TTP2
Lecture 4

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November 6, 2023

4 Non-abelian gauge theories

We have talked at length about the theory of electromagnetism – Quantum Electrodynamics or QED. Let us remind ourselves about how this theory is constructed. We start by writing the Dirac Lagrangin

$$L = \bar{\psi}(x)(i\partial_\mu \gamma^\mu - m)\psi(x) \tag{4.1}$$

This Lagrangian is invariant under *global* (i.e. x^μ-independent) $U(1)$-transformations. Indeed, writing

$$\psi(x) = e^{i\alpha}\psi_1(x), \quad \bar{\psi}(x) = \bar{\psi}_1(x)e^{-i\alpha}, \tag{4.2}$$

we find that

$$L[\bar{\psi}, \psi] = L[\bar{\psi}_1, \psi_1]. \tag{4.3}$$

Suppose we would like this transformation to be *local*, i.e. x^μ-dependent. Then,

$$\psi(x) = e^{i\alpha(x)}\psi_1(x), \quad \bar{\psi}(x) = \bar{\psi}_1(x)e^{-i\alpha(x)}, \tag{4.4}$$

It is easy to see that in this case

$$L[\bar{\psi}, \psi] = L[\bar{\psi}_1, \psi_1] - (\partial_\mu \alpha)\, \bar{\psi}_1(x)\gamma^\mu \psi_1(x). \tag{4.5}$$

To maintain the invariance of the Lagrangian under local $U(1)$ transformations, we can add a vector field to it and let the vector field also transform under local $U(1)$ transformation. We write

$$L = \bar{\psi}(iD_\mu \gamma^\mu - m)\psi, \tag{4.6}$$

where $D_\mu = \partial_\mu + ieA_\mu$. Assuming that

$$A^\mu = A_1^\mu - e^{-1}\partial^\mu \alpha(x), \tag{4.7}$$

under a gauge transformation, we find

$$L[\bar{\psi}, \psi, A_\mu] = L[\bar{\psi}_1, \psi_1, A_1]. \tag{4.8}$$

Hence, if we require that Lagrangians should be invariant under local gauge transformations, the apeparance of gauge vector fields becomes unavoidable. Moreover, these vector fields are massless since a term $A_\mu A^\mu$ is inconsistent with the gauge invariance. Furthermore, the kinetic energy of the field is described by the following term

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{4.9}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. $F_{\mu\nu}$ is the field-strength tensor; in electrodynamics, its elements are the electric and magnetic fields. For the following discussion, it is convenient to write $F_{\mu\nu}$ as the commutator of two covariant derivatives

$$[D_\mu, D_\nu] = ieF_{\mu\nu}. \tag{4.10}$$

The usefulness of this representation is that it allows us to easily write an expression for the field-strength tensor in a non-abelian case which we will discuss now.

To this end, consider a theory that is slightly more complex than QED. The theory contains two fermion fields instead of one. These fermions have equal masses. We write the Dirac Lagrangian as a sum of two Lagrangians

$$L = \bar{\psi}_a (i\gamma_\mu \partial^\mu - m)\psi_a + \bar{\psi}_b (i\gamma_\mu \partial^\mu - m)\psi_b. \tag{4.11}$$

To make the above formula look more compact, we combine the two fermion fields into a new field Ψ

$$\Psi(x) = \begin{pmatrix} \psi_a(x) \\ \psi_b(x) \end{pmatrix}. \tag{4.12}$$

Note that $\Psi(x)$ has eight components since $\psi_{a,b}$ are four-component spinor fields. Then,

$$L = \bar{\Psi}(x)(i\gamma_\mu \partial^\mu - m)\Psi(x). \tag{4.13}$$

The Lagrangian L is invariant under $SU(2)$ rotations of the spinor $\Psi(x)$

$$\Psi(x) = \hat{U}\Psi_1(x), \tag{4.14}$$

where U is an x^μ-independent $SU(2)$ matrix. This transformation means that, instead of $\psi_{a,b}$ we may decide to consider linear combinations of these spinor fields.

The matrix U can be written in the following form

$$U = e^{i\sum\limits_{i=1}^{3} \tau_a \alpha_a}, \tag{4.15}$$

where α_i are (real-valued) parameters and $\vec{\tau}_{1,2,3}$ are generators of the $SU(2)$ algebra. They satisfy the commutation relation

$$[\tau^a, \tau^b] = if^{abc}\tau^c. \tag{4.16}$$

In principle, these commutation relations are general; in case of $SU(2)$ $\tau^a = \sigma^a/2$ where $\sigma^{1,2,3}$ are Pauli matrices, and $f^{abc} = \epsilon^{abc}$ is a Levi-Cevita tensor.

We will now try to make the transformation in Eq. (4.14) local. Similar to the QED case, the invariance of the Lagrangian is achieved by introducing vector fields and requiring that changes of these fields compensate unwanted changes in the Lagrangian once a local $SU(2)$ rotation of the spinor Ψ is performed. We write

$$L = \bar{\Psi}(x)(i\gamma_\mu D^\mu - m)\Psi(x), \tag{4.17}$$

where

$$D^\mu = \partial^\mu - ig_s\hat{A}^\mu(x). \tag{4.18}$$

and $\hat{A}^\mu$ is a two-by-two matrix that is yet to be determined.

We note that if the field $\hat{A}_\mu(x)$ transforms as

$$A^\mu(x) = U(x)\ A_1^\mu(x)\ U^+(x) - \frac{i}{g_s}(\partial^\mu U U^+, \tag{4.19}$$

then

$$L[\bar{\Psi}, \Psi, \hat{A}_\mu] = L[\bar{\Psi}_1, \Psi_1, \hat{A}_{1,\mu}] \tag{4.20}$$

As we already said $\hat{A}^\mu(x)$ is a two-by-two matrix. We would like to argue that if this matrix is chosen to belong to the *algebra* of the group $SU(2)$, meaning that it can be written as a linear combinaton of the $SU(2)$ generators,

$$\hat{A}_\mu(x) = \sum_{a=1}^{3} A_\mu^a(x)\tau^a, \tag{4.21}$$

then gauge transformations do not change this property. Note that this implies that there are $N^2 - 1$ $A_\mu^{(a)}$ gauge fields in a theory with the group $SU(N)$.

To show this, we first consider infinitesimal gauge transformations

$$U(x) \approx 1 + i\epsilon^a(x)\tau^a, \tag{4.22}$$

where summation over index a is assumed. We then find

$$\hat{A}_{1,\mu} \approx \hat{A}_\mu - i\epsilon^a \left[\tau^a, \hat{A}_\mu\right] - \frac{1}{g_s}(\partial_\mu \epsilon^a)\tau^a + \mathcal{O}(\epsilon^2). \tag{4.23}$$

Hence, since

$$[\tau^a, \tau^b] = i f^{abc} \tau^c, \tag{4.24}$$

if the original field $\hat{A}_\mu$ belongs to the Lie algebra, the field obtained as the result of infinitesimal transformations also belongs to the Lie algebra. Since any transformation that belongs to the $SU(2)$ group can be obtained as a sequence of infinitesimal transformations, we conclude that Eq. (4.21) always holds and $\hat{A}^\mu(x)$ belongs to the Lie algebra of $SU(2)$.

We also need the kinetic term for the field $\hat{A}_\mu$. We have seen that in the abelian case the kinetic term can be obtained from a commutator of two covariant derivatives. We can try to do the same for the non-abelian case and define

$$\hat{F}_{\mu\nu} = \frac{i}{g_s}[D_\mu, D_\nu], \tag{4.25}$$

where the covariant derivative is given in Eq. (4.18). It is straighforward to compute this commutator. We find

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i g_s [\hat{A}_\mu, \hat{A}_\nu]. \tag{4.26}$$

Since the gauge field belongs to the Lie algebra of the group $SU(2)$, we can write

$$[\hat{A}_\mu, \hat{A}_\nu] = i\, f^{abc} A^a_\mu A^b_\nu \tau^c, \tag{4.27}$$

so that $\hat{F}_{\mu\nu} = \tau^a F^{(a)}_{\mu\nu}$ where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu. \tag{4.28}$$

We will use $\hat{F}_{\mu\nu}$ to write the kinetic term as

$$L_{\text{kin}} = -\frac{1}{2} \text{Tr}\left[\hat{F}_{\mu\nu}\, \hat{F}^{\mu\nu}\right]. \tag{4.29}$$

The coefficient is chosen in such a way that in the limit $g_s \to 0$, we recover a kinetic term for each of the $A^{(a)}_\mu$ fields.[1]

[1] This assumes the standard normalization of the generators $\text{Tr}[\tau^a \tau^b] = 1/2\delta^{ab}$.

To show that L_{kin} is invariant under the gauge transformations, we need to understand how either $F_{\mu\nu}$ or the covariant derivatives transform. This can be easily done. Indeed, we start with the covariant derivative and change the gauge field; then we find

$$\begin{aligned} D_\mu &= \partial_\mu - i g_s \hat{A}_\mu = \partial_\mu - i g_s \left(U \hat{A}_{1,\mu} U^+ - \frac{i}{g_s} (\partial_\mu U) U^+ \right) \\ &= U \left(\partial_\mu - i g_s \hat{A}_{1,\mu} \right) U^+ = U D_{1,\mu} U^+. \end{aligned} \tag{4.30}$$

It follows that the field-strength tensor transforms in the same way

$$F_{\mu\nu} \to U \hat{F}_{\mu\nu} U^+, \tag{4.31}$$

and this makes the kinetic energy part of the Lagrangian invariant under gauge transformations.

It is to be noted that L_{kin} *alone* defines a highly non-trivial theory in a non-abelian case. This is because L_{kin} contains both triple and quartic terms of the form $g_s (\partial_\mu A_\nu^a) f^{abc} A_\mu^b A_\nu^c$ and $g_s^2 f^{abc} A_\mu^b A_\nu^c f^{ab_1c_1} A_\mu^{b_1} A_\nu^{c_1}$. These interaction terms imply that in non-abelian gauge theories gauge fields carry charges and can interact with each other. These interactions have very important consequences for physics. One of the most important consequences is the so-called *confinement* of these non-abelian charges which means that particles with non-abelian charges cannot be observed as free particles in Nature. Although there is clear empirical evidence that supports this statement, no formal mathematical proof is currently known.

The most important non-abelian physical theories are Quantum Chromodynamics (QCD) and the electroweak Standard Model. The Standard Model is somewhat special because gauge symmetry in that case is broken by the Higgs mechanism. Quantum Chromodynamics describes the strong force that keeps nuclei together and, thus, is responsible for interactions between protons, neutrons and pions, among other thing. However, the Lagrangian of QCD is a Lagrangian of a non-abelian theory with gauge group $SU(3)$; it is formulated using the language of quarks and gluons. Gluons are the "photons" of the strong field. Similar to quarks, they are not observable. We will continue to talk about QCD in the following lectures.

The discussion of the $SU(2)$ theory that we had so far involved fermions in the *fundamental* representation of the gauge group. However, since there are infinitely many representations of the gauge group, there is more than one way to construct a non-abelian theory even if the gauge group is fixed. Consider, for example, a scalar field ϕ that transforms under the so-called *adjoint* representation of $SU(2)$. We write

$$\hat{\phi} = \sum_{a=1}^{3} \phi^{(a)} \tau^a, \tag{4.32}$$

so that similar to the gauge field, ϕ belongs to the $SU(2)$ algebra. The transformation rule for the field ϕ is

$$\hat{\phi} \to U(x) \hat{\phi} U^{+}(x). \tag{4.33}$$

The kinetic term that involves the Lagrangian of the field ϕ reads

$$L = \mathrm{Tr}[\partial_\mu \phi \, \partial^\mu \phi] = \sum_{a=1}^{3} \frac{1}{2} \partial_\mu \phi^{(a)} \partial^\mu \phi^{(a)}. \tag{4.34}$$

Clearly, the Lagrangian in Eq. (4.34) is not invariant under local gauge transformations; to ensure that, we need to introduce a covariant derivative. We then write

$$\partial_\mu \hat{\phi} \to [D_\mu, \hat{\phi}]. \tag{4.35}$$

Since both D_μ and $\hat{\phi}$ transform in the same way, we easily find a transformation rule for the covariant derivative acting on the field $\hat{\phi}$

$$[D_\mu, \hat{\phi}] \to U [D_\mu, \hat{\phi}] U^{+}. \tag{4.36}$$

Hence, if we write the kinetic term of the Lagrangian as

$$L = \mathrm{Tr}\left[\left[D_\mu, \hat{\phi} \right] [D_\mu, \hat{\phi}] \right], \tag{4.37}$$

it will be invariant under gauge transformations. We note that similar to all other quantities that we considered in this lecture, the "kinetic" term in Eq. (4.37) contains interactions terms that define the coupling between the gauge fields and the scalar fields.

The first time gauge transformations appear in physics courses is when the covariant formulation of electrodynamics is introduced. There we express the electric and magnetic fields through the generalized vector potential A^μ, and say that Maxwell's equations do not change if we replace A^μ with $A^\mu + \partial^\mu f(x)$, where $f(x)$ is an atrbitrary function. We then argue that this freedom allows us to impose certain conditions on the vector potential or, as we say, *fix the gauge*.

There are different ways to do that; on of the most famous ones is to require that

$$\partial_\mu A^\mu = 0, \tag{4.38}$$

which is the Lorentz gauge condition. To see why Eq. (4.38) is possible, imagine that we have a generic vector potential A_1^μ. We then redefine the vector potential by writing

$$A_1^\mu = A^\mu + \partial^\mu f. \tag{4.39}$$

We compute the divergence of both parts of the above equation and choose f to be

$$f(x) = -\frac{1}{\partial^2}\,\partial_\mu A_1^\mu. \tag{4.40}$$

With this choice, the vector potential A^μ satisfies the Lorentz condition.

Another popular choice is the so-called Coulomb or radiation gauge

$$\vec{\nabla}\cdot\vec{A} = 0. \tag{4.41}$$

In this case, function $f(x)$ is computed in the same way as in the case of the Lorentz gauge but we calculate a three- rather than four-dimensional divergence.

Let us clarify the meaning of these gauge fixing equations. When we say that we fix the gauge, we mean that there are no two vector potentials A_1^μ and A_2^μ that satisfy the gauge fixing condition (say the Coulomb one)

$$\vec{\nabla}\cdot\vec{A}_{1,2} = 0, \tag{4.42}$$

and, at the same time, are related to each other by a gauge transformation, i.e.

$$A_2^\mu = A_1^\mu + \partial^\mu f(x) \tag{4.43}$$

To see this, we compute the divergence of both sides of the above equation and find

$$0 = \nabla^2 f(x). \tag{4.44}$$

This is the Laplace equation. We need a solution that vanishes (or at least is not singular) at infinity. This implies that $f(x) = 0$.

We now discuss to what extent the same gauge-fixing procedure can be carried through in non-abelian theories. To this end we impose the analog of the Coulomb gauge

$$\vec{\nabla} \cdot \vec{A}^{(a)} = 0, \tag{4.45}$$

on the non-abelian vector potential. The simplest "gauge configuration" where this condition is satisfied is the "vacuum" configuration, with $A^a_\mu = 0$. Obviously, this choice of the vector potential corresponds to $\hat{F}_{\mu\nu} = 0$; for this value of $\hat{F}_{\mu\nu}$, the Lagrangian and the action S vanish.

We would like to investigate whether there are other time-independent field configurations that are a) gauge transformations of $A^a_\mu = 0$ and b) satisfy Eq. (4.45). A gauge transformation of $A_\mu = 0$ is

$$\hat{A}_\mu = \frac{i}{g_s} U^+(x)\partial_\mu U(x). \tag{4.46}$$

We would like to consider time-independent field-configurations, so that $U(x)$ does not depend on x_0. Then, $A_0 = 0$ and the above equation simplifies to

$$\hat{A}_i = \frac{i}{g_s} U^+ \partial_i U. \tag{4.47}$$

We are interested in understanding whether or not the equation

$$0 = \partial_i \hat{A}_i \to \partial_i \left[U^+ \partial_i U \right] = 0, \tag{4.48}$$

has solutions that are not infinite at $|\vec{x}| = \infty$ and anywhere else. In general,

$$U(x) = e^{i\theta(x)\vec{n}^a \tau^a}. \tag{4.49}$$

For the group $SU(2)$ we can take $\theta(x) = f(r)$ and $\vec{n} = \vec{r}/r$, since there are three generators of the $SU(2)$ group. Hence, we mapped a sphere in the coordinate space onto a sphere in the $SU(2)$ space. It is straightforward to compute $\hat{A}_i$ by taking the derivatives. We find

$$\begin{aligned}\hat{A}_i = -\frac{1}{g_s}\Big[&\vec{n}_i f'(r)\vec{n}\cdot\vec{\tau} + \frac{\sin(f(r))}{r}\left(\tau_i - \vec{n}_i(\vec{n}\cdot\vec{\tau})\right) \\ &+ \frac{1-\cos(f(r))}{r}\epsilon_{ibc}\tau_b\vec{n}_c\Big].\end{aligned} \tag{4.50}$$

To ensure that A_i satisfies the gauge condition, we compute the divergence of A_i and requiring that it vanishes; we find that it is possible to do so provided that the function $f(r)$ satisfies the following equation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) - 2\sin[f] = 0. \tag{4.51}$$

To solve this equation, we need boundary conditions. Since we would like the vector potential to be non-singular everywhere, we require $f(0) = 0$. It is then possible to construct the series solution of the above equation at $r = 0$. We find

$$f(r) = f'r - \frac{1}{30}(f'r)^3 + \frac{1}{560}(f'r)^5 + ..., \tag{4.52}$$

where $f' = \mathrm{d}f(r)/\mathrm{d}r$ at $r = 0$. It is clear from the solution that f' controls the spatial extend of the solution.

To understand how this solution extends to $r \to \infty$, it is convenient to change variables and write $r = e^s$. Since $r \in [0, \infty]$, then $s \in [-\infty, \infty]$. To write an equation using the s-coordinate, we note that

$$\frac{\mathrm{d}}{\mathrm{d}r} = e^{-s}\frac{\mathrm{d}}{\mathrm{d}s}, \tag{4.53}$$

and find

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) = e^{-s}\frac{\mathrm{d}}{\mathrm{d}s}e^s\frac{\mathrm{d}f}{\mathrm{d}s} = \frac{\mathrm{d}^2 f(s)}{\mathrm{d}s^2} + \frac{\mathrm{d}f}{\mathrm{d}s}. \tag{4.54}$$

The differential equation then becomes

$$\frac{\mathrm{d}^2 f}{\mathrm{d}s^2} + \frac{\mathrm{d}f}{\mathrm{d}s} - 2sin[f] = 0. \tag{4.55}$$

If we interpret s as time and f as a dispacement, the above equation describes a motion of particle with the mass $m = 1$ in the potential $V(f) = 2\cos f$ subject to a friction force.

The small-r solution that we have found implies that the at $s = -\infty$ the particle starts at a maximum of the potential ($f = 0$) and moves either to the left and to the right, dependning on the sign of f'. Because of the friction, the particle never makes it to the next maximum (say $f = 2\pi$) and starts moving back instead. After a few oscillations, the particle looses all its energy and finally ends up at the minimum of the potential at $f = \pi$. Hence, $f \to \pi$ at $r \to \infty$.

One can construct an approximate solutions for $f(r)$ at $r \to \infty$. Using the explicit solutions and Eq. (4.50) one finds that at large values of r the vector potential becomes

$$\lim_{r\to\infty} A_i \to \frac{2}{g_s r}\epsilon_{inc}\vec{n}_b\tau_c. \tag{4.56}$$

This analysis shows that the gauge fixing in non-Abelian gauge theories is not complete, at least for some gauge fixing choices. However, fields that escape the gauge fixing conditions are inversly proportional to the coupling constant i.e $A_i \sim 1/g_s$; so if the gauge coupling is small, then these fields are very large.

In the next lectures we will study how to perform perturbative quantization of QCD. An essential part of this procedure will be the gauge fixing which will allow us to remove equivalent configurations from the so-called path integral. Although, as we just discussed, the gauge-fixing is not complete in non-Abelian theories, for the purpose of perturbative expansion we deal with fields which are close to $A_\mu = 0$ only and for such fields the gauge fixing procedure is valid.