TTP2
Lecture 12

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12 One-loop renormalization in QCD

The goal of this lecture is to discuss the renormalization of QCD at one loop. The Lagrangian reads

$$L = -\frac{1}{2}\mathrm{Tr}\left[\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\right]+\bar{\psi}i\hat{D}\psi+\bar{c}^a(-\partial_\mu D^\mu_{ab})c^b-\frac{1}{\xi}\mathrm{Tr}\left[(\partial_\mu\hat{A}^\mu)(\partial_\nu\hat{A}^\nu)\right]. \quad (12.1)$$

Although all the quantities that appear in this Lagrangian have already been introduced in the previous lectures, we will repeat their definitions one more time

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc}A^b_\mu A^c_\nu, \\ D_\mu &= \partial_\mu - i g_s A^a_\mu t^a, \\ -\partial_\mu D^\mu_{ab} &= -\partial^2\delta_{ab} - g_s\partial^\mu f^{acb}A^c_\mu. \end{aligned} \quad (12.2)$$

We use Eq. (12.2) to expose the dependence of the Lagrangian on fields and couplings; at this point we interpret the fields and the coupling as bare quantities. We then replace bare fields with the renormalized fields and bare couplings with renormalized couplings, and find

$$L = L_{\mathrm{QCD}} + L_{\mathrm{QCD,ct}}, \quad (12.3)$$

where

$$L_{\mathrm{QCD}} = -\frac{1}{2}\mathrm{Tr}\left[\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\right] + \bar{\psi}i\hat{D}\psi + \bar{c}^a(-\partial_\mu D^\mu_{ab}c^b) - \frac{1}{\xi}\mathrm{Tr}\left[(\partial_\mu\hat{A}^\mu)(\partial_\nu\hat{A}^\nu)\right], \quad (12.4)$$

and

$$\begin{aligned} L^{\mathrm{ct}}_{\mathrm{QCD}} &= (Z_3-1)\left[-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a}\right] + (Z_2-1)\bar{\psi}i\hat{\partial}\psi + (Z_c-1)\bar{c}^a(-\partial^2)c^a \\ &+ g_s(Z_g Z_2 Z_3^{1/2}-1)A^a_\mu\bar{\psi}\gamma^\mu t^a\psi - g_s(Z_g Z_3^{3/2}-1)f^{abc}(\partial_\mu A^a_\nu)A^b_\mu A^c_\nu \\ &- \frac{g_s^2}{4}(Z_g^2 Z_3^2-1)f^{eab}A^a_\mu A^b_\nu\, f^{ecd}A^c_\mu A^d_\nu - g_s(Z_g Z_2^{1/2}Z_c-1)f^{abc}\partial^\mu(A^b_\mu c^c). \end{aligned} \quad (12.5)$$

The renormalization constants that appear in the counter-term Lagrangian are defined as follows[1]

$$g_0 = Z_g g_s, \quad A^{a,0}_\mu = Z_3^{1/2}A^a_\mu, \quad \psi^{(0)} = Z_2^{1/2}\psi, \quad c^a_0 = Z_c^{1/2}c^a, \quad \xi_0 = Z_3\xi. \quad (12.6)$$

[1] Note that the relation between g_0 and g_s shown below will change, c.f. Eq. (12.29).

The last equation allows us to keep the gauge fixing term in the QCD Lagrangian but avoid having the corresponding term in the counter-term Lagrangian. Below we will explain why this is a sensible thing to do.

We also see from these equations that we only have four renormalization constants in our disposal. However, the number of divergent Green's function is significantly larger. To find them, we will compute the superficial degree of divergence of the various Green's functions in QCD.

Suppose, we have an L-loop diagram with with L_q internal quark lines, L_g internal boson lines, V_{3g} three-gluon vertices and V_q $q\bar{q}g$ vertices. The artificial degree of divergence is

$$D = 4L - L_q - 2L_g + V_{3g}, \tag{12.7}$$

since the three-gluon vertex is linear in the loop momentum. In addition, the following equations are satisfied

$$\begin{aligned} &L = L_g + L_q - V_{3g} - V_q + 1, \\ &V_q = L_q + \frac{1}{2}N_q, \\ &3V_{3g} + V_q = 2L_g + N_g. \end{aligned} \tag{12.8}$$

Using these equations (start with replacing L with the r.h.s.), we find

$$D = 4 - \frac{3}{2}N_q - N_g. \tag{12.9}$$

We have not talked about ghosts where a slightly more careful analysis needs to be performed. The net effect is that the number of external ghost lines has to be added to N_q in the above formula to make it correct, and the final result reads

$$D = 4 - \frac{3}{2}(N_q + N_{\mathrm{gh}}) - N_g. \tag{12.10}$$

This equation implies that Green's functions with up to four external gluons, one gluon and $q\bar{q}$, one gluon and two ghosts, quark self-energy and ghost self-energy are divergent. These Green's functions are not independent; in fact they are related by the gauge invariance of the theory so that if the *regularized* theory remains invariant under gauge transformations (and it does, if we use dimensional regularization) then proper relations between Green's functions keep being maintained. Hence, we expect that if we determine the necessary renormalization constants from some Green's functions, other divergent Green's functions will also become finite.

In the previous lecture we have computed sufficient number of Green's functions to determine *all* the renormalization constants. However, there is one aspect of this procedure that is different in comparison to the earlier discussion of renormalization. We will discuss it taking the quark self-energy as an example. The result of the one-loop computation is given in Eq. (**??**). The counter-term contribution is

$$i\hat{p}\,(Z_2 - 1)\,\delta_{ij}. \tag{12.11}$$

Combining the two equations, we find

$$i\hat{p}\delta_{ij}\left[\frac{g_s^2\Gamma(1+\epsilon)}{(4\pi)^{d/2}\epsilon}\,\delta_{ij}C_F + (Z_2 - 1)\right]. \tag{12.12}$$

In our earlier discussion, we have determined the renormalization constants from the requirement that Green's function assume certain values for particular values of external momenta; these momenta values were motivated by physics considerations. For example, we have required that the electron self-energy and its first derivative vanish at electron's mass shell. Although we can still impose similar requirement on the quark self-energy, it would probably not make sense physically because quarks are confined and cannot be observed as free particles. For this reason, instead of emphasizing the physical meaning of the renormalization, we can view it as a technical device whose aim is to give us an opportunity to conveniently compute finite Green's functions in QCD and to express them through *some* renormalized parameters.

A scheme that takes this idea to an extreme is known as the *minimal subtraction* scheme. In the original version of this scheme the counter-terms were defined in such a way that they remove the $1/\epsilon$ terms in the Laurant expansion of Green's functions in ϵ. Later, it was realized that such an expansion always generates additional finite terms associated with the Euler constant in the expansion of $\Gamma(1+\epsilon) \approx 1 - \gamma_E\epsilon$ and $\ln(4\pi)$ that originates from the expansion of $(4\pi)^{-\epsilon}$ terms. It was agreed to include these terms into the counter-terms together with the $1/\epsilon$ terms. Hence, we attempt to make an expression in Eq. (12.12) ϵ-finite by choosing

$$Z_2 = 1 - \frac{C_F g_s^2 e^{-\gamma_E\epsilon}}{(4\pi)^{d/2}\epsilon} + \ldots \tag{12.13}$$

There is an additional consideration that is actually useful. Similar to the entire theory, the bare coupling is defined in d dimensions. In four dimensions, the QCD coupling g_s is dimensionless. For this reason we would like to choose the renormalized coupling g_s to be dimensionless at any d since in this case the dimensionality of the *renormalized* coupling does not change if $\epsilon \to 0$ limit is taken.

To see why this is useful, consider an example of a renormalized quark self-energy. It reads

$$\begin{aligned} i\Sigma_q^{\rm ren} &= \frac{i g_s^2 \Gamma(1+\epsilon)}{(4\pi)^{d/2}(-p^2)^\epsilon\,\epsilon}\delta_{ij} C_F\ \hat{p}\ \left(1+\mathcal{O}(\epsilon)\right) + i\hat{p}\delta_{ij}(Z_2-1) \\ &= -i\delta_{ij}\hat{p}C_F\frac{g_s^2}{(4\pi)^2}\ln(-p^2)+..., \end{aligned} \tag{12.14}$$

where in the last step we used Eq. (12.13). Since it is very annoying to work with quantities where logarithms of dimensionful quantities appear, it is useful to redefine the coupling constant in such a way that its dependence on the dimension is taken care of by an additional mass paramenter that we will refer to as μ. It is easy to see that this is accomplished if we change

$$g_s \to g_s \mu^\epsilon, \tag{12.15}$$

in the QCD and the counter-term Lagrangians. Similarly, to account for $\overline{\rm MS}$ renormalizaion automatically, we include $e^{-\gamma_E/2\epsilon}/(4\pi)^{-\epsilon/2}$ into the definition of g_s as well. Hence, the ultimate replacement to make everywhere in $L_{\rm QCD}$ and $L_{\rm QCD}^{\rm ct}$ is

$$g_s \to \frac{g_s \mu^\epsilon e^{\epsilon\gamma_E/2}}{(4\pi)^{\epsilon/2}}. \tag{12.16}$$

Then, the un-renormalzied self-energy becomes

$$i\Sigma_q(p) = i\delta_{ij}\hat{p}C_F\frac{g_s^2\mu^{2\epsilon}\Gamma(1+\epsilon)e^{\gamma_E\epsilon}(-p^2)^{-\epsilon}}{(4\pi)^2}\ \frac{1}{\epsilon}\left(1+\mathcal{O}(\epsilon)\right), \tag{12.17}$$

the factor Z_2

$$Z_2 = 1 - \frac{C_F g_s^2}{(4\pi)^2\epsilon} + \mathcal{O}(g_s^4), \tag{12.18}$$

and the expansion in ϵ of the renormalized self-energy returns

$$i\Sigma_q^{\text{ren}}(p) = -i\delta_{ij}\ \hat{p}\ C_F \frac{g_s^2}{(4\pi)^2} \ln\left(\frac{-p^2}{\mu^2}\right) + \ldots \tag{12.19}$$

Note that nothing changes in the renormalziation constant Z_2 since it is supposed to contain the $1/\epsilon$ poles only.

As the next step, we obtain the (gluon) wave-function renormalization constant Z_3 using the gluon vacuum polarization. Again, we determine these constants in the $\overline{\text{MS}}$ renormalization scheme where we only require that $1/\epsilon$ poles are cancelled. Since we do not have a gauge-fixing term in the counter-term Lagrangian (thanks to the tuned renormalization of ξ and A_μ), the counter-term contribution is

$$(Z_3 - 1)(-i)(p^2 g_{\mu\nu} - p^\mu p^\nu)\delta^{ab}. \tag{12.20}$$

The gluon vacuum polarization is given in Eq. (**??**); we see that that result is indeed proportional to the transversal Lorentz structure $p^2 g_{\mu\nu} - p_\mu p_\nu$ which justifies our choice for the renormalization of the gauge parameter.

We present the divergent part of the gluon propagator here one more time, for completeness

$$i\Pi^{ab,\mu\nu} \approx \frac{i g_s^2 \Gamma(1+\epsilon)}{(4\pi)^{d/2}\epsilon} \delta^{ab} \left(g_{\mu\nu}p^2 - p_\mu p_\nu\right) \left[\frac{5}{3}C_A - \frac{4}{3}n_f T_R\right]. \tag{12.21}$$

Then

$$i(p^2 g_{\mu\nu} - p^\mu p^\nu)\delta^{ab} \left[1 - Z_3 + \frac{g_s^2 \Gamma(1+\epsilon)}{(4\pi)^{d/2}\epsilon} \left(\frac{5}{3}C_A - \frac{4}{3}n_f T_R\right)\right]. \tag{12.22}$$

We find

$$Z_3 = 1 + \frac{g_s^2}{(4\pi)^2\epsilon} \left(\frac{5}{3}C_A - \frac{4}{3}n_f T_R\right). \tag{12.23}$$

We then use the computation of the $q\bar{q}g$ interaction vertex in the previous section to determine the strong coupling renormalization constant. The divergent contribution to the quark gluon interaction vertex is given in Eq. (**??**) and we repeat it here for convenience

$$\hat{V}^\mu_{a,ij} = i \frac{g_s \mu^\epsilon e^{\gamma_E \epsilon/2}}{(4\pi)^{\epsilon/2}} t^a_{ij} \gamma^\mu \left(C_F + C_A\right) \frac{g_s^2 \mu^{2\epsilon} e^{\gamma_E \epsilon} \Gamma(1+\epsilon)}{(4\pi)^2 \epsilon}. \tag{12.24}$$

The counter-term contribution to quark gluon interaction vertex is

$$i g_s t^a_{ij}(\left(Z_g Z_2 Z_3^{1/2} - 1 \right). \tag{12.25}$$

All the renormalization constants in that expression except for Z_g have been already computed. Hence, requiring that the sum of the vertex correction and the counter-term does not contain $1/\epsilon$ divergences, we find

$$Z_g Z_2 Z_3^{1/2} - 1 + (C_F + C_A) \frac{g_s^2}{(4\pi)^2 \epsilon} = \mathcal{O}(\epsilon^0). \tag{12.26}$$

To first order in g_s^2, we can linearize this formula, use Eqs. (**??**) and (12.23) to compute $1 - Z_2$ and $1 - Z_3^{1/2}$ and determine Z_g

$$Z_g = 1 + (1 - Z_2) + (1 - Z_3^{1/2}) - (C_F + C_A) \frac{g_s^2 e^{-\gamma_E \epsilon}}{(4\pi)^{d/2} \epsilon}. \tag{12.27}$$

We find

$$Z_g = 1 + \frac{g_s^2}{(4\pi)^2 \epsilon} \left(-\frac{11}{6} C_A + \frac{2}{3} n_f T_R \right). \tag{12.28}$$

We will now discuss a very important consequence of Eq. (12.28). First, we note that it implies the following relation between the bare and the renormalized coupling constants

$$g_0 = g_s \, \frac{e^{\gamma_E \epsilon/2}}{(4\pi)^{\epsilon/2}} \, \mu^\epsilon Z_g. \tag{12.29}$$

It is technically more convenient to define an analog of the QED fine structure constant, i.e.

$$\alpha_s = \frac{g_s^2}{4\pi}. \tag{12.30}$$

Then

$$\alpha_s^{(0)} = \alpha_s \, \frac{e^{\gamma_E \epsilon}}{(4\pi)^\epsilon} \, \mu^{2\epsilon} \, Z_{\alpha_s}, \tag{12.31}$$

and Z_{α_s} is Z_g^2.

One can make the following observation about the structure of Eqs. (12.29) and 12.31). The left-hand sides of these equations contain bare coupling constants which are independent of μ. The right-hand sides of these equations have terms that explicitly depend on μ and terms where μ-dependencies should appear implicitly to compensate for explicit μ-dependence. We can uncover the rules that govern these implicit dependencies by differentiating both sides of e.g. Eq. (12.31) with respect to μ.

We find

$$0 = \mu \frac{\mathrm{d}\alpha_s}{\mathrm{d}\mu} \mu^{2\epsilon}\, Z_{\alpha_s} + 2\epsilon \alpha_s \mu^{2\epsilon}\, Z_{\alpha_s} + \alpha_s \mu^{2\epsilon}\, \frac{\partial Z_{\alpha_s}}{\partial \alpha_s} \mu \frac{\mathrm{d}\alpha_s}{\mathrm{d}\mu}, \tag{12.32}$$

where we made use of the fact that Z_{α_s} only depends on the strong coupling constant and does not contain any other terms that contain μ-dependencies.

It follows that

$$\mu \frac{\mathrm{d}\alpha_s}{\mathrm{d}\mu} = \frac{-2\epsilon\ \alpha_s}{1 + \alpha_s \frac{1}{Z_{\alpha_s}} \frac{\partial Z_{\alpha_s}}{\partial \alpha_s}}. \tag{12.33}$$

To proceed further, we need to write an Ansatz for the renormalization constants Z_{α_s}. Since this renormalization constant removes ultraviolet $1/\epsilon$ poles from the Green's functions, we write

$$Z_{\alpha_s} = 1 + \sum_{n=1}^{\infty} \frac{A_n}{\epsilon^n}. \tag{12.34}$$

The coefficients A_n are functions of the strong coupling constant α_s.

We then write equation Eq. (12.33) as

$$\mu \frac{\mathrm{d}\alpha_s}{\mathrm{d}\mu} \left(1 + \sum_{n=1}^{\infty} \frac{A_n + \alpha_s A_n'}{\epsilon^n} \right) = -2\epsilon \alpha_s \left(1 + \sum_{n=1}^{\infty} \frac{A_n}{\epsilon^n} \right), \tag{12.35}$$

where $A_n' = \partial A_n / \partial \alpha_s$.

We can solve this equation under the assumption that the coupling constant α_s has now singular $1/\epsilon$ contributions which is what we expect from the renormalized coupling. Then we find that the following set of equations must be fulfilled

$$\begin{aligned} &\mu \frac{\mathrm{d}\alpha_s}{\mathrm{d}\mu} = -2\epsilon \alpha_s + 2\alpha_s^2 A_1', \\ &A_2' = \alpha_s (A_1')^2 + A_1 A_1', \\ &A_3' = \alpha_s A_1' A_2' + A_2 A_1', \\ &A_4' = \alpha_s A_3' A_1' + A_3 A_1', \\ &\ldots \end{aligned} \tag{12.36}$$

To understand what these formulas imply, imagine that coefficients A_n are computed in perturbation theory. Then

$$A_1 = \sum_{n=0}^{\infty} a_{1n}\, \alpha_s^n. \tag{12.37}$$

It follows from Eq. (12.36) that

$$A_2' \sim \alpha_s, \tag{12.38}$$

so that A_2 contains terms $\mathcal{O}(\alpha_s^k)$ with $k \geq 2$. From the analysis of other equations, we find that A_n contains terms $\mathcal{O}(\alpha_s^k)$ with $k \geq n$. It is also important that A_n, $n > 1$, can be fully determined once A_1 is known. Hence, full information about the dependence of the coupling constant α_s on μ is contained in the single $1/\epsilon$ pole of Z_{α_s}.

We now return to the first equation in Eq. (12.36). This equation describes the depenendence of the strong coupling constant on the renormalization scale μ. We write this equation as

$$\mu \frac{\mathrm{d}\alpha_s}{\mathrm{d}\mu} = \beta(\alpha_s), \tag{12.39}$$

where

$$\beta(\alpha_s) = -2\epsilon\alpha_s + 2\alpha_s^2 A_1' \tag{12.40}$$

is called the (QCD) "beta"-function. Setting $\epsilon = 0$, we find

$$\beta(\alpha_s) = 2a_{11}\alpha_s^2 + 4a_{12}\alpha_s^3 + 6a_{13}\alpha_s^4 + \mathcal{O}(\alpha_s^5). \tag{12.41}$$

Assuming that the perturbative expansion is valid, we can truncate the above equation, nelgecting all the terms beyond the leading and find

$$\mu \frac{\mathrm{d}\alpha_s}{\mathrm{d}\mu} = 2a_{11}\alpha_s^2. \tag{12.42}$$

This is a differential equation whose solution reads

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 - a_{11}\alpha_s(\mu_0)\ln\frac{\mu^2}{\mu_0^2}}. \tag{12.43}$$

The meaning of this equation is clear. We need to provide the coupling $\alpha_s(\mu_0)$; after that the above equation allows us to compute the coupling at other values of μ.

We can easily extract a_{11} and the leading order QCD β-function from the result in Eq. (12.28). We find

$$a_{11} = \frac{\alpha_s}{2\pi}\left(-\frac{11}{6} + \frac{2}{3}T_R n_F\right) \tag{12.44}$$

and

$$\beta(\alpha_s) = 2\alpha_s^2 a_{11} = -\frac{\alpha_s^2}{2\pi}\left(\frac{11}{3}C_A - \frac{4}{3}T_R N_f\right) + \mathcal{O}(\alpha_s^3). \tag{12.45}$$

where N_f is the number of quark spiecies and $T_R = 1/2$.

Note that $\beta(\alpha_s) < 0$ in QCD because the term with C_A gives a larger contribution than the term with N_f but it would have been the other way around in QED where C_A must be set to zero. The sign of the β-function is important. Indeed, it follows from Eq. (12.39) that, for *negative* $\beta(\alpha_s)$ the coupling becomes smaller (larger) for larger values of μ. Theories with negative β functions are called *asymptotically-free theories*. QCD is an asymptotically free theory because the number of quarks is not too large to compensate the effect of gluons.

It is instructive to go back to Eq. (12.46) and write it as

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 + b_0 \frac{\alpha_s(\mu_0)}{2\pi} \ln \frac{\mu^2}{\mu_0^2}}, \tag{12.46}$$

where

$$b_0 = \frac{11}{6}C_A - \frac{2T_R}{3}N_f. \tag{12.47}$$

It is easy to see that this equation can be written as follows

$$\alpha_s(\mu) = \frac{2\pi}{b_0 \ln \frac{\mu^2}{\Lambda_{\rm QCD}^2}}, \tag{12.48}$$

where

$$\Lambda_{\rm QCD} = \mu_0 e^{-\frac{\pi}{\alpha_s(\mu_0) b_0}}. \tag{12.49}$$

An immediate consequence of this equation is that the coupling constant becomes large (infinite) at a scale $\mu = \Lambda_{\rm QCD}$. This scale is the *non-perturbative* scale of strong interactions and is a true parameter of QCD. For energy scales that exceed $\Lambda_{\rm QCD}$, α_s is small and perturbative computations are valid (except that quarks and gluons do not exist so the meaning of such non-perturbative computations has to be put into the right context). For energy scales comparable to $\Lambda_{\rm QCD}$, non-abelian interactions become strong and one cannot study them using perturbation theory.

To estimate $\Lambda_{\rm QCD}$ we need to know the value of the strong coupling constant α_s at some point. A convenient reference place is $mu = M_Z \approx$ 90 GeV, which is the mass of the Z-boson where α_s is measured to be $\alpha_s(M_Z) \approx 0.12$. Taking $N_f = 4$ and $C_A = 3$ gives $b_0 = 4.2$ and we find

$$\Lambda_{\rm QCD} = 0.16 \text{ GeV}. \tag{12.50}$$