TTP2
Lecture 7

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7 Path integral and gauge invariance: the QED case

We have seen that integrating over all values of classical fields allows us to compute Green's functions in Quantum Field Theory. Consider now Quantum Electrodynamics. We will not consider fermion fields since they are not needed for this discussion. Hence, in order to compute Green's functions in QED, we will need to study a path integral of the form

$$Z[J] = \int \mathcal{D}A_\mu \; e^{iS[A] + \int \mathrm{d}^4x \; J_\mu A^\mu}, \tag{7.1}$$

where $\mathcal{D}A_\mu$ means that we have to integrate over four components of the field A_μ at each space-time point and

$$S[A] = \int \mathrm{d}^4x \; L, \tag{7.2}$$

with

$$L = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}. \tag{7.3}$$

The quantity N is the normalization factor. It plays no role in the computation of Green's functions. The current J_μ is supposed to be convserved.

The path integral is clearly not the most mathematically transparent quantity in general but it is easy to see that $Z[J]$, as defined above, is especially problematic. The reason it is problematic is related to *gauge invariance*. Indeed, thanks to gauge invariance, for each value of the field $A^\mu(x)$, there are infinitely many fields related to A_μ by gauge transformations which do not change the action $S[A]$. $Z[J]$ is (very!) infinite and no further computations are possible.

To make Z somewhat more sensible, one can argue that the integration in Eq. (7.1) should only include fields that are unrelated by gauge transformations. Hence, as the first step, we would like to rewrite the path integral in such a way that integration over fields that are *not related by gauge transformations and fields that are gauge-equivalent are separated*.

To see how this can be done, suppose that we choose the generalized Lorentz gauge, i.e. we write

$$\partial_\mu A^\mu(x) = f(x), \tag{7.4}$$

where $f(x)$ is an arbitrary function. For each point x, we can write

$$A^\mu = A^\mu_\perp + \partial^{-2}\,\partial^\mu f(x) \tag{7.5}$$

where

$$A^\mu_\perp = (g^{\mu\nu} - \partial^{-2}\partial^\mu\partial^\nu)A_\nu. \tag{7.6}$$

The field $A^\mu_\perp$ has three independent components to integrate over and the longitudinal component of the field A^μ is completely determined by the gauge-fixing condition and does not need to be integrated over. Hence, what we would like to do is to restrict the integration measure in the path integral to $A_{\perp,\mu}$, effectively.

We will now describe a smart and very general way to do that which is also generalizable to non-abelian gauge theories. We will consider a general gauge condition

$$G[A] = 0. \tag{7.7}$$

The quantity G can be a function of A_μ and also include differential operators. Now, for each A_μ that satisfies the above equation, we consider other fields related to it by a gauge transformation,

$$A^\chi_\mu = A_\mu + \partial_\mu\chi. \tag{7.8}$$

We then write unity in a complicated way, by integrating over all possible functions χ

$$1 = \int \mathcal{D}\chi\ \ \delta(G[A^\chi])\ \det\left(\frac{\mathrm{d}G[A^\chi]}{\mathrm{d}\chi}\right). \tag{7.9}$$

To understand this formula, consider the following n-dimensional integral

$$I = \int \mathrm{d}^n\vec{a}\,\delta^{(n)}(\vec{g}(\vec{a})). \tag{7.10}$$

Suppose that $\vec{g}(\vec{a}_0) = 0$. Then, to compute I, we write $\vec{a} = \vec{a}_0 + \vec{\xi}$ and obtain

$$I = \int \mathrm{d}^n\vec{\xi}\,\delta^{(n)}\left(\hat{A}\vec{\xi}\right), \tag{7.11}$$

where $\hat{A}_{ij} = \partial g_i/\partial a_j$, computed at the point $\vec{a} = \vec{a}_0$. Next, we write

$$\delta^{(n)}\left(\hat{A}\vec{\xi}\right) = \int \frac{\mathrm{d}^n\vec{r}}{(2\pi)^n} e^{i\,\vec{r}\hat{A}\vec{\xi}}, \tag{7.12}$$

and find

$$I = \int \frac{\mathrm{d}^n\vec{\xi}\,\mathrm{d}^n\vec{r}}{(2\pi)^n}\; e^{i\,\vec{r}\,\hat{A}\,\vec{\xi}}. \tag{7.13}$$

To finalize this integration, we assume that the matrix $\vec{A}$ has an orthonormal set of n eigenvectors, that we use to define a reference frame for both vectors $\vec{\xi}$ and $\vec{r}$. We then find

$$I = \prod \lambda_m^{-1} = \frac{1}{\det[\hat{A}]}. \tag{7.14}$$

Hence, if we want the integral to be equal to one, we have to write

$$1 = \int \mathrm{d}^n\vec{a}\,\delta^{(n)}(\vec{g}(\vec{a}))\det\left[\frac{\partial g_i}{\partial a_j}\right]. \tag{7.15}$$

Eq. (7.9) is a generalization of the above equation to the case of an infinitely-dimensional vector space.

We now insert Eq. (7.9) into the functional integral in Eq. (7.1). We find

$$Z = \int \mathcal{D}A_\mu\,\mathcal{D}\chi\;\;\delta(G[A^\chi])\det\left(\frac{\mathrm{d}G[A^\chi]}{\mathrm{d}\chi}\right)e^{iS[A]}. \tag{7.16}$$

Next, we note that

$$A^\chi_\mu = A_\mu + \partial_\mu\chi. \tag{7.17}$$

Hence, we can replace the integration over A_μ with the integration over A^χ_μ ($\mathcal{D}A_\mu = \mathcal{D}A^\chi_\mu$). Furthermore, since S is gauge-invariant, the following equation holds $S[A^\chi_\mu] = S[A_\mu]$. Combing all these equations, we find

$$Z = \int \mathcal{D}\chi\;\int \mathcal{D}A^\chi_\mu\;\;\delta(G[A^\chi])\det\left(\frac{\mathrm{d}G[A^\chi]}{\mathrm{d}\chi}\right)e^{iS[A^\chi]}, \tag{7.18}$$

which shows that integration over all possible functions χ is separated from the integration over fields A^μ none of which are gauge-equaivalent.

Next, to be specific, we go back to fixing the gauge by using the Lorentz gauge condition

$$G[A] = \partial_\mu A^\mu - f(x). \tag{7.19}$$

Then,

$$G[A^\chi] = \partial_\mu A^\mu + \partial^2\chi - f(x), \tag{7.20}$$

so that

$$\det\left(\frac{\mathrm{d}G[A^\chi]}{\mathrm{d}\chi}\right) = \det\left(\partial^2\right). \tag{7.21}$$

Since ∂^2 is independent of A_μ, we can take it outside the integration over A^χ. Then, renaming $A^\chi \to A$, we obtain

$$Z = \int \mathcal{D}\chi \det\left(\partial^2\right) \int \mathcal{D}A_\mu\, \delta(\partial_\mu A^\mu - f(x))\, e^{iS[A]}. \tag{7.22}$$

The integration over χ and $\det(\partial^2)$ can be included into a normalization factor $\mathcal{N}$. The remaining integral is independent of the function $f(x)$ (by construction, the path integral is independent of the gauge choice) and the δ-function selects fields that cannot be related to each other by the gauge transformation.

There is an additional useful trick that we can use. Consider "reduced" functional integral

$$Z = \int \mathcal{D}A_\mu\, \delta(\partial_\mu A^\mu - f(x)) e^{iS[A]}. \tag{7.23}$$

Since this integral is independent of $f(x)$, we can integrate over all possible functions f with an arbitrary weight

$$Z = \int \mathcal{D}f\ e^{-i\int \mathrm{d}^4x\ \frac{f(x)^2}{2\xi}} \int \mathcal{D}A_\mu\, \delta(\partial_\mu A^\mu - f(x))\, e^{iS[A]}. \tag{7.24}$$

We can now interchange the order of integration and remove the function $f(x)$ completely, since $f = \partial_\mu A^\mu$. Hence, our final result for Z reads

$$Z = \int \mathcal{D}A_\mu\ e^{iS_{QED}[A]}, \tag{7.25}$$

where

$$S_{QED}[A] = \int \mathrm{d}^4x\ \left[-\frac{1}{4}F_{\mu\nu}\, F^{\mu\nu} - \frac{(\partial_\mu A^\mu)^2}{2\xi}\right]. \tag{7.26}$$

We see that the net effect of these complicated manipulations is the appearance of the ξ-dependent term in the action. Let us discuss the significance of this term. To this end, we extend the above computation to write the vacuum-to-vacuum transition in the presence of the source

$$\langle 0|0\rangle_J = \frac{Z[J]}{Z[0]}, \tag{7.27}$$

where

$$Z[J] = \int \mathcal{D}A_\mu \; e^{iS_{QED}[A] + i\int d^4x J_\mu A^\mu}. \tag{7.28}$$

We are interested in finding the propagator for the field A^μ. To this end, we rewrite the action by integrating by parts and find

$$S_{QED}[A] = \int d^4x \, \frac{1}{2} \, A^\mu \, T_{\mu\nu} \, A^\nu \tag{7.29}$$

where

$$T_{\mu\nu} = \left(\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu\right) + \frac{1}{\xi}\partial_\mu\partial_\nu. \tag{7.30}$$

Similar to scalar and fermion cases, we change the integration variable $A_\mu \to B_\mu$ where

$$A^\mu = B^\mu - (T^{-1})^{\mu\nu} J_\nu, \tag{7.31}$$

and find

$$\langle 0|0\rangle_J = e^{-\frac{i}{2}\int d^4x_1 d^4x_2 \; J^\mu(x_1) T^{-1}_{\mu\nu} J^\nu(x_2)}. \tag{7.32}$$

The photon propagator is then

$$\langle 0|T A_\mu(x_1) A_\nu(x_2)|0\rangle = \frac{\delta^2 \langle 0|0\rangle_J}{i^2 \delta J^\mu(x_1)\delta J^\nu(x_2)}|_{J=0} = iT^{-1}_{\mu\nu}(x_1, x_2). \tag{7.33}$$

To find the inverse of the operator $T^{\mu\nu}$, we switch to the momentum space and write

$$iT^{-1}_{\mu\nu}(x_1, x_2) = \int \frac{d^4k}{(2\pi)^4} D_{\mu\nu}(k) e^{-ik_\alpha(x_1^\alpha - x_2^\alpha)}. \tag{7.34}$$

Then

$$T^{\mu\nu}(k) \; D_{\nu\rho}(k) = i g^\mu_\rho, \tag{7.35}$$

where

$$T_{\mu\nu}(k) = -k^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right) k_\mu k_\nu. \tag{7.36}$$

It is easy to see why we need a gauge-fixing term in the QED action to find a photon propagator. Indeed, the original QED action corresponds to the $\xi \to \infty$ limit of Eqs. (7.35,7.36). However, in that case $T_{\mu\nu}(k)$ posseses an eignevector (k^μ) with zero eignevalue. As the result $T^{\mu\nu}(k)$ cannot be inverted and the photon propagator cannot be computed. The presence of the gauge-fixing term solves this problem. After some algebra, we find

$$D^{\mu\nu} = \frac{-i}{k^2 + i0}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right). \tag{7.37}$$

The Feynman gauge propagator corresonds to the choice $\xi = 1$ but there are many other choices that are legitimate.