# TTP2
# Lecture 6

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# 6 Path integral in field theory

In the previous lecture, we talked about path integral formulation of Quantum Mechanics. We will now show how a small modification of that discussion will give as a description of quantum field theory. We consider a chain of identical masses connected to each other by identical springs. The Hamiltonian reads

$$H = \sum_{a=1}^{N} \left[ \frac{p_a^2}{2m} + \frac{k}{2}(q_a - q_{a-1})^2 \right]. \tag{6.1}$$

We assume that the equilibrium distance between two neighbors is $l$ and $q_a$ describes a displacement of a particle $a$ from its equilibrium position. We will identify $q_0$ with zero.

The computation of the time evolution operator has to be modified but only slightly. Indeed, the position operator is now a vector $\vec{q} = (q_1, ..., q_N)$. We are interested in a transition from an eigenstate of the operator $\vec{q}$, that we denote as $\vec{x}_i$, at $t = t_i$ to an eigenstate of the operator $\vec{q}$ that we denote as $\vec{x}_f$, at $t = t_f$. Specifically,

$$\vec{q}|\vec{x}_{i,f}\rangle = \vec{x}_{i,f}|\vec{x}_{i,f}\rangle. \tag{6.2}$$

We then write the matrix element of the evolution operator as

$$U(\vec{x}_f, \vec{x}_i; t_f, t_i) = \langle \vec{x}_f | e^{-iH(t_f - t_i)/\hbar} | \vec{x}_i \rangle. \tag{6.3}$$

To re-write the quantity $U(\vec{x}_f, \vec{x}_i; t_f, t_i)$ we proceed in exactly the same way as in the previous lecture, i.e. we split the time interval into segments, insert identity operators into strategic places and replace them with integrals over coordinates using completeness relations

$$1 = \int \mathrm{d}\vec{x}\; |\vec{x}\rangle\langle\vec{x}|. \tag{6.4}$$

It is then obvious that when all is said and done, we obtain

$$U(\vec{x}_f, \vec{x}_i; t_f, t_i) = \langle \vec{x}_f | e^{-iH(t_f - t_i)/\hbar} | \vec{x}_i \rangle = \int [\mathcal{D}\vec{x}(t)] e^{iS/\hbar}|_{\vec{x}(t_i)=\vec{x}_i, \vec{x}(t_f)=\vec{x}_f}. \tag{6.5}$$

The action reads

$$S = \int_{t_i}^{t_f} \mathrm{d}\tau L(\vec{x}(\tau), \dot{\vec{x}}(\tau)) = \int_{t_i}^{t_f} \mathrm{d}\tau \sum_{a=1}^{N} \left[ \frac{m\dot{x}_a^2}{2} - \frac{k(x_a - x_{a-1})^2}{2} \right]. \tag{6.6}$$

We would like to take the limit $N \to \infty$, $l \to 0$, keeping $lN = L$ fixed. We then write

$$x_a(t) = \varphi(t, \xi), \tag{6.7}$$

where $\xi = al$ is the equilibrium position of a particle "$a$" along the chain. We replace the sum with the integral

$$\sum_{a=1}^{N} \to \int_0^L \frac{\mathrm{d}\xi}{l}, \tag{6.8}$$

and write

$$\begin{aligned} &\lim_{l\to 0} \sum_{a=1}^{N} \left[ \frac{m\dot{x}_a^2}{2} - \frac{k(x_a - x_{a-1})^2}{2} \right] \\ &= \lim_{l\to 0} \int_l^L \frac{\mathrm{d}\xi}{l} \left[ \frac{m}{2} \left( \frac{\partial\varphi(t,\xi)}{\partial t} \right)^2 - \frac{k}{2} \left( \varphi(t,\xi) - \varphi(t,\xi - l) \right)^2 \right]. \end{aligned} \tag{6.9}$$

The last term on the right hand side reads

$$\lim_{l\to 0} \left( \varphi(t,\xi) - \varphi(t,\xi - l) \right) \to l\, \frac{\partial\varphi(t,\xi)}{\partial\xi}. \tag{6.10}$$

We obtain

$$S = \int_{t_i}^{t_f} \mathrm{d}\tau \int_0^L \mathrm{d}\xi \left[ \frac{m}{2l} \left( \frac{\partial\varphi(t,\xi)}{\partial t} \right)^2 - \frac{kl}{2} \left( \frac{\partial\varphi(t,\xi)}{\partial\xi} \right)^2 \right]. \tag{6.11}$$

We remove the prefactor in a term with time derivatives by redefining $\varphi$

$$\varphi \to \sqrt{\frac{l}{m}}\varphi \tag{6.12}$$

and find

$$S = \int_{t_i}^{t_f} \mathrm{d}\tau \int_0^L \mathrm{d}\xi \left[ \frac{1}{2} \left( \frac{\partial\varphi(t,\xi)}{\partial t} \right)^2 - \frac{kl^2}{2m} \left( \frac{\partial\varphi(t,\xi)}{\partial\xi} \right)^2 \right]. \tag{6.13}$$

The combination of parameters $kl^2/m$ has the dimension of velocity squared; we denote it as $kl^2/m = c^2$. Taking in addition the $L \to \infty$ limit, we find

$$S = \int_{t_i}^{t_f} \mathrm{d}\tau \int \mathrm{d}\xi \left[ \frac{1}{2} \left( \frac{\partial \varphi(t,\xi)}{\partial t} \right)^2 - \frac{c^2}{2} \left( \frac{\partial \varphi(t,\xi)}{\partial \xi} \right)^2 \right]. \tag{6.14}$$

We see that in the continuous limit, our system of oscillators is described by the "field" $\varphi(t,\xi)$; this field parameterizes a displacement of a particle at the point $\xi$ and at time $t$ from its equilibrium position. The quantum transition amplitude from a quantum state with the definite value of the filed $\varphi_i$ at at $t = t_i$ to a quantum field with the definite value of the field $\varphi = \varphi_f$ at $t = t_f$ is given by a path integral

$$\langle \varphi_f(\xi), t_f | \varphi_i(\xi), t_i \rangle = U(\varphi_f, \varphi_i; t_f, t_i) = \int [\mathcal{D}\varphi] \; e^{\frac{iS}{\hbar}} |_{\varphi(t_{f_i}) = \varphi_{i,f}}, \tag{6.15}$$

where the integration goes over all fields with the following boundary condition $\varphi(t_{f,i}, x) = \varphi_{f,i}(x)$. Note that Eq.(6.15) implies that in our quantum theory the field $\varphi(\xi)$ is a quantum operator, just like a position operator $\vec{q} = (q_1, ...., q_N)$ in $N$-body quantum mechanics used to be. Since we went from the latter to the former by taking the $N \to \infty$ limit, quantum field theory is quantum mechanics with infinitely many degrees of freedom.

It is clear that the above discussion generalizes to a four-dimensional Minkowski space. In particular, for a scalar field theory defined by the action

$$S = \int \mathrm{d}^4 x \, \left( \frac{1}{2} \partial_\mu \varphi \partial^\mu \, \varphi - \frac{m^2}{2} \varphi^2 + V(\varphi) \right) \tag{6.16}$$

the vacuum-to-vacuum transition in the presence of a source $J(x)$ is given by[1]

$$\langle 0 | 0 \rangle_J = \int [\mathcal{D}\varphi] e^{iS[\varphi] + i \int \mathrm{d}^4 x J(x) \varphi(x)}. \tag{6.17}$$

The Green's functions in that theory are then obtained by taking functional derivatives w.r.t. the source $J(x)$.

There are a few things that are worth discussing in connection with the path integral. For example, let us compute the propagator in a Quantum Field

[1] From now on, we will again use relativistic units, so that $\hbar = 1$ and $c = 1$.

Theory of a free scalar field. First, we take the $J \to 0$ limit in Eq. (6.17). Since $\langle 0|0\rangle = 1$, we find

$$1 = \int [\mathcal{D}\varphi e^{iS[\varphi]}. \tag{6.18}$$

To make sure this is ensured automatically, we rewrite Eq. (6.17) as follows

$$\langle 0|0\rangle_J = \frac{Z[j]}{Z[0]}, \tag{6.19}$$

where

$$Z[j] = \int [\mathcal{D}\varphi]\; e^{iS[\varphi]+i\int \mathrm{d}^4x J(x)\varphi(x)}. \tag{6.20}$$

Next, consider the free theory, i.e. $V(\varphi) = 0$. To compute $Z[j]$, we can do the following. First, we rewrite the action $S$ by integrating by parts and discarding contributions at infinity. We find

$$S[\varphi] + \int \mathrm{d}^4x J\varphi = \int \mathrm{d}^4x\; \left[-\frac{1}{2}\varphi(\Box + m^2)\varphi + J\varphi\right] \tag{6.21}$$

Let us change integration variables in the path integral, i.e. instead of integrating over the field $\varphi$, we will integrate over the field $\xi$ defined as

$$\varphi = \xi + (\Box + m^2)^{-1}J. \tag{6.22}$$

Clearly, $[\mathcal{D}\varphi] = [\mathcal{D}\xi]$ and

$$-\frac{1}{2}\varphi(\Box + m^2)\varphi + J\varphi = -\frac{1}{2}\xi(\Box + m^2)\xi + \frac{1}{2}J(\Box + m^2)^{-1}J. \tag{6.23}$$

Hence, we find

$$Z[j] = \int [\mathcal{D}\xi]\; e^{iS[\xi]}\; e^{i\int\; \frac{1}{2}J(\Box+m^2)^{-1}J} = Z[0]\; e^{i\int\; \frac{1}{2}J(\Box+m^2)^{-1}J}. \tag{6.24}$$

It follows that

$$\langle 0|0\rangle_J = e^{i\int \frac{1}{2}J(\Box+m^2)^{-1}J}. \tag{6.25}$$

We are now in position to compute the propagator of a scalar particle in a free field theory. From the discussion about Quantum Mechanics, we know that

$$\langle 0|T\phi(x)\phi(y)|0\rangle = \frac{\delta^2}{i^2\delta J(x)\delta J(y)}\langle 0|0\rangle_J\Big|_{J=0}. \tag{6.26}$$

A simple computation gives

$$\langle 0|T\phi(x)\phi(y)|0\rangle = G(x,y), \tag{6.27}$$

where

$$G(x,y) = -i(\Box + m^2)^{-1}(x,y). \tag{6.28}$$

The function $G(x,y)$ satisfies the following equation

$$(\Box + m^2)G(x,y) = -i\delta(x-y). \tag{6.29}$$

To find the function $G(x,y)$ we use the momentum-space representation and obtain the usual expression for the propagator

$$G(x,y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{i}{p^2 - m^2}\; e^{-ip(x-y)}. \tag{6.30}$$

Note that in this set up the $i0$ prescription that we need to introduce to recover the true Feynman propagator can be thought of as an addition of a small imaginary part to the mass $m^2 \to m^2 - i0$ and the sign of this term ensures the *convergence* of the functional integral by damping the exponential.

Scalar fields are not the only fields that we need to construct realistic theories of Nature; we also require vector bosons (spin-one particles) and fermions (spin one half). A naive extension of the above formalism to vector bosons is, in principle, straightforward; all we need to do is to integrate over *four* fields $A^\mu$, $\mu = 0,1,2,3$ at each point. If one looks at this problem more carefully, one uncovers subtleties similar to what we saw when we discussed quantization of QED last semester. However, for fermions, problems arise at a much earlier stage because proper quantization of fermion fields requires us to declare them to be *anticommuting* operators. Obviously if we write

$$\langle 0|T\psi(x)\bar{\psi}(y)|0\rangle \int [\mathcal{D}\bar{\psi}][\mathcal{D}\psi]\; \psi(x)\bar{\psi}(y)e^{iS_D}, \tag{6.31}$$

and assume that $\psi$ and $\bar{\psi}$ in the path integral are classical functions, this anticommutativity property seems impossible to realize. To see how this can be done we need to introduce the so-called *Grassmann numbers*.

Ordinary numbers or $c$-numbers commute. That is, if $x$ and $y$ are two $c$-numbers, then

$$xy = yx. \tag{6.32}$$

Grassmann number *anticommute*. If $\theta_1$ and $\theta_2$ are two Grassmann numbers, then

$$\theta_1\theta_2 = -\theta_2\theta_1. \tag{6.33}$$

This assumption has many consequences which we will now explore. The first one is that a square of any Grassmann number is zero

$$\theta^2 = 0. \tag{6.34}$$

Grassmann numbers can be added or multiplied by a constant, so this is no different in comparison with ordinary numbers.

The anticommutativity makes functions of Grassmann numbers almost trivial. Indeed, any function of a Grassmann number $f(\theta)$, that can be Taylor-expanded, reads

$$f(\theta) = A + B\theta. \tag{6.35}$$

Any function of two Grassmann numbers, that I will denote as $\theta$ and $\bar{\theta}$ contains just four terms

$$f(\theta, \bar{\theta}) = A + B\theta + C\bar{\theta} + D\bar{\theta}\theta. \tag{6.36}$$

The reason we talk about Grassmann variables is that we would like to use them to describe fermionic fields in the path integral. This means that we need to understand how to perform analogs of definite integrals over commuting variables. All the integrals over Grassmann variables that we might be interested in follow from three rules

1. Linearity

$$\int \mathrm{d}\eta \left(Af_1(\eta) + Bf_2(\eta)\right) = \pm A \int \mathrm{d}\eta f_1(\eta) \pm B \int \mathrm{d}\eta f_2(\eta), \tag{6.37}$$

   where $\pm$ depends on whether $A$ and $B$ are Grassmann or $c$-variables. Note that the integration measure $\mathrm{d}\eta$ behaves as a Grassmann variable in the context of commuting or anti-commuting with other variables.

2. Translation invariance

$$\int \mathrm{d}\eta f(\eta) = \int \mathrm{d}\eta \; f(\eta + \theta). \tag{6.38}$$

3. Normalization

$$\int \mathrm{d}\eta \, \mathrm{d}\bar{\eta} \; e^{\bar{\eta}\eta} = 1. \tag{6.39}$$

We now show how to use these requirements to integrate *any* function of Grassmann variables. We will start with one variable. Then, using translational invariance, we find

$$\int \mathrm{d}\eta \, (A + B\eta) = \int \mathrm{d}\eta \, (A + B\eta + B\xi) \, . \tag{6.40}$$

Using linearity, we find

$$B\xi \int \mathrm{d}\eta = 0. \tag{6.41}$$

Since $B$ and $\xi$ are arbitrary, this implies that

$$\int \mathrm{d}\eta = 0. \tag{6.42}$$

From the normalization condition, we find

$$1 = \int \mathrm{d}\eta \, \mathrm{d}\bar{\eta} \, (1 + \bar{\eta}\eta) = \int \mathrm{d}\eta\eta \int \mathrm{d}\bar{\eta}\bar{\eta}. \tag{6.43}$$

We then choose

$$\int \mathrm{d}\eta\eta = \int \mathrm{d}\bar{\eta}\bar{\eta} \; = 1, \tag{6.44}$$

and these rules are sufficient to integrate any function over Grassmann variables.

In particular, consider the following integral

$$I_n(A) = \int \mathrm{d}\eta_1, , , \mathrm{d}\eta_n \, \mathrm{d}\bar{\eta}_1, , , \mathrm{d}\bar{\eta}_n e^{\bar{\eta}_i A_{ij} \eta_j}, \tag{6.45}$$

over $2n$ Grassmann variables $\bar{\eta}_{1,..,N}$ and $\eta_{1,,N}$. The matrix $A_{ij}$ is assumed to be Hermitian and can be diagonalized by a unitary transformation. This integral equals to

$$I_n(A) = \det[A]. \tag{6.46}$$

To prove this result, we change the integration variables using the eigenvectors of the matrix $A$. Writing

$$\eta_i = \sum_{k=1}^{n} c_k v_i^{(k)}, \tag{6.47}$$

where $\{c_k\}$ are the new Grassman variables and $v_i^{(k)}$ is the $i$-th component of the eigenvector $v^{(k)}$ of the matrix $A$. These vectors satisfy the following equations

$$A_{ij} v_j^{(k)} = \lambda_k v_i^{(k)}, \quad k \in \{1...N\}. \tag{6.48}$$

Then[2]

$$I_n(A) = \int \mathrm{d}c_1,,,\mathrm{d}c_n\,\mathrm{d}\bar{c}_1,,,\mathrm{d}\bar{c}_n e^{\sum\limits_{k=1}^{N} \lambda_k \bar{c}_k c_k}. \tag{6.49}$$

There is only one term in the expansion of the exponential function that does not integrate to zero – namely, the one where each variable $c_k$ and each variable $\bar{c}_k$ appear exactly once. Then

$$I_n(A) = \prod \lambda_k = \det A. \tag{6.50}$$

The last thing we need, in addition to the integration, are the rules for taking derivatives with respect to Grassmann variables. The first rule is that these derivatives anticommute, i.e.

$$\frac{\partial^2}{\partial\theta\partial\xi} = -\frac{\partial^2}{\partial\xi\partial\eta}. \tag{6.51}$$

The second rule is that

$$\frac{\partial}{\partial\theta_i}\theta_j = \delta_{ij}. \tag{6.52}$$

We will now use the Grassmann variables to illustrate the quantization of the Dirac field. We write

$$Z[\bar{\eta}, \eta] = \int [\mathcal{D}\psi][\mathcal{D}\bar{\psi}] e^{iS[\bar{\psi},\psi] + i\int \mathrm{d}^4x(\bar{\eta}\psi + \bar{\psi}\eta)}, \tag{6.53}$$

where $\psi, \bar{\psi}, \eta, \bar{\eta}$ are four-component Grassmann variables and

$$S = \int \mathrm{d}^4x\, \bar{\psi} A \psi, \tag{6.54}$$

where

$$A = i\gamma_\mu \partial^\mu - m, \tag{6.55}$$

[2]One has to prove that the measure is invariant under if the unitary transformation of integration variables is performed.

is the Dirac operator. The Green's functions are computed by calculating $\eta$ and $\bar{\eta}$ derivatives of

$$\langle 0|0\rangle_{\bar{\eta},\eta} = \frac{Z[\bar{\eta},\eta]}{Z[0,0]}. \qquad (6.56)$$

To write $\langle 0|0\rangle_{\bar{\eta},\eta}$ in a useful form, we shift the integration variables as follows

$$\psi = \xi - A^{-1}\eta, \quad \bar{\psi} = \bar{\xi} - \bar{\eta}A^{-1}. \qquad (6.57)$$

Then,

$$S[\bar{\psi},\psi] + \int \mathrm{d}^4x(\bar{\eta}\psi + \bar{\psi}\eta) = S[\bar{\xi},\xi] - \int \mathrm{d}^4x\; \bar{\eta}\; A^{-1}\; \eta, \qquad (6.58)$$

and

$$\langle 0|0\rangle_{\bar{\eta},\eta} = e^{-i\int \mathrm{d}^4x\; \bar{\eta}\; A^{-1}\; \eta}. \qquad (6.59)$$

The Green's functions of the free Dirac field are then obtained by taking the derivatives w.r.t. $\eta$ and $\bar{\eta}$. The anti-commuting property of the field operators is ensured because derivatives w.r.t. $\eta$ and $\bar{\eta}$ anticommute.