TTP2
Lecture 11

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11 One-loop computations in QCD

We will continue with some examples of the one-loop computations in QCD. As we will see, some of these calculations are similar to what we have discussed in QED and some are different. We will also mostly focus on studying general properties of the results and exploring the ultraviolet divergences to set up a stage for a discussion of the renormalization of QCD. We will use the dimensional regularization throughout this lecture. We will also work in the Feynman gauge, i.e. $\xi = 1$.

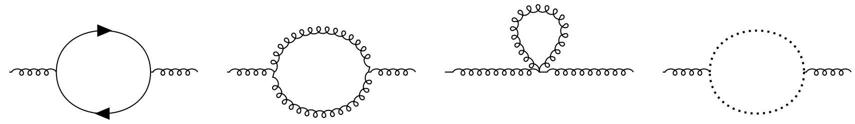


Figure 1: One-loop vacuum polarization diagrams in QCD.

In QED, the one-loop corrections to the gluon propagator are related to vacuum polarization diagrams with leptons. In QCD there are gluon-quark, three-gluon, four-gluon and ghost-gluon vertices. Hence, there are *four* one-loop vacuum polarization diagrams in QCD. We note that a diagram with a four-gluon vertex vanishes in dimensional regularization because its loop part is a scaleless integral

$$\int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\mathrm{Num}}{p^2 + i0} = 0. \tag{11.1}$$

The next simplification comes from the fact that the vacuum polarization diagram with quarks can be easily read off from the calculation in QED. The main difference is that quarks and gluons carry color and that quarks (for our purposes here) are massless.

We first address the issue of color. The quark-gluon vertex involves $SU(3)$ generators; hence, denoting color indices of the two external gluon lines in the two-point function as a and b, we find that the following "color factor" has to be computed

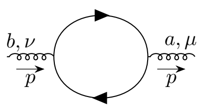
$$t^a_{ki} \delta_{km} \delta_{ij} t^b_{jm} = \mathrm{Tr}\left[t^a t^b\right] = T_R \delta^{ab}, \tag{11.2}$$

where $T_R = 1/2$ is a normalization factor for $SU(3)$ generators in the fundamental representation. Apart from this, the calculation is identical to that of QED. Therefore, we take Eq. (**??**), set $m^2 \to 0$, replace e^2 with g_s^2 and obtain

$$i\Pi_q^{ab}(p) = -\frac{ig_s^2\Gamma(1+\epsilon)(-p^2)^{-\epsilon}}{(4\pi)^{d/2}\,\epsilon}\,T_R\delta^{ab}\,\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\frac{4(1-\epsilon)}{(3-2\epsilon)(1-2\epsilon)}. \quad (11.3)$$

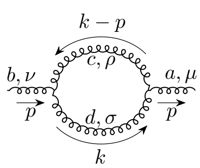
The subscript q indicates that this is a quark-loop contribution to the gluon vacuum polarization.

Restoring dependence on gluon's polarization indices, we write



$$= i\Pi_q^{ab,\mu\nu}(p) = (p^2 g^{\mu\nu} - p^\mu p^\mu)\; i\Pi^{ab}(p). \quad (11.4)$$

As the next step, we consider the one-loop diagram which describes the contribution of the gluon loop to the gluon vacuum polarization. Note that these diagrams exist because gluons interact with gluons which is a special feature of non-abelian (as opposed to abelian, e.g. QED) gauge theories. The corresponding expression reads



$$= i\Pi_g^{ab,\mu\nu} = -\frac{g_s^2}{2}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2}\frac{1}{(k-p)^2}N^{\mu\nu}(k,p)f^{acd}f^{bdc}, \quad (11.5)$$

where the factor $1/2$ is the symmetry factor,

$$N_\nu^\mu = V_{3g}^{\mu\rho\sigma}(-p, p-k, k)V_{3g,\nu\rho\sigma}(p, -k, k-p), \quad (11.6)$$

and the three-gluon vertex (all momenta incoming) is defined as

$$V_{3g}(p_1^{a_1,\mu_1}, p_2^{a_2,\mu_2}, p_3^{a_3,\mu_3}) = g_s f^{a_1a_2a_3}V_{3g}(p_1^{\mu_1}, p_2^{\mu_2}, p_3^{\mu_3}), \quad (11.7)$$

with

$$V_{3g}(p_1^{\mu_1}, p_2^{\mu_2}, p_3^{\mu_3}) = g^{\mu_1\mu_2}(p_1-p_2)^{\mu_3} + g^{\mu_2\mu_3}(p_2-p_3)^{\mu_1} + g^{\mu_3\mu_1}(p_3-p_1)^{\mu_2}. \quad (11.8)$$

The calculation proceeds in the standard way. First, we deal with the color algebra using

$$f^{acd}f^{bcd} = C_A\delta^{ab}, \quad (11.9)$$

where $C_A = 3$ for the group $SU(3)$.

The calculation proceeds in the standard way. First, we deal with the color algebra using

$$f^{acd} f^{bcd} = C_A \delta^{ab}, \tag{11.10}$$

where $C_A = 3$ for the group $SU(3)$.

Then, we integrate over the loop momentum. To do so, we combine the two denominators using the Feynman parameters

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2} \frac{1}{(k-p)^2} = \int_0^1 \mathrm{d}x \, \frac{1}{((k-px)^2 + p^2 x(1-x))^2}. \tag{11.11}$$

We change the loop momentum $k = l + xp$. Upon doing so, the numerator function turns into

$$N^{\mu\nu} = N_2^{\mu\nu\alpha\beta} l_\alpha l_\beta + N_1^{\mu\nu\alpha} l_\alpha + N_0^{\mu\nu}, \tag{11.12}$$

where tensors $N_{2,1,0}$ on the right hand side are l-independent. We find

$$\begin{aligned} N_2^{\mu\nu\alpha\beta} l_\alpha l_\beta &= -2g^{\mu\nu} l^2 + (6-4d) l^\mu l^\nu, \\ N_0^{\mu\nu} &= -g^{\mu\nu} p^2 ((1+x)^2 + (2-x)^2) \\ &\quad + p^\mu p^\nu ((2-d)(1-2x)^2 + 2(1+x)(2-x)). \end{aligned} \tag{11.13}$$

The integration over l can be easily performed. First wee simplifying the tensor and vector integrals

$$\begin{aligned} \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l_\alpha l_\beta}{(l^2-\Delta)^2} &= \frac{g^{\alpha\beta}}{d} \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^2}{(l^2-\Delta)^2}, \\ \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l_\alpha}{(l^2-\Delta)^2} &= 0. \end{aligned} \tag{11.14}$$

Then, using

$$\begin{aligned} \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{(l^2-\Delta)^2} &= \frac{i\Gamma(1+\epsilon)}{(4\pi)^{d/2}\epsilon} \, \Delta^{-\epsilon} \\ \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^2}{(l^2-\Delta)^2} &= \frac{i\Gamma(1+\epsilon)}{(4\pi)^{d/2}\epsilon} \, \frac{2-\epsilon}{1-\epsilon} \, \Delta^{1-\epsilon}. \end{aligned} \tag{11.15}$$

Using these results and after a fair amount of algebra, we arrive at the following integral representation of the gluon loop contribution to the gluon vacuum polarization

$$
\begin{aligned}
i\Pi_g^{ab,\mu\nu} = & \frac{ig_s^2(-p^2)^{-\epsilon}\Gamma(1+\epsilon)}{(4\pi)^{d/2}\epsilon} C_A \delta^{ab} \int\limits_0^1 \frac{\mathrm{d}x}{(x(1-x))^{\epsilon}} \Big\{ \\
& g^{\mu\nu} p^2 \Big[-\frac{3(3-2\epsilon)}{2(1-\epsilon)} x(1-x) + \frac{1}{2}(2-x)^2 + \frac{1}{2}(1+x)^2 \Big] \\
& + p^\mu p^\nu \left[(1-\epsilon)\left(1-2x\right)^2 - (1+x)(2-x) \right] \Big\}.
\end{aligned}
\tag{11.16}
$$

To compute these integrals, we use

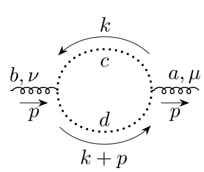
$$
\int\limits_0^1 \mathrm{d}x \; x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{11.17}
$$

We then find

$$
\begin{aligned}
i\Pi_g^{ab,\mu\nu} = & \frac{ig_s^2(-p^2)^{-\epsilon}\Gamma(1+\epsilon)}{(4\pi)^{d/2}\epsilon} \; C_A \delta^{ab} \; \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \\
\times & \frac{1}{(3-2\epsilon)(1-2\epsilon)} \left[g^{\mu\nu} p^2 \left(\frac{19}{4} - 3\epsilon \right) - p^\mu p^\nu \left(\frac{11}{2} - \frac{7}{2}\epsilon \right) \right].
\end{aligned}
\tag{11.18}
$$

It follows from Eq. (11.18) that – in variance to the quark-loop contribution – the gluon-loop contribution to the gluon vacuum polarization is not transversal, i.e. $p_\mu \Pi_g^{ab,\mu\nu} \neq 0$.

However, the transversality is recovered if, in addition to the gluon loop we also consider ghost-loop contribution to the gluon vacuum polarization. Using Feynman rules presented in lecture 8, we write



$$
= i\Pi_{\mathrm{gh}}^{ab,\mu\nu} = -g_s^2 \; f^{acd} f^{bcd} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \; \frac{1}{k^2(k+p)^2} \; k^\mu (k+p)^\nu. \tag{11.19}
$$

Note that there is an additional (-1)-factor that contributes o the above formula; this factor is needed because ghost fields *anti-commute*, so that for closed ghost loops same rules as for fermions apply.

The calculation of $\Pi_{\rm gh}^{ab,\mu\nu}$ proceeds in the same way the calculation of the gluon loop. We first compute the color factor. Then, we combine Feynman propagators, shift the loop momentum, integrate over the shifted momentum and obtain

$$\begin{aligned}\Pi_{\rm gh}^{ab,\mu\nu} &= \frac{ig_s^2(-p^2)^{-\epsilon}\Gamma(1+\epsilon)}{(4\pi)^{d/2}\epsilon}\, C_A\delta^{ab} \int\limits_0^1 \frac{{\rm d}x}{(x(1-x))^\epsilon} \\ &\times x(1-x)\left\{\frac{g^{\mu\nu}p^2}{2(1-\epsilon)} + p^\mu p^\nu\right\}.\end{aligned} \tag{11.20}$$

Integrating over x, we find

$$\Pi_{\rm gh}^{ab,\mu\nu} = \frac{ig_s^2(-p^2)^{-\epsilon}\Gamma(1+\epsilon)}{(4\pi)^{d/2}\epsilon} C_A\delta^{ab}\, \frac{\Gamma^2(2-\epsilon)}{\Gamma(4-2\epsilon)}\left\{\frac{g^{\mu\nu}p^2}{2(1-\epsilon)} + p^\mu p^\nu\right\}. \tag{11.21}$$

This contribution is also not transversal. However, if we add the gluon loop and the ghost loop, we find

$$\begin{aligned}i\Pi_{\rm g+gh}^{ab,\mu\nu} &= \frac{ig_s^2\Gamma(1+\epsilon)}{(4\pi)^{d/2}\epsilon}\,\delta^{ab}C_A(-p^2)^{-\epsilon}\,\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \\ &\times \frac{5-3\epsilon}{(3-2\epsilon)(1-2\epsilon)}(g^{\mu\nu}p^2 - p^\mu p^\nu).\end{aligned} \tag{11.22}$$

The full one-loop contribution to the gluon vacuum polarization is given by the sum of the quark, gluon and ghost contribution. Before adding them together, we note that more than one quark exists in Nature and each of them can contribute to the gluon vacuum polarization. We will treat all quarks as massless and we will denote their number by n_f. Then

$$i\Pi^{ab,\mu\nu} = i\Pi_{g+{\rm gh}}^{ab,\mu\nu} + n_f i\Pi_q^{ab,\mu\nu}. \tag{11.23}$$

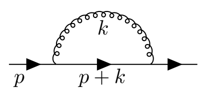
The full result reads

$$\begin{aligned}i\Pi^{ab,\mu\nu} &= \frac{ig_s^2\Gamma(1+\epsilon)}{(4\pi)^{d/2}\epsilon}\delta^{ab}(-p^2)^\epsilon\left(g_{\mu\nu}p^2 - p_\mu p_\nu\right)\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \\ &\times\left[C_A\frac{5-3\epsilon}{(3-2\epsilon)(1-2\epsilon)} - n_f T_R\frac{4(1-\epsilon)}{(3-2\epsilon)(1-2\epsilon)}\right].\end{aligned} \tag{11.24}$$

In the next lecture, we will discuss the one-loop renormalization of QCD; to do this, we will require the $1/\epsilon$ singularities of the above equation. Performing the expansion (and keeping certain terms unexpanded for future convenience), we find

$$i\Pi^{ab,\mu\nu} \approx \frac{ig_s^2\Gamma(1+\epsilon)}{(4\pi)^{d/2}\epsilon}\delta^{ab}\left(g_{\mu\nu}p^2 - p_\mu p_\nu\right)\left[\frac{5}{3}C_A - \frac{4}{3}n_f T_R\right]. \tag{11.25}$$

We continue with the discussion of the quark self-energy. The calculation is similar to what we have done for the electron self-energy in QED; the difference is that we will consider quarks to be massless. Furthermore, we will focus on the UV-divergent contributions to the electron self-energy. I will sketch how the corresponding calculation can be done. The quark self energy reads



$$= i\Sigma_q^{ij} = -g_s^2(t^a t^a)_{ij}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\gamma^\mu(\hat{p}+\hat{k})\gamma_\mu}{k^2(p+k)^2}. \tag{11.26}$$

First we compute the color factor

$$(t^a t^a)_{ij} = C_F\delta_{ij}, \tag{11.27}$$

where $C_F = 4/3$ for the group $SU(3)$.

Next, we simplify the numerator of the integrand in Eq. (11.32). Since we are interested in divergent $1/\epsilon$ contributions only, we can make use of the algebra of γ-matrices in four dimensions. Then

$$\gamma^\mu(\hat{p}+\hat{k})\gamma_\mu = -2(\hat{p}+\hat{k}). \tag{11.28}$$

After that, combining the propagators, shifting the loop momentum $k \to l - xp$ and discarding linear terms in the shifted momentum l, we find

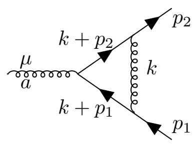
$$i\Sigma_q = g_s^2 C_F\delta_{ij}2\hat{p}\int_0^1 \mathrm{d}x\,(1-x)\int\frac{\mathrm{d}^d l}{(2\pi)^d}\frac{1}{(l^2-\Delta^2)^2}, \tag{11.29}$$

where $\Delta^2 = -p^2x(1-x)$. We have already encountered the remaining integral over l several times. Using earlier results, we obtain

$$i\Sigma_{q,ij} \approx \frac{ig_s^2\Gamma(1+\epsilon)}{(4\pi)^{d/2}\epsilon}\,\delta_{ij}C_F\,\hat{p}, \tag{11.30}$$

where only divergent terms are shown. Note an important fact that, in the case of a massless quark, the self-energy is proportional to $\hat{p}$ and, therefore, if a quark is massless to begin with, its mass cannot be generated perturbatively.

Finally, we will compute divergent contributions to three-point (amputated) Green's functions that describe interactions between a quark and a gluon. The first diagram is QED-like; it reads



$$= \hat{V}^{1,\mu}_{a,ij} = g_s^3 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \, (t^b t^a t^b)_{ij} \, \gamma^\rho \frac{1}{\hat{k}+\hat{p}_2} \gamma^\mu \frac{1}{\hat{k}+\hat{p}_1} \gamma_\rho \, \frac{1}{k^2}. \tag{11.31}$$

We are interested in ultraviolet divergence; it originates from the region where the loop momentum k is very large. Neglecting $p_{2,1}$ in quark propagators, we find

$$\hat{V}^{1,\mu}_{a,ij} \approx g_s^3 \; (t^b t^a t^b)_{ij} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2)^3} \; \gamma^\rho \hat{k} \gamma^\mu \hat{k} \gamma_\rho \frac{1}{k^2}, \tag{11.32}$$

and all terms that were discarded when moving from Eq. (11.31) to Eq. (11.32) give only ϵ-finite contributions. Averaging over directions of k and performing algebra of γ-matrices in four dimensions, we find

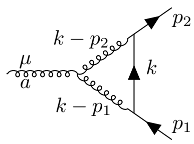
$$\gamma^\rho \hat{k} \gamma^\mu \hat{k} \gamma_\rho \to \frac{k^2}{4} \gamma^\rho \gamma^\alpha \gamma^\mu \gamma_\alpha \gamma_\rho \to k^2 \gamma^\mu. \tag{11.33}$$

Hence, Eq. (11.32) turns into

$$\begin{aligned} \hat{V}^{1,\mu}_{a,ij} &\approx g_s^3 \; (t^b t^a t^b)_{ij} \, \gamma^\mu \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2)^2} \to \\ & \frac{i g_s^3 \Gamma(1+\epsilon) g_s^3}{(4\pi)^{d/2} \epsilon} \gamma^\mu \; (t^b t^a t^b)_{ij} \\ &= i g_s t^a_{ij} \gamma^\mu \; \frac{g_s^2 \Gamma(1+\epsilon) g_s^3}{(4\pi)^{d/2} \epsilon} \, (C_F - C_A/2). \end{aligned} \tag{11.34}$$

In the last step we computed the color factor and factored out the tree-level Green's function $i g_s t^a_{ij} \gamma^\mu$.

The second diagram that we want to consider involves the three-gluon vertex. To understand how the ultraviolet-divergent contribution is generated we count powers of the loop momentum. Then, the three-gluon vertex is linear in the loop momentum, quark-gluon vertex is momentum-independent, each of the two gluon propagators is $1/k^2$ and the quark propagator is $/k$. Hence, this diagram may have a logarithmic divergence; we can capture it if, when writing the integrand, we keep the loop momentum everywhere and neglect all the external momenta. In this approximation, the expression for the diagram reads



$$= \hat{V}^{2,\mu}_{a,ij} \approx i g_s^3 f^{abc} (t^c t^b)_{ij} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2)^2} \times V_{3g}(0^\mu, -k^\sigma, k^\rho)\, \gamma^\rho \frac{1}{\hat{k}} \gamma^\sigma. \tag{11.35}$$

To simplify this expression, we note that

$$i f^{abc} (t^c t^b)_{ij} = \frac{C_A}{2} t^a_{ij}. \tag{11.36}$$

To further simplify the integrand, we write

$$V_{3g}(0^\mu, -k^\sigma, k^\rho) = g^{\mu\rho} k^\sigma - 2 g^{\rho\sigma} k^\mu + g^{\sigma\mu} k^\rho, \tag{11.37}$$

and note that

$$V_{3g}(0^\mu, -k^\sigma, k^\rho) \gamma^\rho \frac{1}{\hat{k}} \gamma^\sigma = \frac{1}{k^2} (2 k^2 \gamma^\mu + 4 \hat{k} k^\mu) \to 3 \gamma^\mu. \tag{11.38}$$

where in the last step we averaged over directions of the loop momentum k. We then find

$$\hat{V}^{2,\mu}_{a,ij} \approx g_s^3 C_A t^a_{ij}\, \gamma^\mu \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{3}{(k^2)^2} = i g_s t^a_{ij} \gamma^\mu\, (\frac{3 C_A}{2}) \frac{g_s^2 \Gamma(1+\epsilon)}{(4\pi)^{d/2} \epsilon}. \tag{11.39}$$

The full (divergent) one-loop contribution to the quark-gluon vertex is given by the sum of Eq. (11.34) and Eq. (11.39). It reads

$$\hat{V}^{\mu}_{a,ij} = \hat{V}^{1,\mu}_{a,ij} + \hat{V}^{2,\mu}_{a,ij} \approx i g_s t^a_{ij} \gamma^\mu\, (C_F + C_A) \frac{g_s^2 \Gamma(1+\epsilon)}{(4\pi)^{d/2} \epsilon}. \tag{11.40}$$

The Green's functions whose divergences we have computed, are not the only Green's functions that exhibit ultraviolet divergences. However, as we explain in the next lecture, we have computed sufficient number of Green's functions to determine all the needed counterterms to renormalize Quantum Chromodynamics at one loop.