TTP2
Lecture 13

Kirill Melnikov
TTP KIT
January 9, 2024

13 Renormalization group

We have seen that we can derive an interesting equation that describes the dependence of the QCD coupling constant on the renormalization scale. This equation follows from an observation that renormalization relates bare quantities and renormalized quantities. Since renormalized quantities (couplings, Green's functions, etc.) depend on a parameter or a set of parameters that uniquely determine the renormalization prescription, whereas bare quantities do not depend on them, the dependence of the renormalized quantities on these parameters cannot be arbitrary. These dependences are described by a set of partial differential equations that are known as the *renormalization group equations*.

It is particularly simple to understand and derive the renormalization group equations if a minimal subtraction $\overline{\rm MS}$ scheme is used. To this end, consider a Green's function in a particular theory (say QCD with massless quarks and gluons, for definiteness) that has n_q quark and anti-quark and n_g gluon fiels, i.e.

$$G_0(x_1, x_2, ...y_1, ..., y_q) = \langle 0|T\psi_0(x_1)...\psi_0(x_{n_q})\ A_0(y_1)...A_0(y_{n_g})|0\rangle. \quad (13.1)$$

Note that we do not distinguish between ψ and $\bar{\psi}$ fields in the above formula; furthermore, we suppressed all Lorentz indices of the gluon fields. The fields in the above equation are considered to be unrenormalized (bare) fields.

Replacing unrenormalized with renormalized fields, we find

$$G_0(x_1, x_2, ...y_1, ..., y_{n_q}) = Z_2^{n_q/2} Z_3^{n_g/2} G_R(x_1, x_2, ..y_{n_g}). \quad (13.2)$$

Instead of working with the position space Green's functions, we switch to the momentum space and write

$$G_0(\{p\}) = Z_2^{n_q/2} Z_3^{n_g/2} G_R(\{p\}). \quad (13.3)$$

Imagine that we work in the $\overline{\rm MS}$-scheme. Then, similar to the renormalization condition for the coupling constant, this equation is peculiar in that the left hand side does not depend on the renormalization parameter μ whereas the various quantities on the right hand side do depend on it. Hence, differentiating both sides of the equation with μ gives

$$\mu \frac{\rm d}{{\rm d}\mu}\ G_0(\{p\}) = 0 \quad \leftrightarrow \quad \mu \frac{\rm d}{{\rm d}\mu} \left[Z_2^{(n_q/2)} Z_3^{(n_g/2)}\ G_R(\{p\}) \right] = 0. \quad (13.4)$$

To proceed further, we rewrite this last equation as follows

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} G_R(\{p\}) + \left(\frac{n_q}{2}\gamma_q + \frac{n_g}{2}\gamma_g \right) G_R(\{p\}) = 0, \tag{13.5}$$

where

$$\gamma_{q,g} = \frac{1}{Z_{q,g}} \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \, Z_{q,g} = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \, \ln Z_{q,g} \tag{13.6}$$

These quantities are called *anomalous dimensions*, for the reason that will become clear shortly.

Since G_R is an ϵ-finite Green's function, anomalous dimensions $\gamma_{q,g}$ should also have finite limits as $\epsilon \to 0$. Since we work in the $\overline{\mathrm{MS}}$ renormalization scheme, we can compute them easily. Indeed, in this scheme renormalization constant depend on μ *only* through the μ-dependence of the coupling constant. Hence, we write ($Z = Z_{q,g}$)

$$Z = 1 + \frac{A_1}{\epsilon} + \frac{A_2}{\epsilon^2} + \ldots \frac{A_n}{\epsilon^n} + \ldots, \tag{13.7}$$

where A_n are series in $\alpha_s(\mu)$. Then,

$$\gamma = \frac{1}{1 + \frac{A_1}{\epsilon} + \frac{A_2}{\epsilon^2} + ..} \left[\frac{A_1'}{\epsilon} + \frac{A_2'}{\epsilon^2} + \ldots \frac{A_n'}{\epsilon^n} + \ldots \right] \mu \frac{\mathrm{d}\alpha_s(\mu)}{\mathrm{d}\mu}, \tag{13.8}$$

where $A_n' = \partial A_i / \partial \alpha_s$. The QCD coupling constant reads

$$\mu \frac{\mathrm{d}\alpha_s(\mu)}{\mathrm{d}\mu} = -2\epsilon\alpha_s + \beta(\alpha_s). \tag{13.9}$$

Using this result in Eq. (13.8), we find

$$\gamma(\alpha_s) = -2\alpha_s \frac{\partial A_1(\alpha_s)}{\partial \alpha_s}, \tag{13.10}$$

and Eq. (13.8) can be used to predict higher $1/\epsilon$ poles that appear in the wave function renormalization constants.

We return to Eq. (13.5). In variance with the renormalization constants in the $\overline{\mathrm{MS}}$ scheme , the renormalized Green's function G_R *does depend* on the parameter μ explicitly in addition to its dependence on $\alpha_s(\mu)$. Hence,

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} G_R(\{p\}) = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} G_R(\{p\}, \mu, \alpha_s(\mu)) = \mu \frac{\partial}{\partial \mu} G_R + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} G_R. \tag{13.11}$$

Therefore, we obtain a partial *homogenious* differential equation

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(\alpha_s)\frac{\partial}{\partial\alpha_s}+\gamma_q(\alpha_s)\frac{n_q}{2}+\gamma_g(\alpha_s)\frac{n_g}{2}\right)G_R(\{p\},\mu,\alpha_s(\mu))=0, \tag{13.12}$$

that relates the dependencies of the Green's function on μ and on $\alpha_s(\mu)$.

We will now explain how to find solutions to the above equation that contain important physics information. Suppose we know the Green's function $G_R(p,\mu,\alpha_s(\mu))$ for some $\{p\}$ and μ. We would like to determine the Green's function G_R for the same value of μ but for values of momenta re-scaled by a parameter λ, i.e. $G_R(\{\lambda p\},\alpha_s,\mu)$. Each Green's function has a well-defined canonical mass dimension which evaluates to $n=3/2n_q+n_g+4-4n_g-4n_q$. We extract this mass dimension and write

$$G_R(\{p\},\mu,\alpha_s(\mu))=\mu^n\widetilde{G}_R\left(\frac{\{p\}}{\mu},\alpha_s(\mu)\right). \tag{13.13}$$

Since

$$\begin{aligned}&\mu\frac{\partial}{\partial\mu}G_R(\{\lambda p\},\mu,\alpha_s(\mu))=\\&\mu^n\left[n\widetilde{G}_R\left(\frac{\{\lambda p\}}{\mu},\alpha_s(\mu)\right)+\mu\frac{\partial}{\partial\mu}\widetilde{G}_R\left(\frac{\{\lambda p\}}{\mu},\alpha_s(\mu)\right)\right],\end{aligned} \tag{13.14}$$

and

$$\left(\mu\frac{\partial}{\partial\mu}+\lambda\frac{\partial}{\partial\lambda}\right)\widetilde{G}_R\left(\frac{\{\lambda p\}}{\mu},\alpha_s\right)=0, \tag{13.15}$$

$\widetilde{G}_R$ satisfies the following equation

$$\left(\lambda\frac{\partial}{\partial\lambda}-\beta(\alpha_s)\frac{\partial}{\partial\alpha_s}-\gamma(\alpha_s)-n\right)\widetilde{G}_R\left(\frac{\{\lambda p\}}{\mu},\alpha_s\right)=0, \tag{13.16}$$

where

$$\gamma=\gamma_q(\alpha_s)\frac{n_q}{2}+\gamma_g(\alpha_s)\frac{n_g}{2}. \tag{13.17}$$

Note that the μ-dependence of α_s at this point is not relevant for solving Eq. (13.16) because this equation does not involve explicit μ-derivatives. This means that, as far as solving Eq. (13.16) is concerned, $\alpha_s(\mu)$ there should be considered as a given constant.

To solve Eq. (13.16), we remove terms without derivatives from there by writing

$$\widetilde{G}_R = \lambda^n e^{-\int\limits_0^{\alpha_s} \mathrm{d}x\, \frac{\gamma(x)}{\beta(x)}} F\left(\frac{\{\lambda p\}}{\mu}, \alpha_s\right). \tag{13.18}$$

The function F satisfies the following differential equation

$$\left(\lambda\frac{\partial}{\partial\lambda} - \beta(\alpha_s)\frac{\partial}{\partial\alpha_s}\right) F\left(\frac{\{\lambda p\}}{\mu}, \alpha_s\right) = 0. \tag{13.19}$$

To solve this equation we note that any function that depends on λ and α_s in the combination

$$\Psi(\lambda, \alpha_s) = \ln\lambda + \int^{\alpha_s} \frac{\mathrm{d}x}{\beta_x}, \tag{13.20}$$

is the solution of Eq. (13.19). To make this quantity Ψ look "nicer", we define an "auxiliary" quantity $\bar{\alpha}_s(\lambda, \alpha_s)$ so that

$$\Psi(\lambda, \alpha_s) = \int^{\bar{\alpha}_s(\lambda,\alpha_s)} \frac{\mathrm{d}x}{\beta(x)}. \tag{13.21}$$

Combining the two previous equations, we find

$$\int\limits_{\alpha_s}^{\bar{\alpha}_s(\lambda,\alpha_s)} \frac{\mathrm{d}x}{\beta(x)} = \ln\lambda. \tag{13.22}$$

The function $\bar{\alpha}_s(\lambda, \alpha_s)$ is *the running coupling constant* that we have seen before. Indeed, taking a derivative of Eq. (13.22) w.r.t. λ, we find

$$\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\bar{\alpha}_s = \beta(\bar{\alpha}_s), \tag{13.23}$$

and the boundary condition for this equation is $\bar{\alpha}_s = \alpha_s$ at $\lambda = 1$.

Hence, we conclude that the solution of Eq. (13.19) can be written as

$$F\left(\frac{\{\lambda p\}}{\mu}, \alpha_s\right) = F\left(\frac{\{p\}}{\mu}, \bar{\alpha}_s(\lambda, \alpha_s)\right). \tag{13.24}$$

Note that since $\bar{\alpha}_s(\lambda, \alpha_s) = \alpha_s$ at $\lambda = 1$, the above equation is consistent with the boundary condition.

Finally, we can combine the intermediate steps and write the following expression for the Green's function

$$G_R(\{\lambda p\}, \mu, \alpha_s) = \lambda^n e^{\int\limits_{\alpha_s}^{\bar{\alpha}_s(\lambda,\alpha_s)} \frac{\mathrm{d}x\, \gamma(x)}{\beta(x)}} G_R(p, \mu, \bar{\alpha}_s(\lambda, \alpha_s)). \tag{13.25}$$

We will now discuss the meaning of the above equation. Suppose that $\{p\}$, μ and $\alpha_s(\mu)$ are fixed and we are interested in computing the Green's function on the left hand side for some (large) value of λ. This corresponds to increasing all momenta in the Green's function by the same amount. Also we imagine that we deal with QCD, so that the running coupling $\overline{\alpha}_s$ becomes *small* as λ increases. Since the coupling constant becomes smaller and smaller, the Green's function on the right-hand side of Eq. (13.25) can be well approximated by the free Green's function. Then we can write

$$G_R(\{\lambda p\}, \mu, \alpha_s) \approx e^{\int\limits_{\alpha_s}^{\bar{\alpha}_s(\lambda,\alpha_s)} \frac{\mathrm{d}x\, \gamma(x)}{\beta(x)}} G_R^{\mathrm{tree}}(\{\lambda p\}). \tag{13.26}$$

Suppose that we can compute $\beta(x)$ and $\gamma(x)$ in perturbation theory. Then,

$$\beta(x) = -\frac{x^2}{\pi} b_0, \quad \gamma(x) = -\gamma_0 \frac{x}{\pi}. \tag{13.27}$$

As the result,

$$e^{\int\limits_{\alpha_s}^{\bar{\alpha}_s(\lambda,\alpha_s)} \frac{\mathrm{d}x\, \gamma(x)}{\beta(x)}} \approx e^{\frac{\gamma_0}{b_0} \ln \frac{\bar{\alpha}_s}{\alpha_s}} \approx e^{-\frac{\gamma_0}{b_0} \ln\left(1 + \frac{\alpha_s}{2\pi} b_0 \ln \lambda^2\right)}. \tag{13.28}$$

Hence,

$$G_R(\{\lambda p\}, \alpha_s) = e^{-\frac{\gamma_0}{b_0} \ln\left(1 + \frac{\alpha_s}{2\pi} b_0 \ln \lambda^2\right)} G_R^{\mathrm{tree}}(\{\lambda p\}). \tag{13.29}$$

It is useful to think about the meaning of this equation from the point of view of ordinary perturbation theory where we compute the Green's function order by order in the perturbative expansion. To see the connection, we imagine that α_s is small enough so that expansion of Eq. (13.29) becomes possible. We find

$$G_R(\{\lambda p\}, \alpha_s) = \sum_{n=0}^{\infty} a_n \alpha_s^n L^n \; G_R^{\mathrm{tree}}(\{\lambda p\}), \tag{13.30}$$

where $L = \ln \lambda^2$. Hence, renormalization group equations *re-sum* contributions to the perturbative expansion of Green's functions which contain one power of $\ln \lambda$ for one power of α_s, as the leading approximation. If $\alpha_s \ln \lambda^2 \sim \alpha_s$, these terms are no different from all other terms in the perturbative expansion; in this case, improvements related to the renormalization group analysis make little sense. However, if we deal with the case $\lambda \gg 1$, so that $\alpha_s \ln \lambda^2 \geq 1$ but $\alpha_s \ll 1$, then perturbative expansion in α_s is not valid but the renormalization group equations allow us to re-sum offending terms to all orders in α_s.