TTP2
Lecture 14

Kirill Melnikov
TTP KIT
January 11, 2024

14 Chiral anomalies

When we say "anomaly" when talking about quantum field theory, we usually mean *a symmetry of a theory that exists at the level of a classical field theory but disappears at a quantum level*. Since gauge symmetries are essential for the consistency of any gauge quantum field theory (e.g. they ensure that a limited number of renormalization constants suffice to make a large number of Green's functions finite), it is important to understand how symmetries become anomalous and make sure this does not happen to gauge symmetries.

It is also interesting that it is possible to make statements about anomalies that are *exact*; as such, they can be used to make (some) first-principles statements in strongly interacting theories which will not be possible otherwise. A classical example of an anomaly is the anomaly of the *singlet axial vector current* that we now consider.

Consider a theory of a massless fermion ψ interacting with an abelian gauge field

$$\mathcal{L} = \bar{\psi} i \hat{D} \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad D_\mu = \partial_\mu + ieA_\mu. \tag{14.1}$$

The field ψ is split into left- and right components

$$\psi_{L,R} = \frac{1 \pm \gamma_5}{2} \psi, \tag{14.2}$$

and the Lagrangian becomes

$$\mathcal{L} = \bar{\psi}_L i \hat{D} \psi_L + \bar{\psi}_R i \hat{D} \psi_R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{14.3}$$

We can separately change phases of left and right components of the fermion field ψ without changing the Lagrangian; by Noether's theorem this means that both left and right currents

$$j_L^\mu = \bar{\psi}_L \gamma^\mu \psi_L, \quad \text{and} \quad j_R^\mu = \bar{\psi}_R \gamma^\mu \psi_R, \tag{14.4}$$

are conserved

$$\partial_\mu j_{L,R}^\mu = 0. \tag{14.5}$$

We can take the linear combinations of these currents and form vector and axial currents

$$J^\mu = j_L^\mu + j_R^\mu = \bar{\psi} \gamma_\mu \psi, \quad J_5^\mu = j_L^\mu - j_R^\mu = \bar{\psi} \gamma^\mu \gamma_5 \psi. \tag{14.6}$$

As the consequence of Eq.(14.5) these currents are also conserved

$$\partial_\mu J^\mu = 0, \qquad \partial_\mu J_5^\mu = 0. \tag{14.7}$$

Note that the current J^μ couples to the gauge field A^μ whereas the axial current J_5^μ does not couple to the gauge field.

We will now show that *it is only possible to satisfy one (out of two) equations in Eq.(14.7) once quantum effects are taken into account. Interestingly, we can choose which equation remains valid and which one is violated.* One of the two symmetries that led to Eq. (14.7) and which we choose to give up at the quantum level is called "an anomalous symmetry".

Since it is inconvenient to work with operators (currents are operators), we will consider a matrix element of the axial current between the vacuum state and the two photons and write it as

$$\langle \gamma(k_1)\gamma(k_2)|J_{5,\mu}|0\rangle = -e^2 \epsilon_1^{\alpha,*}\epsilon_2^{\beta,*} T_{\mu;\alpha\beta}(k_1,k_2), \tag{14.8}$$

factoring out the coupling constant e and the polarization vectors of the two photons. Tensor $T_{\mu;\alpha\beta}(k_1,k_2)$ is obtained from the vacuum expectation value of the time-ordered product

$$T_{\mu;\alpha\beta}(k_1,k_2) = \int \mathrm{d}^4x \mathrm{d}^4y \; e^{ik_1x} e^{ik_2y} \;\; \langle 0|T J_\alpha(x) J_\beta(y) J_{5\mu}(0)|0\rangle. \tag{14.9}$$

The current conservation Eq.(14.7) implies the transversality of $T_{\mu;\alpha\beta}$ w.r.t. *all* of its three indices

$$T_{\mu;\alpha\beta}k_1^\alpha = 0, \quad T_{\mu;\alpha\beta}k_2^\beta = 0 \quad q^\mu T_{\mu;\alpha\beta} = 0. \tag{14.10}$$

Here $q = k_1 + k_2$.

To compute $T_{\mu;\alpha\beta}$, we need to calculate two diagrams shown in Fig. 1. We write

$$T_{\mu;\alpha\beta} = i\int \frac{\mathrm{d}^4 l}{(2\pi)^4} \mathrm{Tr}\left[\gamma_\mu\gamma_5 \frac{1}{\hat{l}-\hat{q}}\gamma_\alpha \frac{1}{\hat{l}-\hat{k}_2}\gamma_\beta \frac{1}{\hat{l}} + \gamma_\mu\gamma_5 \frac{1}{\hat{l}-\hat{q}}\gamma_\beta \frac{1}{\hat{l}-\hat{k}_1}\gamma_\alpha \frac{1}{\hat{l}}\right]. \tag{14.11}$$

An important feature of this expression is that it is *ill-defined because of a linear divergence of the two contributions at large values of l*. This is not a

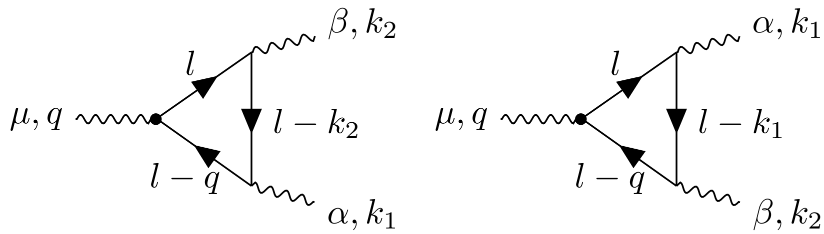

Figure 1: Two graphs that contribute to the correlator of an axial current and two vector currents.

problem of this particular Green's function since many quantities in QFT are ill-defined. What we need to do therefore is to introduce a regularization in Eq. (14.11), perform the computation to the end and then see if the dependence on the regulator can be removed. In what follows, we will first discuss a slightly different approach to this problem which will allow us to illustrate some interesting features.

A peculiar feature of *linearly-divergent unregularized integrals* that will play a crucial role in our discussion is that *a choice of a different momentum flow in such an integral may lead to different result*. To see this, imagine that we compute a difference of two identical integrals where the only difference is that the loop momentum is shifted by a constant vector in one of them. Hence, we write

$$I = \int \frac{d^4 l}{(2\pi)^4} \left[F(l) - F(l-p) \right]. \tag{14.12}$$

Normally, we would treat the two integrals separately, shift the loop momentum $l \to l + p$ in the second one and conclude that the result is zero. However, mathematically, such manipulations are only valid if $\int d^4 l/(2\pi)^4 \; F(l)$ converges (at least, it should not diverge faster than a logarithm.). If shifts are not allowed, we can compute a derivative of I with respect to p^μ. We write

$$\frac{\partial I}{\partial p^\mu} = \int \frac{d^4 l}{(2\pi)^4} \frac{\partial F(l-p)}{\partial l^\mu} \tag{14.13}$$

In this integral, the linear divergence is not present anymore, so we can shift the momentum and find

$$\frac{\partial\, I}{\partial p^\mu} = \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{\partial F(l)}{\partial l^\mu}. \tag{14.14}$$

We now have an integral of the total derivative and it can be computed using the Stokes theorem. To apply the Stokes theorem to these integrals, we peform the Wick rotation and write $l^0 = i l_E^0$, $\vec{l} \to \vec{l}_E$. Then

$$\frac{\partial F}{\partial l^\mu} = \frac{1}{i^{\delta_{\mu 0}}} \frac{\partial F}{\partial l_E^\mu}, \tag{14.15}$$

and $\mathrm{d}^4 l = i \mathrm{d}^4 l_E$. Hence,

$$\frac{\partial\, I}{\partial p^\mu} = i \int \frac{\mathrm{d}^4 l_E}{(2\pi)^4} \frac{1}{i^{\delta_{\mu 0}}} \frac{\partial F}{\partial l_E^\mu} = \lim_{|l_E| \to \infty} \frac{i}{(2\pi)^4} \frac{1}{i^{\delta_{\mu 0}}} \int\limits_{|l_E|} \mathrm{d}S_{3,\mu}\, F(l_E) \tag{14.16}$$

The surface of a three-dimensional sphere in four-dimensional space with the radius l_E scales as $|l_E|^3$; hence, if $F(l)$ scales as $1/l_E^3$, the integral I does not vanish and gives a finite contribution. Going back to the original definition of the integral I Eq. (14.12), $F(l) \sim l^{-3}$ implies that individual integrals in Eq.(14.12) diverge *linearly*.

It follows from Eq. (14.16) that to determine I we need to know the function $F(l)$ at large values of the loop momentum; we will assume that this dependence read

$$\lim_{l \to \infty} F(l) \approx \frac{a_\mu l^\mu}{(l^2)^2}. \tag{14.17}$$

We now perform the Wick rotation and write

$$\lim_{l \to \infty} F(l_E) \approx -\frac{a_{E,\mu} l_E^\mu}{(l_E^2)^2} = -\frac{a_E^\nu n_E^\nu}{l_E^3}, \tag{14.18}$$

where n_E^μ is a unit radial vector in the Eucledian four-dimensional space. Next, using $\mathrm{d}S_3^\mu = \mathrm{d}\Omega^{(4)} n_E^\mu l_E^3$ and $\Omega^{(4)} = 2\pi^2$, we find

$$\int\limits_{|l_E|} \mathrm{d}S_{3,\mu}\, F(l_E) = -\int \mathrm{d}\Omega^{(4)} n_E^\mu\, a_E^\nu n_E^\nu = -a_E^\nu \frac{\delta^{\mu\nu}}{4} \Omega^{(4)} = -a_E^\mu \frac{\Omega^{(4)}}{4}. \tag{14.19}$$

Since $a_E^\mu = i^{\delta_{\mu 0}} a^\mu$, we find

$$\frac{\partial\, I}{\partial p^\mu} = \frac{i}{(2\pi)^4}\frac{1}{i^{\delta_{\mu 0}}}\,(-1) i^{\delta_{\mu 0}} a^\mu \frac{\Omega^{(4)}}{4} = -\frac{i a_\mu}{32\pi^2}. \tag{14.20}$$

We can now integrate this expression using the fact that the integral I vanishes at $p = 0$. We therefore find

$$I = \int \frac{\mathrm{d}l^4}{(2\pi)^4}\left[F(l) - F(l-p)\right] = -\frac{i a_\mu p^\mu}{32\pi^2}. \tag{14.21}$$

We now return to to the computation of the correlator of three currents and check the behavior of the integrand at large values of the loop momentum. Consider the first diagram in Fig. 1. The integrand reads

$$\mathrm{Tr}\left[\gamma_\mu\gamma_5\frac{1}{\hat{l}-\hat{q}}\gamma_\alpha\frac{1}{\hat{l}-\hat{k}_2}\gamma_\beta\frac{1}{\hat{l}}\right] \approx \frac{1}{(l^2)^3}\mathrm{Tr}\left[\gamma_\mu\gamma_5\hat{l}\gamma_\alpha\hat{l}\gamma_\beta\hat{l}\right]. \tag{14.22}$$

To simplify this trace, we write

$$\hat{l}\gamma_\alpha\hat{l} = 2l_\alpha\hat{l} - \gamma_\alpha l^2. \tag{14.23}$$

Since

$$\mathrm{Tr}\left[\gamma_5\gamma_\mu\gamma_\alpha\gamma_\beta\gamma_\rho\right] = 4i\epsilon_{\mu\alpha\beta\rho}, \tag{14.24}$$

we find that

$$\mathrm{Tr}\left[\gamma_\mu\gamma_5\hat{l}\gamma_\beta\hat{l}\right] = 0, \tag{14.25}$$

so that the first term on the right hand side in Eq. (14.23) vanishes. We find

$$\mathrm{Tr}\left[\gamma_\mu\gamma_5\frac{1}{\hat{l}-\hat{q}}\gamma_\alpha\frac{1}{\hat{l}-\hat{k}_2}\gamma_\beta\frac{1}{\hat{l}}\right] \approx \frac{4i\epsilon_{\mu\alpha\beta\sigma}l^\sigma}{(l^2)^2}. \tag{14.26}$$

Hence, the "vector" a^μ that appears in the asymptotic formula of the function F (c.f. Eq. (14.17)) reads in this case

$$a_\sigma = 4i\epsilon_{\mu\alpha\beta\sigma} \tag{14.27}$$

Suppose that we would like to compute the difference in the contribution of the first diagram to $T_{\mu;\alpha\beta}$ due to different choices of the loop momentum. We find

$$T^{(1)}_{\mu;\alpha\beta}[l] - T^{(1)}_{\mu;\alpha\beta}[l-r_1] = -i\frac{\epsilon_{\mu\alpha\beta\sigma}r_1^\sigma}{8\pi^2}. \tag{14.28}$$

Performing a similar computation for the second diagram that contributes to $T_{\mu;\alpha\beta}$, we find

$$T^{(2)}_{\mu;\alpha\beta}[l] - T^{(2)}_{\mu;\alpha\beta}[l - r_2] = i\frac{\epsilon_{\mu\alpha\beta\sigma} r_2^{\sigma}}{8\pi^2}, \tag{14.29}$$

where the sign change is related to the fact that matrices γ^{α} and γ^{β} appear in a different order in the expression for trace of the second diagram.

This implies that the result for the correlator $T_{\mu;\alpha\beta}$ is ambiguous up to

$$T_{\mu;\alpha\beta}|_{\text{shift}} = T_{\mu;\alpha\beta} + \frac{i}{8\pi^2}\epsilon_{\mu\alpha\beta\delta}\left(r_1^{\delta} - r_2^{\delta}\right), \tag{14.30}$$

where $r_{1,2}$ reflect the freedom of momenta choices in the two diagrams.

Since we can shift the loop momenta in each of the two diagrams independently, we can ask *if it is possible to choose these shifts in such a way that the transversality conditions Eq. (14.10) for both vector and axial currents are satisfied*. The answer to this question is that it is in fact *not possible*.

To see how this conclusion is reached, we restrict the class of shifts that we apply. One of the symmetries that we would like to keep intact is the symmetry between the two photons, i.e. the Bose-symmetry. Hence, we require that we only consider correlators that satisfy the following property

$$T_{\mu;\alpha\beta}(k_1, k_2) = T_{\mu;\beta\alpha}(k_2, k_1). \tag{14.31}$$

Therefore, if we parameterize the shifts as $r_1 = a_1 k_1 + a_2 k_2$ and $r_2 = b_1 k_1 + b_2 k_2$, and require that shifts do not violate the Bose symmetry, we obtain the following condition.

$$a_1 - b_1 = b_2 - a_2. \tag{14.32}$$

The non-trivial solution occurs if we satisfy Eq.(14.32) by choosing $b_2 = a_1$ and $b_1 = a_2$. Then, the allowed shifts are

$$r_1 = a_1 k_1 + a_2 k_2, \quad r_2 = a_2 k_1 + a_1 k_2, \tag{14.33}$$

and Eq.(14.30) becomes

$$T_{\mu;\alpha\beta}|_{\text{shift}} = T_{\mu;\alpha\beta} + i\frac{(a_1 - a_2)}{8\pi^2}\epsilon_{\mu\alpha\beta\delta}(k_1^{\delta} - k_2^{\delta}). \tag{14.34}$$

The transversality conditions now read[1]

$$\begin{aligned} k_1^\alpha T_{\mu,\alpha\beta}|_{\rm shift} &= k_1^\alpha T_{\mu;\alpha\beta} + i\frac{(a_1-a_2)}{8\pi^2}\epsilon_{\mu\beta\alpha\sigma}k_1^\alpha k_2^\sigma, \\ q^\mu T_{\mu,\alpha\beta}|_{\rm shift} &= q^\mu T_{\mu;\alpha\beta} - i\frac{(a_1-a_2)}{4\pi^2}\epsilon_{\alpha\beta\mu\sigma}k_1^\mu k_2^\sigma. \end{aligned} \tag{14.35}$$

We will discuss the computation of the vector current Ward identity and will quote the result for the axial one. Computing $k_1^\alpha T_{\mu,\alpha\beta}$ we find

$$k_1^\alpha T_{\mu,\alpha\beta} = -i\int \frac{{\rm d}^4 l}{(2\pi)^4}\left(F_V(l) - F_V(l-k_1)\right), \tag{14.36}$$

where

$$F_V = \frac{{\rm Tr}\left[\gamma_\mu\gamma_5(\hat{l}-\hat{k}_2)\gamma_\beta\hat{l}\right]}{(l-k_2)^2 l^2}. \tag{14.37}$$

The integral in Eq. (14.36) can be unambigously computed using earlier discussion. We find

$$F_V(l) \approx \frac{4i\epsilon_{\mu\alpha\beta\sigma}k_2^\alpha l^\sigma}{(l^2)^2}. \tag{14.38}$$

Extracting the relevant vector a^μ and using Eq. (14.21), we find

$$k_1^\alpha T_{\mu;\alpha\beta} = -\frac{i}{8\pi^2}\epsilon_{\mu\beta\sigma\rho}k_1^\sigma k_2^\rho. \tag{14.39}$$

A similr computation for the axial current gives

$$q^\mu T_{\mu;\alpha\beta} = -\frac{i}{4\pi^2}\epsilon_{\alpha\beta\sigma\rho}k_1^\sigma k_2^\rho. \tag{14.40}$$

Hence, we find

$$\begin{aligned} k_1^\alpha T_{\mu;\alpha\beta}|_{\rm shift} &= -\frac{i}{8\pi^2}\epsilon_{\mu\beta\alpha\sigma}k_1^\alpha k_2^\sigma(1-(a_1-a_2)), \\ q^\mu T_{\mu;\alpha\beta}|_{\rm shift} &= \quad \frac{i}{4\pi^2}\epsilon_{\alpha\beta\rho\sigma}k_1^\rho k_2^\sigma(1+(a_1-a_2)). \end{aligned} \tag{14.41}$$

Eq.(14.41) contains *a very important result*: it shows that there is no choice of momentum flow in two triangle diagrams that enforces *simultaneous*

[1] We display one equation for the vector current; the second equation that is obtained by contracting $T_{\mu;\alpha\beta}$ with k_2^β contains no new information.

conservation of *both* the axial and the vector currents; the choice *which of the two currents is conserved* is up to us to make. We make this choice based on the observation that the vector current may couple to a gauge field (this is a typical situation in QED and QCD) and so its conservation is essential for the well-being of the theory. The conservation of the axial current was a nice feature of the theory to have, but if we cannot maintain this feature at the quantum level, so be it. Hence, we fix the momentum routing by choosing $a_1 - a_2 = 1$ and, from Eq.(14.41), obtain two results

$$\begin{aligned} k_1^\alpha T_{\mu;\alpha\beta} &= 0, \\ q^\mu T_{\mu;\alpha\beta} &= \frac{i}{2\pi^2}\epsilon_{\alpha\beta\rho\sigma}k_1^\rho k_2^\sigma. \end{aligned} \tag{14.42}$$

We now use this result in Eq.(14.8) and find

$$q^\mu \langle \gamma(k_1)\gamma(k_2)|J_{5\mu}|0\rangle = -\frac{ie^2}{2\pi^2}\epsilon_{\alpha\beta\rho\sigma}\epsilon_1^\alpha \epsilon_2^\beta k_1^\rho k_2^\sigma. \tag{14.43}$$

We can rewrite the r.h.s of this equation by introducing momentum representation of the field-strength tensor

$$f_{\mu\nu} = \epsilon_1^\mu k_1^\nu - \epsilon_1^\nu k_1^\mu. \tag{14.44}$$

We also introduce the dual tensor

$$\tilde{f}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}f^{\alpha\beta}. \tag{14.45}$$

Putting everything together, we obtain

$$\langle \gamma(k_1)\gamma(k_2)|q^\mu\ J_{5\mu}|0\rangle = \frac{ie^2}{4\pi^2}f_{1,\mu\nu}\tilde{f}_2^{\mu\nu}. \tag{14.46}$$

The above equation implies that the divergence of the axial current can be written as

$$\partial_\mu J_5^\mu = \frac{\alpha}{2\pi}F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{14.47}$$

where $\tilde{F}_{\mu\nu}$ is the dual field-strength tensor.

Comments:

- It should be clear from this discussion that understanding anomalies requires us to deal with divergent, poorly defined quantities. We have described one way to resolve this ambiguity. However, other approaches are possible. Two most popular ones are the Pauli-Villars and dimensional regularizations. Pauli-Villars automatically conserves vector current and violates conservation of the axial current. Dimensional regularization requires us to define matrix γ_5 in d-dimensions; one can do this in such a way that the vector current is conserved and the axial current is not.

- Eq.(14.47) is the *exact operator equation*; there are no higher order corrections to it (Adler-Bardeen-Jackiw theorem).

- In our discussion anomaly arises as the consequence of different *ultraviolet* behavior of correlators that involve vector and axial currents. However, anomaly can also be viewed as an *infrared effect*. To this end, imagine that we attempt to compute the imaginary part of the matrix element of the axial current through the dispersion relations. Then, the intermediate massless quarks are on the mass shell and no integration over the loop momentum is involved. Hence, computing $q^\mu \mathrm{Im} T_{\mu;\alpha\beta}$ gives zero which would imply that the imaginary part vanishes. In reality, the imaginary part is actually proportional to a delta function $\delta(q^2)$.

- We have made the choice to conserve the vector current and let the axial current to become anomalous because vector currents couple to gauge fields so that their conservation is essential. In more complex theories, such as the Standard Model of particle physics axial currents are gauged as well. It then becomes important to check if *gauged* axial currents are anomalous or not.