TTP2
Lecture 10

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10 Quark-gluon scattering in QCD

The spinor-helicity methods that we discussed in the previous lecture can be used for an efficient computation of the scattering amplitudes. To this end, consider the amplitude for quark-gluon scattering process which we write in a somewhat strange way, as a decay of an empty space into $q\bar{q}gg$ final state

$$0 \to q(p_1, i) + \bar{q}(p_4, j) + g(p_2, a) + g(p_3, b), \tag{10.1}$$

where a, b, i, j refer to color indices of the corresponding particles.

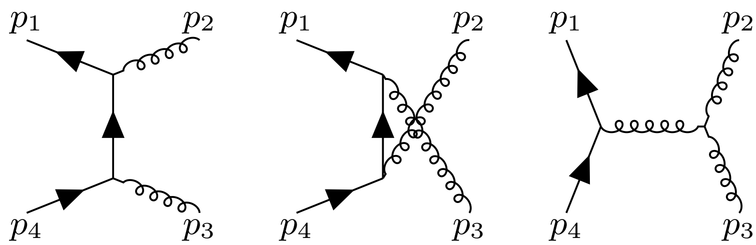

Figure 1: The three Feynman diagrams of the process $0 \to q(p_1) + \bar{q}(p_4) + g(p_2) + g(p_3)$.

There are three diagrams that contribute to this process; two "abelian" and one "non-abelian", that involves triple gluon couplings. We will take the left-handed spinor $\langle 1|$ for the outgoing quark with momentum p_1 and the left-handed spinor $|4]$ for the outgoing (right-handed) anti-quark with momentum p_4. The expression for the matrix element is

$$\begin{aligned} iM = &-ig^2\langle 1| \left\{ \frac{\hat{\epsilon}_2(\hat{p}_1+\hat{p}_2)\hat{\epsilon}_3}{s_{12}} \left(t^a t^b\right)_{ij} + \frac{\hat{\epsilon}_3(\hat{p}_1+\hat{p}_3)\hat{\epsilon}_2}{s_{13}} \left(t^b t^a\right)_{ij} \right\} |4] \\ &- g^2 f^{abc} t^c_{ij} \frac{\langle 1\gamma^\lambda 4]}{s_{14}} \left(\epsilon_2\cdot\epsilon_3(p_2-p_3)_\lambda + \epsilon_{3\lambda}(2p_3+p_2)\cdot\epsilon_2 + \epsilon_{2\lambda}(-2p_2-p_3)\epsilon_3\right). \end{aligned} \tag{10.2}$$

Here, $t^{a,b}$ are the $SU(3)$ Lie algebra generators in the fundamental representation and f^{abc} are the $SU(3)$ structure constants.

The $SU(3)$ generators are normalized $\mathrm{Tr}\left[t^a t^b\right] = \delta^{ab}/2$ and, as Lie algebra generators, they satisfy the commutation relation

$$t^a t^b - t^b t^a = i f^{abc} t^c. \tag{10.3}$$

We can use this relation to remove the $SU(3)$ structure constants from the expression for the amplitude. Also, we rescale $t^a = T^a/\sqrt{2}$, to have $\mathrm{Tr}\left[T^a T^b\right] = \delta^{ab}$. As the result of this, the amplitude is written as the sum of two terms

$$M = \left(\frac{g}{\sqrt{2}}\right)^2 \left(M_1 (T^q T^b)_{ij} + M_2 (T^b T^a)_{ij}\right), \tag{10.4}$$

where

$$\begin{aligned} M_1 = -\Bigg[& \frac{\langle 1|\hat{\epsilon}_2(\hat{1}+\hat{2})\hat{\epsilon}_3|4]}{s_{12}} - \frac{\langle 1\gamma^\mu 4]}{s_{14}} \left(\epsilon_2 \cdot \epsilon_3 (p_2 - p_3)_\mu \right. \\ & \left. + \epsilon_{3\mu}(2p_3 + p_2) \cdot \epsilon_2 + \epsilon_{2\mu}(-2p_2 - p_3)\epsilon_3\right) \Bigg]. \\ M_2 = -\Bigg[& \frac{\langle 1|\hat{\epsilon}_3(\hat{1}+\hat{3})\hat{\epsilon}_2|4]}{s_{13}} - \frac{\langle 1\gamma^\mu 4]}{s_{14}} \left(\epsilon_2 \cdot \epsilon_3 (p_3 - p_2)_\mu \right. \\ & \left. + \epsilon_{3\mu}(-2p_3 - p_2) \cdot \epsilon_2 + \epsilon_{2\mu}(2p_2 + p_3)\epsilon_3\right) \Bigg]. \end{aligned} \tag{10.5}$$

Next, we note an interesting property of the above amplitude. If we write $M_1 = M(1,2,3,4)$, then $M_2 = M(1,3,2,4)$, so it is sufficient to compute one function of external momenta to get the full result. We note that out of three diagrams that contribute to the amplitude M only *two* contribute to the function M_1. The diagram that does not contribute has its external particles arranged in such a way that they can not be ordered (clockwise) as p_1, p_2, p_3, p_4.

The amplitude $M(1,2,3,4)$ is called "color-ordered". It is transversal (i.e. gauge-invariant) and independent of color indices of the colliding partons; because of this it is a simpler object to compute than the full amplitude. One can show that one can arrange the QCD Feynman rules in such a way that a direct computation of color-ordered amplitudes for any process becomes possible.

We now calculate the color-ordered amplitude $M(1_L, 2, 3, 4_L)$. As the first step, we consider equal photon helicities, starting from the right-handed gluons. The relevant formula reads

$$\hat{\epsilon}_R = \gamma_\mu \epsilon_R^\mu = \frac{\sqrt{2}}{\langle rk\rangle}\left(|k]\langle r| + |r\rangle[k|\right), \tag{10.6}$$

so that

$$\langle 1|\hat{\epsilon}_{3R} = \frac{\sqrt{2}}{\langle r_3 3\rangle}\langle 1 r_3\rangle[3| \qquad (10.7)$$
$$\langle 1|\hat{\epsilon}_{2R} = \frac{\sqrt{2}}{\langle r_2 2\rangle}\langle 1 r_2\rangle[2|.$$

Furthemore, we have seen that the scalar products of polarization vectors with *same* helicities vanishes if the two vectors have identical reference momenta

$$\epsilon_{3R}\cdot\epsilon_{2R} \sim \langle r_2 r_3\rangle. \qquad (10.8)$$

It is then easy to see that if we choose $r_2 = r_3 = p_1$, the amplitude vanishes

$$M_1(q_{1_L}, g_{2_R}, g_{3_R}, \bar{q}_{4_L}) = 0. \qquad (10.9)$$

Similar argument can be used to prove that amplitude $M_1(q_{1_L}, g_{2_L}, g_{3_L}, \bar{q}_{4_L})$ vanishes as well. Indeed,

$$\hat{\epsilon}_L = -\frac{\sqrt{2}}{[rk]}\left(|r]\langle k| + |k\rangle[r|\right), \qquad (10.10)$$

so that

$$\hat{\epsilon}_{3L}|4] = -\frac{\sqrt{2}}{[r_3 3]}|3\rangle[r_3 4], \qquad (10.11)$$
$$\hat{\epsilon}_{2L}|4] = -\frac{\sqrt{2}}{[r_2 2]}|2\rangle[r_2 4].$$

So, we choose $r_2 = r_3 = p_4$ and find $M_1(q_{1_L}, g_{2_L}, g_{3_L}, \bar{q}_{4_L}) = 0$.

Next, we will study the color-ordered amplitude where the two photon polarizations are different. Specifically, we consider $M(q_{1_L}, g_{2_R}, g_{3_L}, \bar{q}_{4_L})$. The explicit expression for the amplitude reads

$$M = -\left[\frac{\langle 1|\epsilon_{2R}(\hat{1}+\hat{2})\epsilon_{3L}|4]}{s_{12}} - \frac{\langle 1\gamma^\mu 4]}{s_{14}}\left(\epsilon_{2R}\cdot\epsilon_{3L}(p_2-p_3)_\mu\right.\right. \qquad (10.12)$$
$$\left.\left. +\epsilon_{3L\mu}(2p_3+p_2)\cdot\epsilon_{2R} + \epsilon_{2R\mu}(-2p_2-p_3)\epsilon_{3L}\right)\right].$$

To understand how to simplify computations, we will study contributing terms in Eq.(10.12) separately. The first term reads

$$\langle 1|\hat{\epsilon}_{2R}(\hat{1}+\hat{2})\hat{\epsilon}_{3L}|4] = -\frac{2\langle 1r_2\rangle[2|(\hat{1}+\hat{2})|3\rangle[r_34]}{\langle r_22\rangle[r_33]} = -\frac{2\langle 1r_2\rangle[21]\langle 13\rangle[r_34]}{\langle r_22\rangle[r_33]}. \tag{10.13}$$

The third and the fourth terms in Eq.(10.12) contain the following spinor products

$$\begin{aligned} \langle 1\gamma^\mu 4]\epsilon_{3L\mu} &= \langle 1|\hat{\epsilon}_{3L}4] = -\frac{\sqrt{2}\langle 13\rangle[r_34]}{[r_33]}, \\ \langle 1\gamma^\mu 4]\epsilon_{2R\mu} &= \langle 1|\hat{\epsilon}_{2R}4] = \frac{\sqrt{2}\langle 1r_2\rangle[24]}{\langle r_22\rangle}, \end{aligned} \tag{10.14}$$

Hence, we conclude that if we choose $r_3 = p_4$ and $r_2 = p_1$ all contributions in Eq.(10.13) and Eq.(10.14) vanish; therfore, only the second term in Eq.(10.12) contributes. We find

$$M_1(q_{1_L}, g_{2_L}, g_{3_L}, \bar{q}_{4_L}) = \frac{\langle 1|(\hat{2}-\hat{3})|4]}{s_{14}}\,\epsilon_{2R}\cdot\epsilon_{3L}. \tag{10.15}$$

To simplify it further, we use momentum conseration

$$\langle 1|(\hat{2}-\hat{3})|4] = -2\langle 13\rangle[34], \tag{10.16}$$

and compute the product of two polarization vectors

$$\epsilon_{2R}\cdot\epsilon_{3L} = \frac{\langle r_2\gamma^\mu 2]}{\sqrt{2}\langle r_22\rangle}\frac{(-1)[r_3\gamma_\mu 3\rangle}{\sqrt{2}[r_33]} = -\frac{\langle 1\gamma^\mu 2][4\gamma_\mu 3\rangle}{2\langle 12\rangle[43]} = -\frac{\langle 13\rangle[42]}{\langle 12\rangle[43]}. \tag{10.17}$$

We therefore find (use $s_{14} = s_{23} = -\langle 23\rangle[23]$)

$$M(q_{1_L}, g_{2_R}, g_{3_L}, \bar{q}_{4_L}) = -\frac{2\langle 13\rangle[34]}{\langle 23\rangle[23]}\,\frac{\langle 13\rangle[42]}{\langle 12\rangle[43]} = \frac{2\langle 13\rangle^2[42]}{\langle 12\rangle\langle 23\rangle[23]}. \tag{10.18}$$

We can simplify this expression by multiplying it by $1 = \langle 13\rangle/\langle 13\rangle$. It follows from momentum conservation that

$$\langle 13\rangle[32] = \langle 1|\hat{3}|2] = \langle 1\left(-\hat{1}-\hat{2}-\hat{4}\right)2] = -\langle 14\rangle[42]. \tag{10.19}$$

Then,

$$M_1(q_{1_L}, g_{2_R}, g_{3_L}, \bar{q}_{4_L}) = -\frac{2\langle 13\rangle^3\langle 43\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}. \tag{10.20}$$

Next, we will compute the second color-ordered amplitude $M(q_{1_L}, g_{2_L}, g_{3_R}, \bar{q}_{4_L})$. We will again go through the same exercise of trying to force as many terms as possible to vanish. We will do it slightly differently this time. The amplitude reads

$$\begin{aligned} M = -\Bigg[& \frac{\langle 1|\hat{\epsilon}_{2L}(\hat{1}+\hat{2})\hat{\epsilon}_{3R}|4]}{s_{12}} - \frac{\langle 1\gamma^\mu 4]}{s_{14}}\left(\epsilon_{2L}\cdot\epsilon_{3R}(p_2-p_3)_\mu\right. \\ & \left. +\epsilon_{3R\mu}(2p_3+p_2)\cdot\epsilon_{2L} + \epsilon_{2L\mu}(-2p_2-p_3)\epsilon_{3R}\right)\Bigg]. \end{aligned} \tag{10.21}$$

Lets focus on the "non-abelian" contribution to this amplitude. There are three terms that involve

$$p_3\cdot\epsilon_{2L}, \quad p_2\cdot\epsilon_{3R}, \quad \epsilon_{2L}\cdot\epsilon_{3R}. \tag{10.22}$$

Since

$$\epsilon_{2L}\cdot\epsilon_{3R} \sim [r_2 3]\langle r_3 2\rangle, \tag{10.23}$$

we can ensure that all terms in Eq.(10.22) vanish if $r_2 \sim p_3$ and $r_3 \sim p_2$. With these choices of reference vectors, we find

$$\begin{aligned} \langle 1|\hat{\epsilon}_{2L} &= -\frac{\sqrt{2}\langle 12\rangle[3|}{[32]}, \\ \hat{\epsilon}_{3R}|4] &= \frac{\sqrt{2}|2\rangle[34]}{\langle 23\rangle}. \end{aligned} \tag{10.24}$$

Therefore, we find

$$M(q_{1_L}, g_{2_L}, g_{3_R}, \bar{q}_{4_L}) = \frac{2\langle 12\rangle[34][31]\langle 12\rangle}{s_{12}[32]\langle 23\rangle} = \frac{2[34][31]\langle 12\rangle}{[21][32]\langle 23\rangle}. \tag{10.25}$$

For further simplifications, multiply both numerator and denominator with $\langle 12\rangle\langle 42\rangle$. Then

$$M(q_{1_L}, g_{2_L}, g_{3_R}, \bar{q}_{4_L}) = \frac{2[34][31]\langle 12\rangle^2\langle 42\rangle}{[21][32]\langle 23\rangle\langle 12\rangle\langle 42\rangle}. \tag{10.26}$$

Now, in the denominator use

$$[32]\langle 42\rangle = -[32]\langle 24\rangle = -[3|\hat{2}|4\rangle = [3|\hat{1}|4\rangle = [31]\langle 14\rangle, \tag{10.27}$$

so that

$$M(q_{1_L}, g_{2_L}, g_{3_R}, \bar{q}_{4_L}) = \frac{2[34]\langle 12\rangle^2\langle 42\rangle}{[21]\langle 14\rangle\langle 23\rangle\langle 12\rangle}. \tag{10.28}$$

To simplify it further, note that since $s_{12} = s_{34}$, we have

$$[34]\langle 43\rangle = [21]\langle 21\rangle \leftrightarrow \frac{[34]}{[21]} = \frac{\langle 12\rangle}{\langle 34\rangle}. \tag{10.29}$$

$$M(q_{1_L}, g_{2_L}, g_{3_R}, \bar{q}_{4_L}) = -\frac{2\langle 12\rangle^3\langle 42\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}. \tag{10.30}$$

Amplitudes for other helicity configurations can be obtained from the computed by replacing all square brackets with angle brackets and vice versa. Note that up to a sign, this replacement also makes ϵ^μ_L out of ϵ^μ_R etc. Hence, for example, it follows from Eq. (10.30) that

$$M(q_{1_R}, g_{2_R}, g_{3_L}, \bar{q}_{4_R}) = -\frac{2[12]^3[42]}{[12][23][34][41]}. \tag{10.31}$$

To compute the amplitudes squared, one needs to square the amplitude, sum over colors and spins. This is a straightforward procedure. The only comment to make is that computing $|\mathcal{M}|^2$ of the helicity amplitude is straightforward. Indeed, consider

$$|M(q_{1_R}, g_{2_R}, g_{3_L}, \bar{q}_{4_R})|^2 = 4\frac{[12]^3[42]}{[12][23][34][41]}\ \frac{([12]^*)^3[42]^*}{[12]^*[23]^*[34]^*[41]^*}. \tag{10.32}$$

However, since

$$[ij][ij]^* = [ij]\langle ji\rangle = 2s_{ij}, \tag{10.33}$$

we find

$$|M(q_{1_R}, g_{2_R}, g_{3_L}, \bar{q}_{4_R})|^2 = \frac{s_{12}s_{24}}{s_{23}^2} = \frac{4st}{u^2}, \tag{10.34}$$

where in the last step we used the Mandelstam variables.