TTP2
Lecture 15

Kirill Melnikov
TTP KIT
January 18, 2024

15 Spontaneous symmetry breaking, Goldstone effect

Consider a theory of a single scalar field

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4}\phi^4. \tag{15.1}$$

As we know, the parameter m is the mass of particle-like excitations of the filed ϕ and λ is the self-coupling. The equation of motion reads

$$\left(\partial_\mu \partial^\mu + m^2\right)\phi = -\lambda \phi^3. \tag{15.2}$$

For small values of λ, we can neglect the right hand side in Eq.(15.2) and describe excitations of the field ϕ as plane waves

$$\phi \sim e^{-i\omega_k t + i\vec{k}\vec{x}}, \quad \text{with} \quad \omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}. \tag{15.3}$$

Plane-wave solution describe particles that propagate in space-time and, if we put the r.h.s. in Eq.(15.2) back into the equation of motion, interact with each other. It assumes, of course, that $m^2 > 0$. The minimal energy that can be stored in the field in this case can be found from the Hamiltonian

$$H = \int \mathrm{d}^3\vec{x} \left[\frac{1}{2}\left(\partial_t \phi\right)^2 + \frac{1}{2}\left(\vec{\nabla}\phi\right)^2 + \frac{m^2\phi^2}{2} + \frac{\lambda}{4}\phi^4\right]. \tag{15.4}$$

Since each term in the above euqation is positive, the minimal value of H corresponds to $\phi = 0$.

What happens if we change the sign of m^2, i.e. we take

$$m^2 = -\mu^2, \tag{15.5}$$

with $\mu^2 > 0$? If we do that, the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - V(\phi), \tag{15.6}$$

with

$$V(\phi) = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4. \tag{15.7}$$

Energy stored in the field is described by the Hamiltonian

$$H = \int \mathrm{d}^3\vec{x} \left[\frac{1}{2}\left(\partial_t \phi\right)^2 + \frac{1}{2}\left(\vec{\nabla}\phi\right)^2 - \frac{\mu^2\phi^2}{2} + \frac{\lambda}{4}\phi^4\right]. \tag{15.8}$$

It is clear that also in this case time- and space-independent fields ϕ still minimize the energy, but the value of ϕ that does this is different from zero. In fact, it corresponds to the minimum of the potential $V(\phi)$. We find it by computing

$$\frac{\partial V(\phi)}{\partial \phi} = 0 \quad \rightarrow \quad \phi_{\rm min} = \pm \phi_{\rm vac}, \quad \phi_{\rm vac} = \sqrt{\frac{\mu^2}{\lambda}}. \tag{15.9}$$

The energy of the vacuum is then

$$E_{\rm vac} = \Omega \left[-\frac{\mu^2}{2}\frac{\mu^2}{\lambda} + \frac{\lambda}{4}\frac{\mu^4}{\lambda^2} \right] = -\frac{\Omega \mu^4}{2\lambda}, \tag{15.10}$$

where $\Omega = \int {\rm d}^3\vec{x}$ is the space volume.

The important point is that if we want to describe small excitations of the field ϕ, we cannot construct such an expansion around $\phi = 0$. This is because, even for small λ, the equations of motion of the field ϕ around $\phi = 0$ is

$$\left(\partial_\mu \partial^\mu - \mu^2\right) \phi = 0. \tag{15.11}$$

The solutions to this equation are $\phi \sim e^{\pm \mu t}$, so that there is an exponentially growing field that "moves away" from $\phi = 0$ rather than oscillates around this value. To have "small oscillations around the vacuum", we need to consider values of the field that are close to $\phi = \pm \phi_{\rm vac}$.

It follows from Eq. (15.10) that there are two values of the field that minimize the field's energy, so an important question is which of the two minima should be considered? If this were quantum mechanics, the answer to this question is "neither" of the two because the ground state of a quantum-mechanical system with two minima is a symmetric wave function with maxima both at the left and at the right minima. The reason for this is the tunneling through a potential barrier; it connects the two minima and forces us to choose a symmetric wave function as a true ground state.

It is very important to understand that in quantum field theory we can choose one of the two ground states and we do not need to care about the tunneling phenomenon. To see why this is so, let us map the quantum field theory problem on a quantum-mechanical problem by considering fields that are $\vec{x}$-independent. Then, the action reads

$$S = \int {\rm d}t \, \left[\frac{\Omega}{2} \left(\partial_t \phi\right)^2 - \Omega V(\phi) \right], \tag{15.12}$$

where $\Omega = \int \mathrm{d}^3\vec{x}$ is the space volume where the field ϕ has a non-vanishing support. If we identify $\phi(t)$ with $x(t)$, we can view Eq.(15.12) as an action of a particle with the mass Ω and and the potential energy $\Omega V(\phi)$.

We can now compute the tunneling amplitude from one vacuum to the other vacuum using the quantum mechanical formulas

$$\mathcal{A}_{\rm tunnel} \sim e^{-\int p\mathrm{d}x}, \tag{15.13}$$

where the integration is performed through a region that is classically forbidden. In Eq. (15.13), $p \sim \sqrt{m|U|} \to \Omega\sqrt{|V(\phi)|}$ and $\mathrm{d}x \to \mathrm{d}\phi$. Hence, in the quantum field theory, the tunneling amplitude reads

$$\mathcal{A}_{\rm tunnel} \sim e^{-\Omega \int\limits_{-\phi_{\rm vac}}^{\phi_{\rm vac}} \sqrt{|V(\phi)|}\,\mathrm{d}\phi}. \tag{15.14}$$

Therefore, if we consider quantum field theory in an infinitely large volume $\Omega \to \infty$ the tunneling amplitude *vanishes*. For this reason, at variance with quantum mechanics, we *must* choose *one and only one* vacuum in a quantum field theory. However, which one it is – the "left" one $\phi = -\phi_{\rm vac}$ or the "right" one $\phi = +\phi_{\rm vac}$, we cannot decide; this happens by "accident" and should not have observable consequences.

Since the original Lagrangian Eq. (15.1) is invairant under $\phi \to -\phi$ symmetry, once one of the two vacua is chosen, the symmetry is broken. We refer to this mechanism of breaking the symmetry as *spontaneous symmetry breaking*. It is important to stress that the spontaneous symmetry breaking implies that the Lagrangian of a theory is symmetric but the ground state is not.

Let us imagine that the system has chosen the "right" vacuum where

$$\langle 0|\phi|0\rangle = \phi_{\rm vac}. \tag{15.15}$$

We then re-write the Lagrangian Eq.(15.6) using a new field χ that is defined as $\phi(x) = \phi_{\rm vac} + \chi(x)$. Since $\partial_\mu \phi_{\rm vac} = 0$, we obtain the new Lagrangian

$$L = \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - V(\phi_{\rm vac}) + \frac{1}{2}\left(\mu^2 - 3\lambda\phi_{\rm vac}^2\right)\chi^2 - \lambda\phi_{\rm vac}\chi^3 - \frac{\lambda}{4}\chi^4. \tag{15.16}$$

We use explicit expression for $\phi_{\rm vac}$ to simplify Eq.(15.16) and find

$$L = \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{2}m_\chi^2\chi^2 - \lambda\phi_{\rm vac}\chi^3 - \frac{\lambda}{4}\chi^4 - V(\phi_{\rm vac}), \tag{15.17}$$

where $m_\chi^2 = 2\mu^2$. Note that Eq.(15.17) describes a theory of a scalar self-interacting field with the mass m_χ^2. At variance to the original theory, there is nothing strange about the theory described by Eq.(15.17) anymore. In particular, the mass of the field χ is *positive*.

As the next step, we extend the original theory by considering larger number of fields that appear in the Lagrangian in a symmetric way. We consider two real fields ϕ_1, ϕ_2 and write them as a two-component vector

$$\vec{\phi} = \left(\begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right). \tag{15.18}$$

The Lagrangian reads

$$L = \frac{1}{2}\partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - V(\vec{\phi}\cdot\vec{\phi}), \tag{15.19}$$

where

$$V(\vec{\phi}\cdot\vec{\phi}) = -\frac{\mu^2}{2}\vec{\phi}\cdot\vec{\phi} + \frac{\lambda}{4}\left(\vec{\phi}\cdot\vec{\phi}\right)^2. \tag{15.20}$$

The Lagrangian has the $\mathcal{O}(2)$ symmetry; if we rotate $\vec{\phi}$ with a 2×2 orthogonal matrices

$$\vec{\phi} = \hat{R}\vec{\phi}', \quad \hat{R}^T R = 1, \tag{15.21}$$

we find

$$L(\vec{\phi}) = L(\vec{\phi}'). \tag{15.22}$$

Since the potential energy $V(\phi)$ depends on the "length" of the vector $\vec{\phi}$ only, we can read off the value of the field that minimizes $V(\vec{\phi}\cdot\vec{\phi})$ from the calculation at the beginning of this lecture. We find

$$\vec{\phi}_{\rm vac}\cdot\vec{\phi}_{\rm vac} = \phi_{1,\rm vac}^2 + \phi_{2,\rm vac}^2 = \frac{\mu^2}{\lambda}. \tag{15.23}$$

It follows from Eq.(15.23) that the "vacuum manifold" is a circle with the radius $|\vec{\phi}_{\rm vac}| = \phi_{\rm vac} = \sqrt{\mu^2/\lambda}$. In contrast to the single-field case, we parameterize the vacuum field by writing

$$\vec{\phi}_{\rm vac} = \phi_{\rm vac}\vec{e}_{\rm vac}, \tag{15.24}$$

where

$$\vec{e}_{\rm vac} = \left(\begin{array}{c} \cos\theta \\ \sin\theta \end{array} \right). \tag{15.25}$$

Eqs.(15.24,15.25) describe a particular choice of the vacuum. To construct an expansion around the vacuum field, we write

$$\vec{\phi} = \vec{\phi}_{\rm vac} + \vec{\chi}. \tag{15.26}$$

Since $\vec{\phi}_{\rm vac}$ is a constant field, it follows

$$\partial_\mu \vec{\phi} = \partial_\mu \vec{\chi}. \tag{15.27}$$

We would like to express the Lagrangian Eq.(15.17) through the field $\vec{\chi}$. To do that, we note that it is convenient to write $\vec{\phi}$ as a sum of two vectors

$$\vec{\chi} = h\vec{e}_{\rm vac} + \chi_\perp \vec{e}_\perp, \tag{15.28}$$

where $\vec{e}_{\rm vac} \cdot \vec{e}_\perp = 0$. Then

$$V(\vec{\phi}\cdot\vec{\phi}) = V((\phi_{\rm vac}+h)^2 + \chi_\perp^2). \tag{15.29}$$

Using explicit form of the potential Eq.(15.20), we find

$$V(\vec{\phi}\cdot\vec{\phi}) = -\frac{\mu^2}{2}\left[(\phi_{\rm vac}+h)^2+\chi_\perp^2\right] + \frac{\lambda}{4}\left((\phi_{\rm vac}+h)^2+\chi_\perp^2\right)^2. \tag{15.30}$$

It is easy to analyze this potential energy to arrive at the following conclusions:

- there are two fields h and $\chi_\perp$ in the Lagrangian after the symmetry breaking;
- the mass of the field h is $2\mu^2$, similar to the single-field case;
- the mass of the field $\chi_\perp$ is zero;
- there are interactions between h and $\chi_\perp$.
- nothing depends on the chosen vacuum state that is characterized by the vector $\vec{e}_{\rm vac}$. The dependence on that vector disappeared completely.

Massless excitations of fields that appeared in the theory after the spontaneous symmetry breaking are known as *Nambu-Goldstone bosons*; in our case these massless appear once we quantize the field $\chi_\perp$. We see that in the theory described by the Lagrangian Eq. (15.19) there is one Goldstone boson after the symmetry breaking.

We will now do the same calculation using a different parameterization of the field $\vec{\phi}$. This is important since choosing a different parameterization offers a different perspective on the Nambu-Goldstone mechanism. We write the field as

$$\vec{\phi}(x) = \rho(x) \begin{pmatrix} \cos\alpha(x) \\ \sin\alpha(x) \end{pmatrix}, \tag{15.31}$$

which means that we have chosen "spherical" coordinates in field space. The potential energy is then

$$V(\vec{\phi}\cdot\vec{\phi}) = V(\rho^2). \tag{15.32}$$

To compute the kinetic energy stored in the field ϕ we calculate the derivative

$$\partial_\mu \vec{\phi} = (\partial_\mu \rho) \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} + \rho(\partial_\mu \alpha) \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}, \tag{15.33}$$

and find

$$\frac{1}{2}\partial_\mu \vec{\phi}\cdot\partial^\mu \vec{\phi} = \frac{1}{2}\partial_\mu \rho \partial^\mu \rho + \frac{\rho^2}{2}\partial_\mu \alpha \partial^\mu \alpha. \tag{15.34}$$

Again, to account for the spontaneous symmetry breaking, we write

$$\rho = \phi_{\text{vac}} + r. \tag{15.35}$$

The Lagrangian becomes

$$L = \frac{1}{2}\partial_\mu r \partial^\mu r + \frac{\phi_{\text{vac}}^2}{2}\partial_\mu \alpha \partial^\mu \alpha - \frac{(2\mu^2)}{2} r^2 + \phi_{\text{vac}} r \partial_\mu \alpha \partial^\mu \alpha + \cdots \tag{15.36}$$

Ellipses in Eq.(15.36) refer to terms that describe interactions between different fields. We observe from Eq.(15.36) that the "angular" variable α describes massless field whose excitations are Goldstone bosons and the "radial variable" r describes a massive field with the mass $2\mu^2$. We note that all terms in the Lagrangian that involve the field α are proportional to $\partial_\mu \alpha$. Since in the momentum space $\partial_\mu \alpha \sim \sum p_\mu \alpha$, this means that the interactions of Goldstone fields are energy-dependent and become *weak* at low energies. This feature is not apparent if the Goldstone field is described by the field $\chi_\perp$, as in the previous example.

We generalize the construction to three fields and a Lagrangian that is symmetric under $SO(3)$ transformations. We again use Eq.(15.19) but this time the field $\vec{\phi}$ is a triplet

$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}. \tag{15.37}$$

Similar to the discussion of the two-component vector, I write

$$\vec{\phi} = (\phi_{\text{vac}} + h)\vec{e}_{\text{vac}} + \vec{\chi}_{\perp}, \tag{15.38}$$

where

$$\vec{e}_{\text{vac}} = \begin{pmatrix} \sin\theta_0 \cos\phi_0 \\ \sin\theta_0 \sin\phi_0 \\ \cos\theta_0 \end{pmatrix}, \tag{15.39}$$

and $\vec{\chi}_{\perp} \cdot \vec{e}_{\text{vac}} = 0$, so that $\vec{\chi}_{\perp}$ is a two-component field. If we use this representation in the formula for the potential energy, we obtain

$$V(\vec{\phi}\cdot\vec{\phi}) = -\frac{\mu^2}{2}\left[(\phi_{\text{vac}} + h)^2 + \vec{\chi}_{\perp}^2\right] + \frac{\lambda}{4}\left((\phi_{\text{vac}} + h)^2 + \vec{\chi}_{\perp}^2\right)^2. \tag{15.40}$$

As we already remarked after Eq.(15.30), this form of the potential energy implies that $\vec{\chi}_{\perp}$ describes two massless fields (Goldstone bosons), whereas h is a massive field with the mass squared being equal to $2\mu^2$. It should be now obvious that if we consider a theory of N fields that is invariant under $SO(N)$ transformations, and potential energy in this theory only depends on the radial component of the vector field, we will get $N-1$ (massless) Goldstone bosons after the symmetry breaking.

To understand how many Goldstone bosons arise in the theory after the symmetry breaking, consider the field $\vec{\phi}$ that describes N fields. After the symmetry breaking, we write it as

$$\vec{\phi} = (\phi_{\text{vac}} + h)\vec{e}_{\text{vac}} + \vec{\chi}_{\perp}, \tag{15.41}$$

where the field $\vec{\chi}_{\perp}$ describes $N-1$ fields that span the $(N-1)$-dimensional space D_{vac} that is orthogonal to $\vec{e}_{\text{vec}}$. Since the potential energy only depends on $\vec{\chi}_{\perp}^2$, the theory is still invariant under $(N-1)$-rotations in D_{vac}. We then say that the symmetry is broken from $SO(N)$ to $SO(N-1)$. We note that group $SO(N)$ allows $G_N = N(N-1)/2$ "independent rotations", that correspond to Lie algebra generators. After the symmetry breaking, the symmetry group becomes $SO(N-1)$, so some of the original symmetry transformations are not symmetry transformations anymore. The number of such "broken" symmetry transformations reads

$$G_N - G_{N-1} = N - 1. \tag{15.42}$$

This is exactly the number of massless particles that we have been finding in our examples.

We will now explain why this isn't a coincidence and that, indeed, the number of Goldstone bosons equals to the number of broken symmetries *in any theory*. To this end, consider a theory with the interaction potential $V(\vec{\phi})$. The theory is invariant under a symmetry that is described by generators T^a, $a = 1..N_a$. Hence, if we consider an infinitesimal transformation

$$\vec{\phi}' = \vec{\phi} + \epsilon_a T^a \vec{\phi}, \tag{15.43}$$

the potential energy computed for $\vec{\phi}'$ and $\vec{\phi}$ should be the same

$$V(\vec{\phi} + \epsilon_a T^a \vec{\phi}) = V(\vec{\phi}). \tag{15.44}$$

Expanding the left hand side to first order in ϵ, we find

$$0 = \epsilon_a \frac{\partial V}{\partial \phi_i} T^a_{ik} \phi_k. \tag{15.45}$$

Since different ϵ_a's parameterize independent symmetry transformations, Eq.(15.45), in fact, splits into N_a independent equations

$$0 = \frac{\partial V}{\partial \phi_i} T^a_{ik} \phi_k, \tag{15.46}$$

one for every symmetry generator.

We now take a derivative of Eq.(15.46) with respect to ϕ_m. We obtain

$$0 = \frac{\partial V}{\partial \phi_i \partial \phi_m} T^a_{ik} \phi_k + \frac{\partial V}{\partial \phi_i} T^a_{im}. \tag{15.47}$$

Eq.(15.47) holds for any $\vec{\phi}$. However, it is instructive to apply it at $\vec{\phi} = \vec{\phi}_{\rm vac}$. Since $\vec{\phi}_{\rm vac}$ minimizes the potential, the last term in Eq.(15.47) vanishes and we obtain

$$0 = \frac{\partial V}{\partial \phi_i \partial \phi_m}\Big|_{\vec{\phi}=\vec{\phi}_{\rm vac}} \; T^a_{ik} \; \phi_{{\rm vac},k}. \tag{15.48}$$

To understand the meaning of this equation, consider a generic Lagrange function

$$L = \frac{1}{2}\partial_\mu\vec{\phi}\partial^\mu\vec{\phi} - V(\vec{\phi}), \tag{15.49}$$

and assume that spontaneous symmetry breaking occurs. We then write $\vec{\phi} = \vec{\phi}_{\rm vac} + \vec{\chi}$ and expand around $\vec{\chi} = 0$. We find

$$L = \frac{1}{2}\partial_\mu\vec{\chi}\partial^\mu\vec{\chi} - V(\vec{\phi}_{\rm vac}) - \frac{\partial V}{\partial\phi_i}|_{\vec{\phi}=\vec{\phi}_{\rm vac}}\,\chi_i - \frac{1}{2}\frac{\partial V}{\partial\phi_i\partial\phi_j}|_{\vec{\phi}=\vec{\phi}_{\rm vac}}\,\chi_i\chi_j + \ldots \tag{15.50}$$

Since the potential $V(\vec{\phi})$ has a minimum at $\vec{\phi} = \vec{\phi}_{\rm vac}$, the right hand side of Eq.(15.50) simplifies. We write

$$L = \frac{1}{2}\partial_\mu\vec{\chi}\partial^\mu\vec{\chi} - V(\vec{\phi}_{\rm vac}) - \frac{1}{2}m_{ij}^2\,\chi_i\chi_j + \ldots, \tag{15.51}$$

where

$$m_{ij}^2 = \frac{\partial V}{\partial\phi_i\partial\phi_j}|_{\vec{\phi}=\vec{\phi}_{\rm vac}} \tag{15.52}$$

is the *mass matrix*. The name comes from the fact that, upon diagonalising it, we get the information about masses of particles that our theory describes.

We note that this matrix also appears in Eq.(15.48) that we write in the following way

$$0 = m_{ij}\xi_j^{(a)}, \tag{15.53}$$

where $\vec{\xi}^{(a)} = T^a\vec{\phi}_{\rm vac}$. Clearly, $\vec{\xi}^{(a)}$ is what you get if you act on a vacuum field by a generator of a symmetry transformation $T^{(a)}$.

According to Eq.(15.53) when the mass matrix multiplies *any* $\vec{\xi}^{(a)}$, the result should be zero, however, this can be realized in two ways. If, for a particular generator T^a, $\vec{\xi}^{(a)} = 0$, Eq.(15.53) does not provide any useful information. However, if $\vec{\xi}^{(a)} \neq 0$, Eq.(15.53) implies that the mass matrix has a non-trivial eigenvector with *zero eigenvalue*, i.e. zero mass squared. The number of such eigenvectors is equivalent to the number of symmetries (number of generators) that *do not* leave the vacuum $\vec{\phi}_{\rm vac}$ unchanged, since $T^a\vec{\phi}_{\rm vac} \neq 0$. Hence, *for each broken symmetry, there is a massless mode that is a Nambu-Goldstone boson*.

Finally, we note that in the above discussion we dealt with classical Lagrangians. We now show that one can reformulate the above discussion in a quantum language. To this end, consider a quantum field theory with a continuous global symmetry. This symmetry implies that there is a number of conserved currents; we will consider one of them and call it $J^\mu(x)$. Since the current is conserved, it satisfies

$$\partial_\mu J^\mu = 0. \tag{15.54}$$

Given the current, we can construct a conserved charge

$$Q(t) = \int \mathrm{d}^3\vec{x}\; J^0(t,\vec{x}), \qquad \frac{\mathrm{d}Q}{\mathrm{d}t} = 0. \tag{15.55}$$

Suppose, we consider a commutator of one of the fields in the theory and the charge Q. Since Q is a generator of symmetry transformations, we will get

$$\phi_a(x) = [Q, \phi_b(x)]. \tag{15.56}$$

We then take the vacumm expectation values of the two sides and assume that the field $\phi_a(x)$ develops a non-vanishing vacuum expectation value

$$v_a = \langle 0|\phi_a(x)|0\rangle = \langle 0|Q\phi_b(x) - \phi_b(x)Q|0\rangle. \tag{15.57}$$

It we now assume that the current is Hermitian, the above equation implies that

$$Q|0\rangle \neq 0, \tag{15.58}$$

which means that *the vacuum state is not invariant under a symmetry transformation that is described by the charge* Q.

Next, consider a correlator

$$\Pi^\mu(q) = -i\int \mathrm{d}^4x e^{iqx}\langle 0|TJ^\mu(x)\phi_b(0)|0\rangle, \tag{15.59}$$

and compute

$$\begin{aligned}
q_\mu\Pi^\mu(q) &= -i\int \mathrm{d}^4x\; q_\mu e^{iqx}\langle 0|TJ^\mu(x)\phi_b(0)|0\rangle \\
&= -\int \mathrm{d}^4x(\partial_\mu e^{iqx})\langle 0|TJ^\mu(x)\phi(0)|0\rangle \\
&= \int \mathrm{d}^4x e^{iqx}\partial_\mu\langle 0|TJ^\mu(x)\phi_b(0)|0\rangle \\
&= \int \mathrm{d}^3x e^{-i\vec{q}\cdot\vec{x}}\langle 0|[J^0(0,\vec{x}),\phi_b(0)]|0\rangle.
\end{aligned} \tag{15.60}$$

The above expression is simplified if we consider the limit $\vec{q} \to 0$. We then use the fact that the integral of J^0 over volume is the charge operator Q and the vacuum expectation value of the commutator of Q with ϕ_b is non-vanishing. Hence,

$$\lim_{q\to 0} q_\mu \Pi^\mu(q) = \langle 0|[Q, \phi_b(0)]|0\rangle = \langle 0|\phi_a(0)|0\rangle = v_a. \tag{15.61}$$

The solution of this equation is

$$\Pi^\mu(q) = \frac{q^\mu v_a}{q^2}. \tag{15.62}$$

The pole at $q^2 = 0$ implies that there is a massless particle in the spectrum. Hence, for each symmetry generator of a continuous global symmetry which is broken spontaneously (which mathematically implies that $Q|0\rangle \neq 0$), there is a massless Goldstone boson in the spectrum of the theory.

To see how Goldstone theorem is used in practice, we consider a QCD Lagrangian with two quark types (flavors): up and down. We assum these quarks to be massless; this actually does not mean that these quarks *must* be massless but, rather, that their masses should be small as compared to $\Lambda_{\rm QCD}$. The Lagrangian reads

$$L = \sum_{i=1}^{2} \bar{q}_i i\hat{D} q_i + \cdots, \tag{15.63}$$

where ellipses describe contributions to the Lagrangian which depend on gauge fields and ghost fields. We will write the two fields as a spinor (of spinors)

$$\Psi = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \tag{15.64}$$

where q_1 describe an up-quark and q_2 – the down-quark fields. The Lagrangian L becomes

$$L = \bar{\Psi} i\hat{D}\Psi + \cdots, \tag{15.65}$$

We can now introduce lef and right projections to write

$$\Psi_L = \frac{(1-\gamma_5)}{2}\Psi_L, \quad \Psi_R = \frac{(1-\gamma_5)}{2}\Psi_R, \tag{15.66}$$

to rewrite the QCD Lagagian as

$$L = \bar{\Psi}_L i\hat{D}\Psi_L + \bar{\Psi}_R i\hat{D}\Psi_R + \cdots \, . \qquad (15.67)$$

This Lagrangian is invariant under independent rotations of left- and right-fields with unitary matrices, so that the full symmetry group of the classical Lagrangian is

$$SU(2)_L \otimes SU(2)_R \otimes U(1)_L \otimes U(1)_R. \qquad (15.68)$$

Using two $U(1)$ symmetries, we can form vector and axial transformations. The "vector" symmetry is the conservation of the so-called baryon (total quark number) number (which means that if we assign a baryon charge 1 u and d quarks and -1 to $\bar{u}$ and $\bar{d}$ quarks, then this "charge" is conserved). The axial symmetry, as we know, is anomalous and, therefore, not a symmetry at all. Finally, it is known experimentally, the $SU(2)_L \otimes SU(2)_R$ symmetry is spontanesouly broken to the diagonal subgroup of $SU(2)_{L+R}$. Since there are six generators of the original group and three of the diagonal subgroup, three symmetry generators are "broken". Hence, one expects to find three massless Goldsone bosons in the spectrum of hadrons. One indeed finds them; they are called "pions" and they are not massless (but light). The assumption then is that their masses are due to the fact that quark masses do not really vanish but are rather small (numerically, a pion has a mass of about 140 MeV whereas up and down quarks have masses of just a few MeV.