} \tag{3.40}$$

Since

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{d_1} = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{d_2}, \tag{3.41}$$

we finally obtain

$$i\Pi(p) = -\frac{4e^2}{(d-1)p^2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\Big(\frac{2m^2+(1-\epsilon)p^2}{d_1 d_2} - \frac{2(1-\epsilon)}{d_1}\Big). \tag{3.42}$$

Since the renormalization condition for the photon vacuum polarization is formualted at $p^2 = 0$, we need to know the value of $\Pi(p)$ at $p^2 = 0$. To compute it, we combine the propagators d_1 and d_2 and integrate over the shifted loop momentum. We find

$$i\Pi(p) = -\frac{4}{(3-2\epsilon)p^2}\frac{ie^2\Gamma(1+\epsilon)}{(4\pi)^{d/2}\epsilon}\int\limits_0^1 \mathrm{d}x\left[\frac{2m^2+(1-\epsilon)p^2}{\Delta^\epsilon} - 2m^{2-2\epsilon}\right], \tag{3.43}$$

where

$$\Delta^\epsilon = m^2 - p^2 x(1-x). \tag{3.44}$$

It is straightforward to compute $\Pi(p)$ at $p^2 = 0$ but, since there is a pre-factor $1/p^2$, it is useful to organize this computation in such a way that such the expansion is easy to obtain. We write

$$i\Pi(p) = -\frac{4}{(3-2\epsilon)}\frac{ie^2\Gamma(1+\epsilon)m^{-2\epsilon}}{(4\pi)^{d/2}\epsilon}\int\limits_0^1 \mathrm{d}x\Big[\frac{2m^2\left(\tilde{\Delta}^{-\epsilon}-1\right)}{p^2} + (1-\epsilon)\tilde{\Delta}^{-\epsilon}\Big], \tag{3.45}$$

where

$$\tilde{\Delta} = 1 - \frac{p^2}{m^2}x(1-x). \tag{3.46}$$

We can now trivially compute $i\Pi(0)$ that we require for the renormalization condition. Indeed, since

$$\lim_{p^2\to 0}\frac{m^2\left(\tilde{\Delta}^{-\epsilon}-1\right)}{p^2} = \epsilon x(1-x). \tag{3.47}$$

we find

$$i\Pi(p^2=0) = -\frac{4}{3\epsilon}\,\frac{ie^2\Gamma(1+\epsilon)m^{-2\epsilon}}{(4\pi)^{d/2}}. \tag{3.48}$$

The full contribution to the photon vacuum polarization is the sum of the loop contribution and the counter-term. Working out the required condition, we find

$$\Pi(p^2=0) - \delta_3 = 0, \tag{3.49}$$

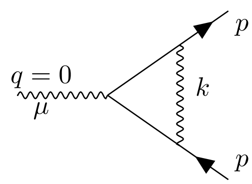


Figure 2: The electron-photon vertex for kinematics used for the electric charge counter-term computation.

which implies

$$Z_3 = 1 - \frac{4}{3\epsilon}\,\frac{e^2\Gamma(1+\epsilon)m^{-2\epsilon}}{(4\pi)^{d/2}}. \tag{3.50}$$

To complete the renormalization program of QED at one loop we require the calculation of δ_e, the charge renormalization constant. This counterterm is determined by the requirement that electron-photon interaction vertex is given by *physical* electric charge e without any corrections at the vanishing value of the photon momentum $q = 0$. Similar to all other computations that we have done in this lecture, to find the counter-term we first require the computation of the interaction vertex at $q = 0$ and for external electrons being on-shell, i.e. $p^2 = m^2$. We make use of the QED Feynman rules and write

$$\Gamma^\mu = -e^3 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\bar{u}_p \gamma^\alpha(\hat{k}+\hat{p}+m)\gamma^\mu(\hat{k}+\hat{p}+m)\gamma_\alpha u_p}{k^2((k+p)^2-m^2)((k+p)^2-m^2)}. \tag{3.51}$$

Combining propagators using the Feynman parameters, we derive the following result

$$\Gamma^\mu = -2e^3 \int [\mathrm{d}x]_3 \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, \frac{\bar{u}_p \gamma^\alpha(\hat{k}+\hat{p}+m)\gamma^\mu(\hat{k}+\hat{p}+m)\gamma_\alpha u_p}{((k+P_{12})^2-P_{12}^2)^3}, \tag{3.52}$$

where $P_{12} = (x_1+x_2)p$, $[\mathrm{d}x]_3 = \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\delta(1-x_1-x_2-x_3)$, and we made use of the on-shell condition for the electron momentum, $p^2 = m^2$. We then shift the loop momentum $k \to k - P_{12}$ and find

$$\Gamma^\mu = -2e^3 \int [\mathrm{d}x]_3 \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, \frac{\bar{u}_p \gamma^\alpha(\hat{k}+\hat{p}x_3+m)\gamma^\mu(\hat{k}+\hat{p}x_3+m)\gamma_\alpha u_p}{(k^2-P_{12}^2)^3}, \tag{3.53}$$

where we have used $P_{12} = p(1 - x_3)$. Next, using spherical symmetry of the integral, we can discard terms that are linear in k and simplify quadratic ones using the following replacement rule

$$k^\alpha k^\beta \to \frac{1}{d} k^2 g^{\alpha\beta}. \tag{3.54}$$

Hence, we find

$$\begin{aligned}\Gamma^\mu = -2e^3 \int [\mathrm{d}x]_3 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - P_{12}^2)^3} \bar{u}_p \Big[\gamma^\alpha \gamma^\rho \gamma^\mu \gamma_\rho \gamma_\alpha \, \frac{k^2}{d} \\ + \gamma^\alpha (\hat{p} x_3 + m) \gamma^\mu (\hat{p} x_3 + m) \gamma_\alpha \Big] u_p.\end{aligned} \tag{3.55}$$

To simplify this formula further, we use

$$\gamma^\alpha \gamma^\mu \gamma_\alpha = (2 - d) \gamma^\mu, \tag{3.56}$$

which gives

$$\gamma^\alpha \gamma^\rho \gamma^\mu \gamma_\rho \gamma_\alpha = (2 - d)^2 \gamma^\mu. \tag{3.57}$$

To simplify the second term, we use

$$\begin{aligned}\gamma^\alpha \hat{P} \gamma^\mu \hat{P} \gamma_\alpha &= (2 - d) \hat{P} \gamma^\mu \hat{P}, \\ \gamma^\alpha (\hat{P} \gamma^\mu + \gamma^\mu \hat{P}) \gamma_\alpha &= d (\gamma^\mu \hat{P} + \hat{P} \gamma^\mu),\end{aligned} \tag{3.58}$$

and find

$$\begin{aligned}&\bar{u}_p \gamma^\alpha (\hat{p} x_3 + m) \gamma^\mu (\hat{p} x_3 + m) \gamma_\alpha u_p \\ &= \bar{u}_p \gamma^\mu u_p \; m^2 \left[(2 - d) x_3^2 + 2 d x_3 + 2 - d \right].\end{aligned} \tag{3.59}$$

Finally, integrating over k, we find

$$\begin{aligned}\Gamma^\mu = -\bar{u}_p \gamma^\mu u_p \, \frac{2i \; e^3 \; \Gamma(1 + \epsilon)}{(4\pi)^{d/2}} \int [\mathrm{d}x]_3 \Big[\frac{(2 - d)^2}{d} \frac{1}{\epsilon \Delta^\epsilon} \\ - \frac{1}{2} \left(\frac{(2 - d)^2}{d} (1 - x_3)^2 + (2 - d) x_3^2 + 2 d x_3 + (2 - d) \right) \frac{m^2}{\Delta^{1+\epsilon}} \Big],\end{aligned} \tag{3.60}$$

where

$$\Delta = m^2 (1 - x_3)^2 \tag{3.61}$$

It is straightforward to integrate over x_3. The integration measure turns into

$$\int [\mathrm{d}x]_3 = \int_0^1 \mathrm{d}x_3 (1 - x_3) \tag{3.62}$$

After that, integration over x is straightforward and we find

$$\Gamma^\mu = -\bar{u}\gamma^\mu u_p \, \frac{ie^3\Gamma(1+\epsilon)m^{-2\epsilon}}{(4\pi)^{d/2}} \frac{(3-2\epsilon)}{\epsilon(1-2\epsilon)}. \tag{3.63}$$

We should add the counter-term contribution to this result and require that the sum vanishes. Hence, we find

$$\delta_e + \frac{e^2\Gamma(1+\epsilon)m^{-2\epsilon}}{(4\pi)^{d/2}} \frac{(3-2\epsilon)}{\epsilon(1-2\epsilon)} = 0. \tag{3.64}$$

We note that the loop contribution in the above equation equals $Z_2 - 1$ (c.f. Eq. (3.27)). Therefore,

$$\delta_e + 1 - Z_2 = 0. \tag{3.65}$$

Then

$$Z_e Z_3^{1/2} Z_2 - 1 + 1 - Z_2 = 0 \Leftrightarrow Z_e = Z_3^{-1/2}, \tag{3.66}$$

where Z_3 is given in Eq. (3.50). The above equality of δ_e and Z_2 is the consequence of the Ward identity for the electron photon vertex that we discussed in the last lecture of the previous semester.