TTP2
Lecture 8

Kirill Melnikov
TTP KIT
November 29, 2023

8 Quantiazation of QCD

We will now discuss quantization of a non-abelian gauge theory, that we will refer to as QCD although our discussion will be sufficiently general.[1] The starting point is again the expression for the path integral[2]

$$Z = \int \mathcal{D}A_\mu^{(a)} e^{iS[A]}. \tag{8.1}$$

The action S is given by

$$L = -\frac{1}{2}\mathrm{Tr}\left[\hat{F}^{\mu\nu}\hat{F}_{\mu\nu}\right], \tag{8.2}$$

where

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - ig_s[\hat{A}_\mu, \hat{A}_\nu]. \tag{8.3}$$

We do not write the source J explicitly; we will easily add it when it becomes necessary.

Similar to the discussion in the previous lecture, the path integral Z is badly divergent because we integrate over all possible QCD fields including those that are connected to each other by gauge transformations and, therefore, have identical action S. Following what we did for the QED case, we would like to separate the integration over gauge-equivalent fields and remove them from path integral. The formula for the "integral representation on unity" holds and we find

$$Z = \int \mathcal{D}A_\mu^{(a)}\ \mathcal{D}\chi^(a)\ \delta(G^{(a)}[A\chi])\ \det\left[\frac{\mathrm{d}G^a[A^\chi]}{\mathrm{d}\chi^b}\right]\ e^{iS[A]}. \tag{8.4}$$

We will consider the Lorentz gauge again

$$G^a[A] = \partial^\mu A_\mu^a - f^a(x). \tag{8.5}$$

We then take the gauge field A_μ^a that satisfies the gauge condition $G[A] = 0$. To compute $\mathrm{d}G[A^\chi]/\mathrm{d}\chi$ at $\chi = 0$, we require an infinitesimal gauge transformation of the field A_μ^a. It reads

$$\hat{A}_\mu^\chi = \hat{A}_\mu + i[\hat{\chi}, \hat{A}_\mu] + \frac{1}{g_s}\partial_\mu\hat{\chi}. \tag{8.6}$$

---

[1] Quantum Chromodynamics or QCD for short is a theory that describes strong interactions; it is a non-abelian gauge theory with the gauge group $SU(3)$.

[2] Note that we integrate over *all* gauge, i.e. four components of the Lorentz vector A_μ and $N^2 - 1$ independent Lie algebra directions, $A^{(a)}$.

Then,

$$G^a[A^\chi] = \partial^\mu A^a_\mu - f^a(x) + \frac{1}{g_s}\partial^\mu\left(\partial_\mu \chi^a - g_s f^{abc}\chi^b A^c_\mu\right). \tag{8.7}$$

Hence,[3]

$$g_s \frac{\mathrm{d}\, G^a[A^\chi]}{\mathrm{d}\chi^b} = \partial^\mu\left(\partial_\mu \delta^{ab} - g_s f^{abc} A^c_\mu\right) = \partial^\mu D^{ab}_\mu. \tag{8.8}$$

We find that in variance with the Abelian case, the determinant of $\mathrm{d}G/\mathrm{d}\chi$ *does depend on the gauge field* and, therefore, cannot be taken outside of the integral over A^μ. In fact, using the rules of integration over Grassmann variables, we can write

$$\det\left[\partial^\mu D^{ab}_\mu\right] \sim \int \mathcal{D}\eta(x)\; \mathcal{D}\bar{\eta}(x)\; e^{iS_{\text{ghost}}[\bar{\eta},\eta,A_\mu]}, \tag{8.9}$$

where $\bar{\eta}^a$ and η^a are *scalar* Grassmann (anti-commuting) fields in the adjoint representation of the gauge group that we refer to as "ghost fields" and

$$S_{\text{ghost}} = \int \mathrm{d}^4x\; \bar{\eta}^a(x)\; \partial^\mu D^{ab}_\mu\; \eta^b(x). \tag{8.10}$$

A peculiar feature of Eq. (8.9) is that once these ghost fields become part of the description of a non-abelian quantum field theory, they will describe quantum fields whose excitations are *scalar* particles in the adjoint representation of the non-Abelian gauge group, *that satisfy Fermi statistics*.

Although this sounds mysterious (since we always say that scalar particles are boson), their role in the construction of a non-abelian quantum field theory is clear. According to Eq. (8.9), ghost fields provide a convenient way to compute a complicated determinant that appeared when we restricted an integral over fields A^a_μ to an integral over gauge-unequivalent field configurations.

The representation of the determinant as an integral over ghost fields is very useful for constructing *perturbative description of non-abelian gauge theories* since contributions of ghosts can be easily incorporated into Feynman diagrams and Green's functions.

Apart from the appearance of these peculiar ghost particles, the rest of the calculation follows the same path as the QED calculation. We introduce an integral over functions $f^a(x)$ to get a gauge-fixing term into a Lagrangian.

[3] Eq. (8.8) should contain a delta-function since $\mathrm{d}\chi(x)/\mathrm{d}\chi(y) = \delta(x-y)$. Since this δ function appears in all terms, I don't display it.

Finally, to write the final expression for the path integral of QCD, we also add fermion fields in the fundamental representation of the gauge group to the theory. We find

$$Z = \int \mathcal{D}A_\mu \ \mathcal{D}\,\eta\mathcal{D}\bar{\eta}\,\mathcal{D}\,\psi\mathcal{D}\bar{\psi}e^{iS_{\rm QCD}[A,\bar{\eta},\eta,\bar{\psi},\psi]}, \tag{8.11}$$

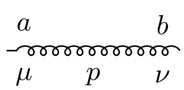
where

$$\begin{aligned} S_{\rm NA} = \int {\rm d}^4x \Bigg[&-\frac{1}{2}{\rm Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] - \frac{1}{\xi}{\rm Tr}\left[(\partial_\mu \hat{A}^\mu)(\partial_\nu \hat{A}^\nu)\right] \\ &+ \bar{\eta}^a\partial^\mu(\delta^{ab}\partial_\mu - g_s f^{abc}A^c_\mu)\eta^b + \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi \Bigg], \end{aligned} \tag{8.12}$$

and $D_\mu = \partial_\mu - ig_s T^a A^{a,\mu}$.

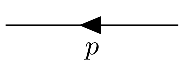
Although the above result is valid for a general gauge theory, we will be primarily interested in QCD. Gauge bosons of QCD are called *gluons* and fermions in the fundamental representation of $SU(3)$ are called *quarks*. QCD Feynman rules follow from the above action in a straightforward way. They read[4]

- the gluon propagator:



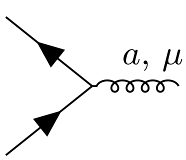
$$= \frac{-i\delta^{ab}}{p^2+i0}\left(g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right).$$

- the quark propagator:



$$= \frac{i}{\hat{p} - m + i0}.$$

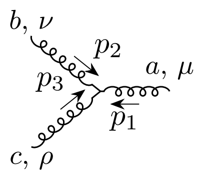
- the gluon-quark vertex:



$$= ig_s\gamma^\mu T^a$$

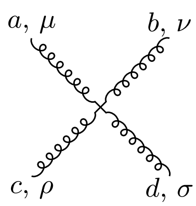
[4]As usual, four-momentum conserving delta-functions are omitted in all vertices. Furthermore, in "real" QCD there is more than one quark field. Quarks have different masses, but their interactions with gluons are identical. Different quarks interact with each other by exchanging gluons.

- the three-gluon vertex:



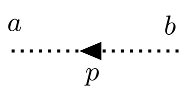
$$= \quad g_s f^{abc}\big[g^{\mu\nu}(p_1 - p_2)^\rho + g^{\nu\rho}(p_2 - p_3)^\mu + g^{\rho\mu}(p_3 - p_1)^\nu\big]$$

- the four-gluon vertex:



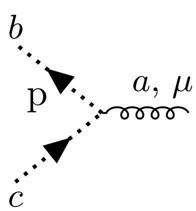
$$= \quad -ig_s^2\big[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})\big]$$

- the ghost propagator:



$$= \quad \frac{i\delta^{ab}}{p^2 + i0}$$

- the gluon-ghost vertex:



$$= \quad g_s f^{abc} p^\mu$$